Patterned non-determinism in communication complexity

(Results and applications)

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In 2022 the speaker changed the English spelling of his first name from the previous russian-odoured form *"Dmitry"* to the Ukrainian *"Dmytro"*.

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- We can often compare the "*strength*" of two communication regimes via presenting a problem with an efficient solution in one, but not in the other. This can lead to non-trivial unconditional *structural separations* – that is to statements that certain tasks are *efficiently solvable* in one regime of communication but not in the other.

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- During *this talk* we will define and investigate a new model of ***non-deterministic communication***, which we will call ***patterned non-determinism*** (*PNP*).

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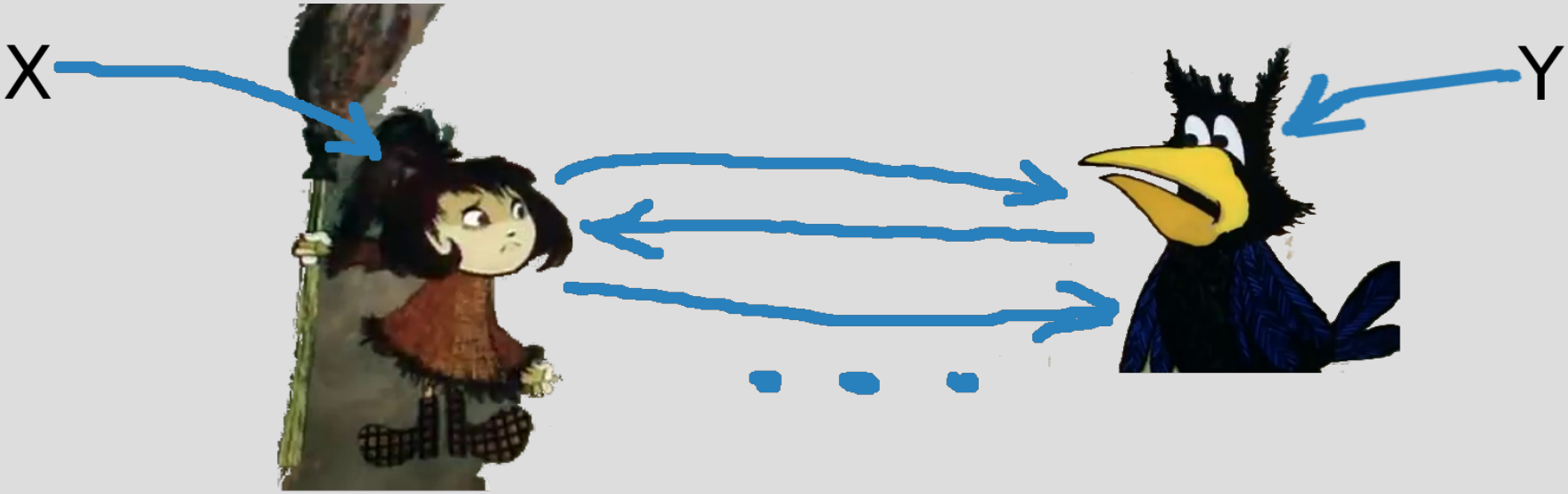

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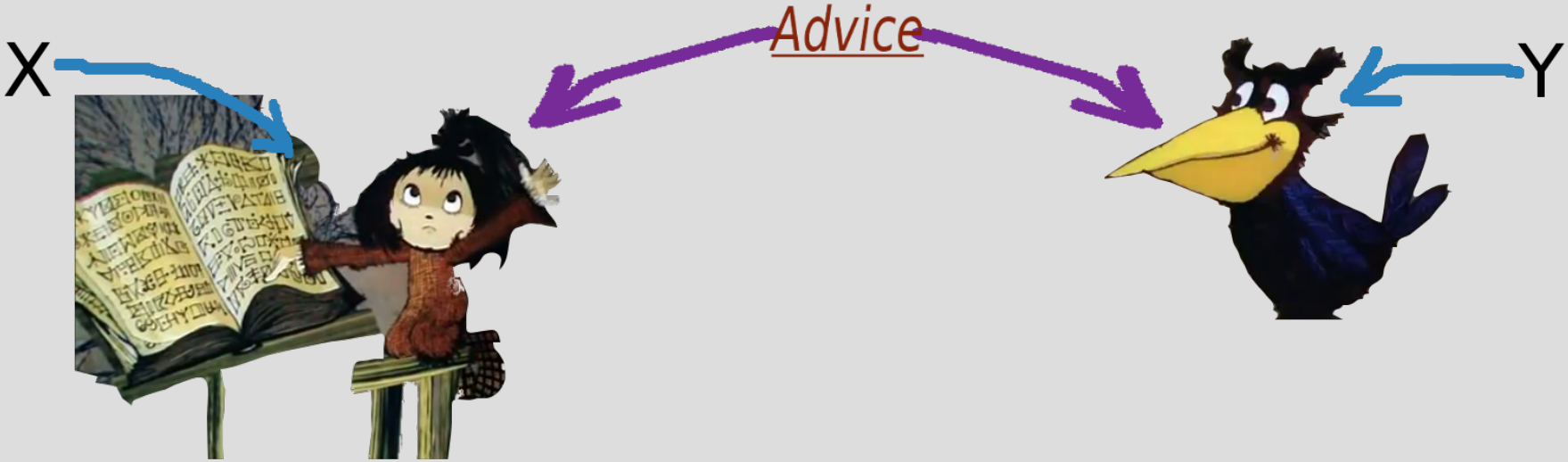

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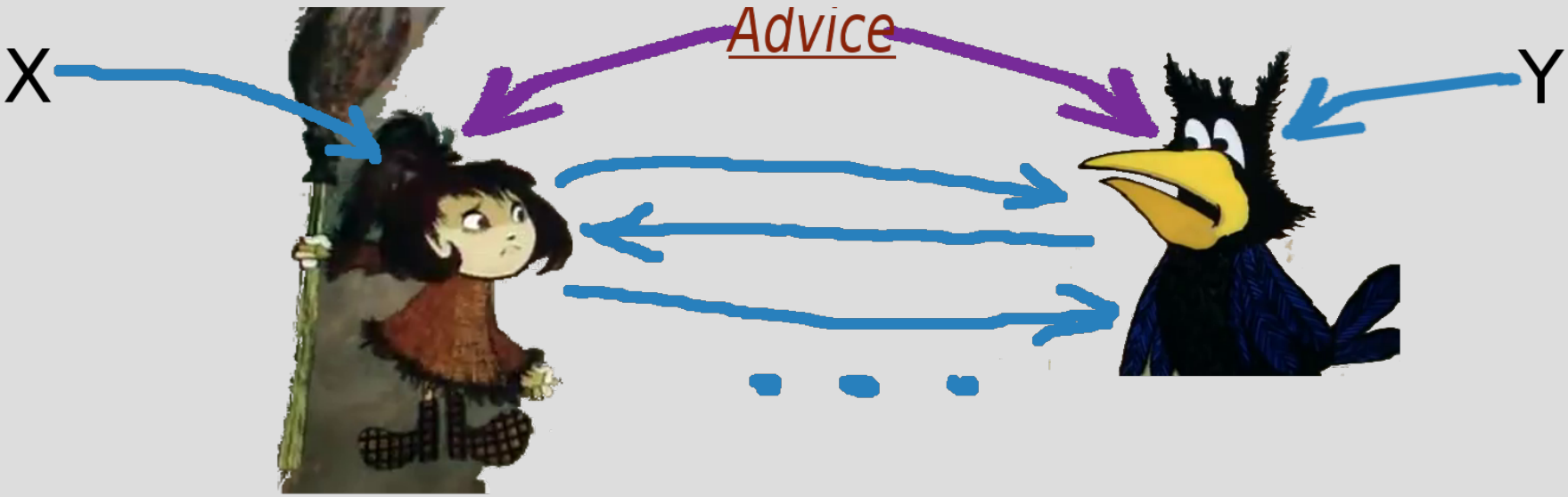

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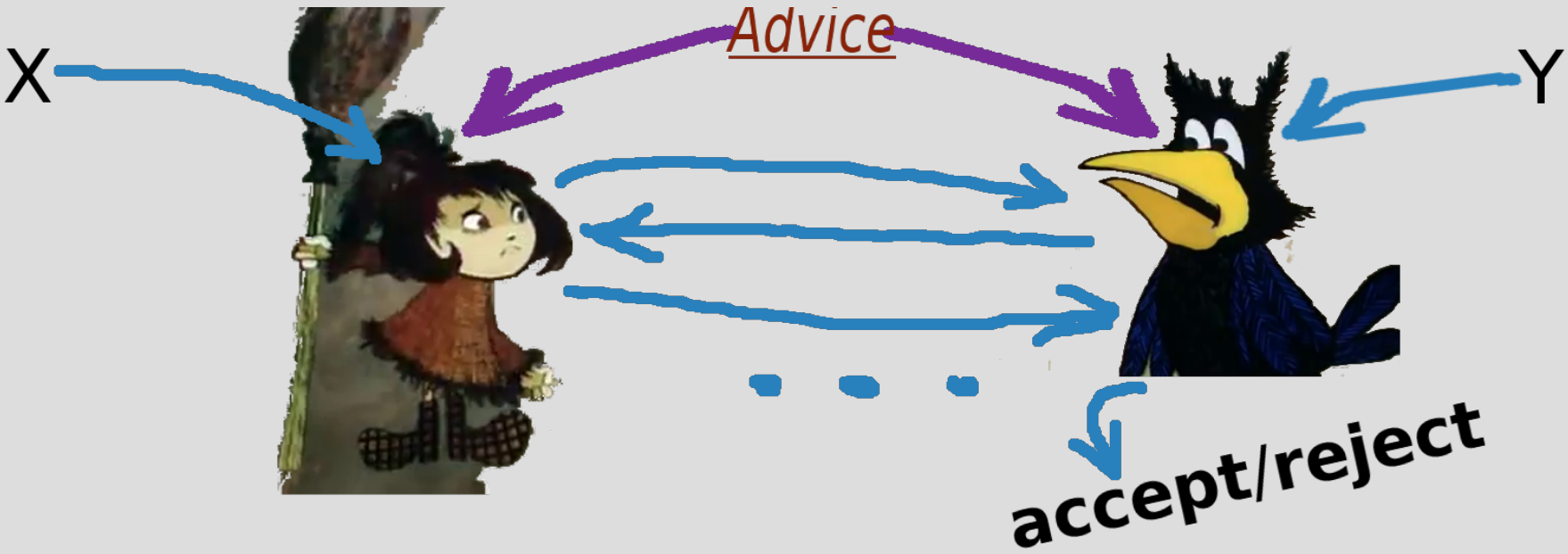

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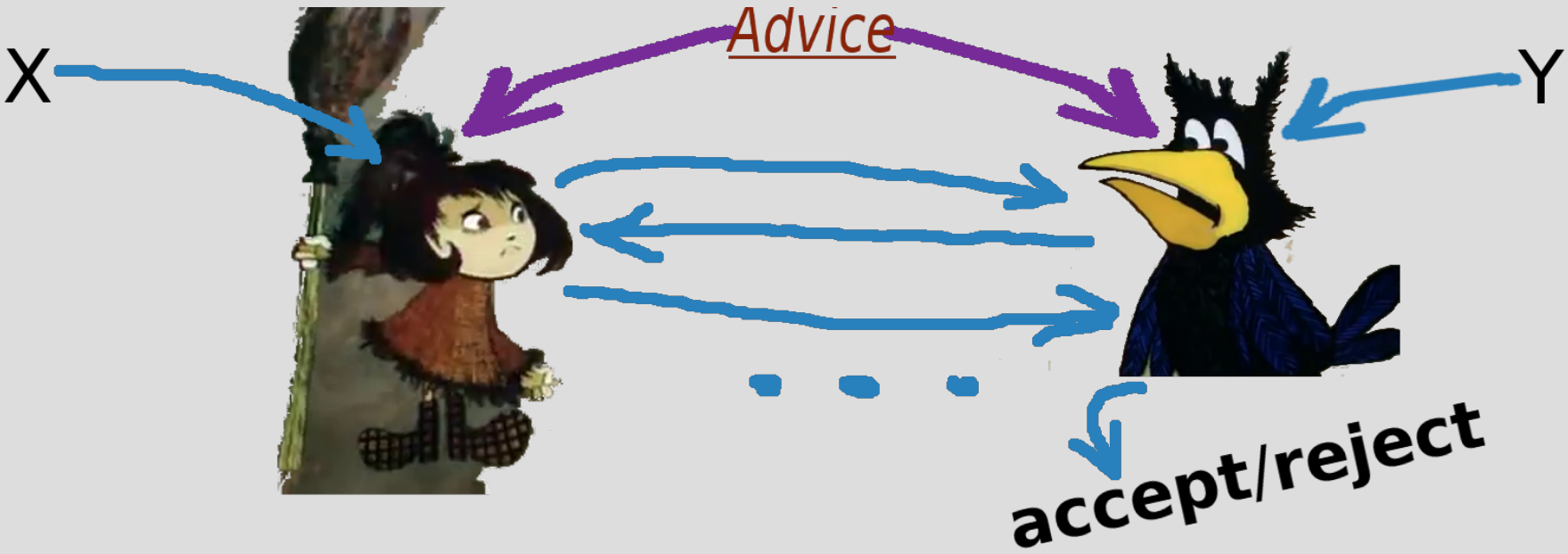

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The input pair is accepted by a non-deterministic protocol if *at least one advice value* leads to its acceptance.

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  $$f(x, y) \equiv \bigvee_{i=1}^{m} f_i(x, y),$$

  where *m is at most quasi-polynomial in n* and *every f_i has an efficient deterministic protocol*: In this case a legitimate *advice* for input (x, y) would be any index i_0 such that $f_{i_0}(x, y) = \top$.

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- Clearly, *non-deterministic protocols are at least as strong as deterministic ones*; on the other hand, there are *functions with very efficient non-deterministic protocols but with deterministic complexity in $\Omega(n)$*.

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  - For $f : \{0,1\}^n \times \{0,1\}^n \to \{\top, \bot\}$ with an NP-decomposition

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    let the corresponding family of **accepting patterns** be defined as

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  - Then $f \in PNP$ if $|\Gamma_f|$ is *at most quasi-polynomial in* n.

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  - *"Totality" is crucial* for these model equivalences: for *partial functions* even $UP \neq P$.

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- We shall see next how the above statement leads to certain (possibly, surprising) model equivalence in *multi-party communication complexity*.

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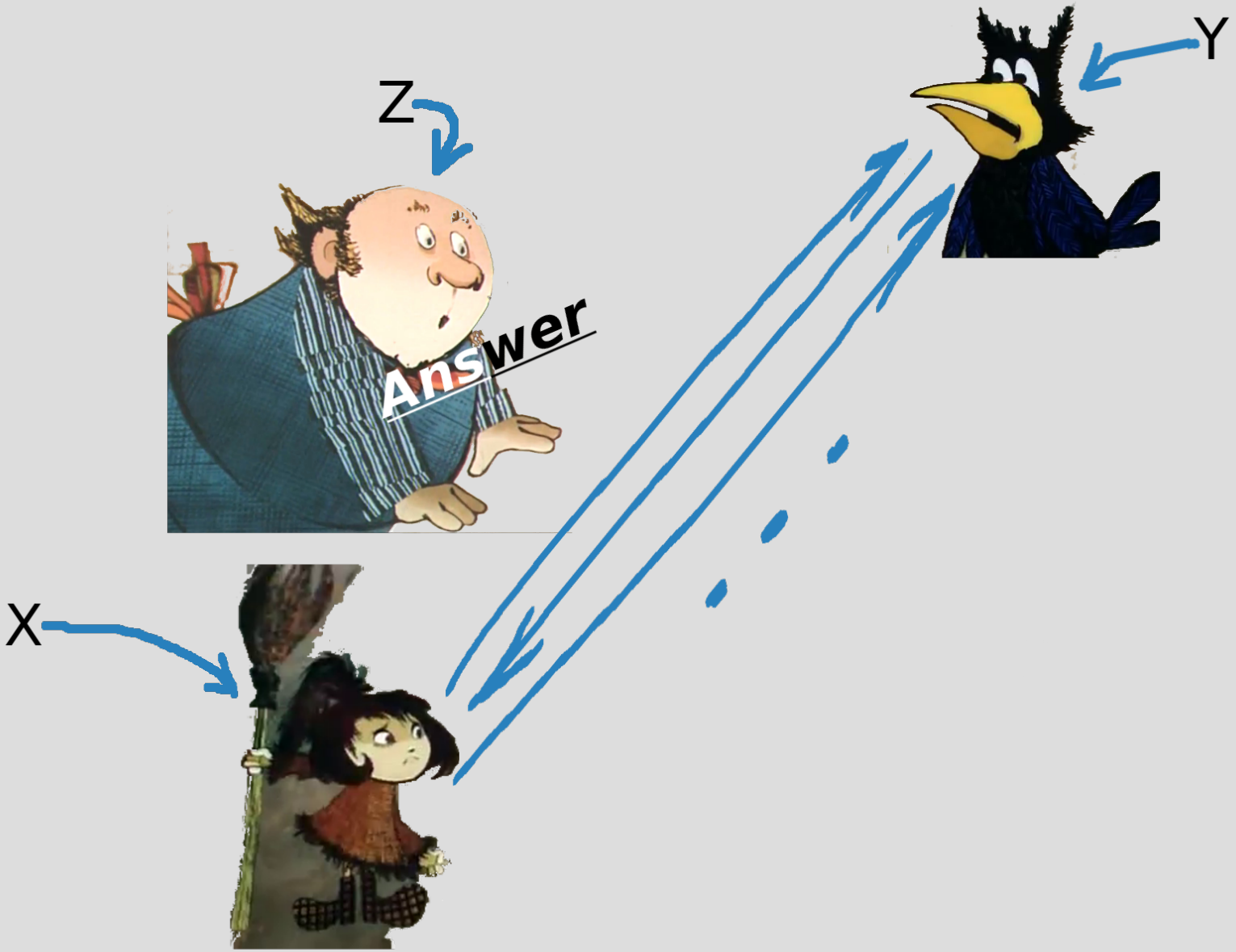

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The argument will be based on the possibility of *efficient pattern searching* in (bipartite) *PNP*.

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Cheating alert: $|\mathcal{Z}|$ can be as large as 2^n (easy to fix).

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- The desired existence of *an efficient protocol for* $f(X, Y, Z)$ *in the model* $[(X \leftrightarrow Y) \rightarrow Z]$ follows.

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