# BOUNDEDNESS OF BIORTHOGONAL SYSTEMS IN BANACH SPACES

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Abstract. We prove that every Banach space that admits a Markushevich basis also admits a bounded Markushevich basis.

## 1. Introduction

A *Markushevich basis* (in short, an *M-basis*) for a Banach space $X$ is a biorthogonal system $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ in $X \times X^*$ such that $\{x_\gamma : \gamma \in \Gamma\}$ is *fundamental*, i.e., linearly dense in $X$, and $\{f_\gamma : \gamma \in \Gamma\}$ is *total*, i.e., $w^*$-linearly dense in $X^*$. The *boundedness constant* of the system is $\sup\{\|x_\gamma\|.\|f_\gamma\| : \gamma \in \Gamma\}$ (eventually $+\infty$). If the boundedness constant of an M-basis is a finite number $K$, we speak of a *$K$-bounded M-basis.* The main results of this note is the construction of a $(2(1+\sqrt{2})+\varepsilon)$-bounded M-basis (for every $\varepsilon > 0$) in every nonseparable Banach space which admits an M-basis.

The boundedness problem for an M-basis (or more generally a biorthogonal system) has received attention in the work of many mathematicians. In the separable case, Davis and Johnson [DJ73] (building up on the work of Singer [S73]) constructed a $(1+\varepsilon)$-bounded fundamental system, an essentially optimal result for fundamental systems (see, e.g., [HMVZ, Corollary 1.26]). An important ingredient in their work was the use of Dvoretzky's theorem on almost Euclidean sections. Their ideas were developed further by Ovsepian and Pełczyński [OP75], who constructed a bounded M-basis in every separable Banach space. Ultimately, Pełczyński [Pe76] and Plichko [Pl77] independently, constructed a $(1+\varepsilon)$-bounded M-basis in every separable Banach space. The existence in every separable Banach space of a 1-bounded M-basis (i.e., an *Auerbach basis*) is still open.

In non-separable spaces, the existence of a bounded M-basis (provided the space has some M-basis) was claimed by Plichko [Pl82]. His method yields a boundedness constant roughly 10 (see, e.g., [HMVZ, Theorem 5.13]). However, the proof of this result in [Pl82] (and its reproduction in [HMVZ], Theorem 5.13) is flawed. The (subtle) troublesome point in the proof (see in [HMVZ] the claim on page 171, line 10 from below; we follow the notation there) is that $\mathrm{span}\{x_\alpha : \alpha \in J_{\gamma+2} \setminus J_{\gamma-1}\}$ is dense in $G_\gamma^\perp \cap X$. This claim (and thus the statement of Plichko's theorem) is true whenever the original M-basis is strong, but it is false in general (see [HMVZ], Proposition 1.35.). Let us recall that an M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ is called *strong* if, for every $x \in X$, $x \in \overline{\mathrm{span}}\{\langle x, f_\gamma\rangle x_\gamma : \gamma \in \Gamma\}$. The class of Banach spaces having a strong M-basis is quite large. For example, *every Banach space belonging to a*

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*$\mathcal{P}$-class has a strong M-basis* [HMVZ, Theorem 5.1]. We recall here that a class $\mathcal{C}$ of Banach spaces is a *$\mathcal{P}$-class* if, for every $X \in \mathcal{C}$, there exists a projectional resolution of the identity $(P_\alpha)_{\omega \leq \alpha \leq \mu}$ (where $\mu$ is the first ordinal with cardinal $\operatorname{dens} X$) such that $(P_{\alpha+1} - P_\alpha)X \in \mathcal{C}$ for all $\alpha \in [\omega, \mu)$. The class of all weakly compactly generated (resp. weakly countably determined, resp. weakly Lindelöf determined) Banach spaces is a $\mathcal{P}$-class.
However, there exists a Banach space with an M-basis admitting no strong M-basis ([HMVZ], Prop. 5.5).

Our approach to the problem uses ideas from several of the above mentioned papers, including [Pl82]. The essential new ingredient is the use of the $\Delta$-system lemma (see Lemma 2), which solves the difficulties in [Pl82]. We are also able to reduce the boundedness constant, by incorporating Dvoretzky's theorem together with the Walsh-matrices-mixing technique used in [OP75].
In the special case of WCG spaces, an adaptation of the proof in the separable case by Plichko leads to a constant $2 + \varepsilon$ (for every $\varepsilon > 0$) [Pl79], which is essentially optimal ([Pl86]).
This alternative approach uses the existence of many projections in the WCG space. In the end of our note we indicate how to obtain a (more or less formal) generalization of the $2 + \varepsilon$ result for wider classes of Banach spaces ($\mathcal{P}$-classes). We refer to [HMVZ] for more results and references related to boundedness of biorthogonal systems.

Our notation is standard. $B_X$ is the closed unit ball of a Banach space $X$, $S_X$ its unit sphere. Given a non-empty subset $S$ of a Banach space, let $\operatorname{span} S$ be the linear span of $S$, and $\operatorname{span}_{\mathbb{Q}} S$ the set of all linear combinations with rational coefficients of elements in $S$. The closed linear span of $S$ is denoted $\overline{\operatorname{span}} S$. Given two subspaces $F$ and $G$ of a Banach space $X$, we put $F \hookrightarrow G$ if $F$ is a subspace of $G$. We denote by $|S|$ the cardinality of a set $S$. The *density character* of $X$, $\operatorname{dens} X$, is the smallest ordinal $\Omega$ such that $X$ has a dense subset with cardinal $|\Omega|$. We identify, as usual, an ordinal number $\Omega$ with the segment $[0, \Omega)$, and a cardinal number with the initial ordinal having this cardinality. The ordinal number of $\mathbb{N}$ is denoted by $\omega$ and its cardinal number by $\aleph_0$. If $\{x_\gamma; f_\gamma\}_{\gamma \in \Gamma}$ is an M-basis for $X$ and $x \in X$, the *support* of $x$ (with respect to the M-basis) is the set $\operatorname{supp}(x) := \{\gamma \in \Gamma : \langle x, f_\gamma \rangle \neq 0\}$. Analogously, if $f \in X^*$, $\operatorname{supp}(f) := \{\gamma \in \Gamma : \langle x_\gamma, f \rangle \neq 0\}$.

For convenience, we formulate the main tools used in the proof of our theorem.

**Theorem 1** (Dvoretzky)**.** *Let $N \in \mathbb{N}$, $\varepsilon > 0$. Then there exists a natural number $K := K(N, \varepsilon)$, such that for every Banach space $(X, \|\cdot\|)$ of dimension at least $K$, there exists a linear space $Y \hookrightarrow X$ of dimension $N$, which is $(1+\varepsilon)$-isomorphic to $\ell_2^N$.*

A family $\{A_\lambda\}_{\lambda \in \Lambda}$ of sets is called a $\Delta$-*system* (with *root* $B$, possibly empty) if $A_\lambda \cap A_\alpha = B$ for all distinct $\lambda, \alpha \in \Lambda$.

**Lemma 2** ($\Delta$-system lemma, see, e.g., [Ju80], Lemma 0.6)**.** *Let $\Lambda > \omega$ be a regular cardinal and $\{A_\lambda\}_{\lambda \in \Lambda}$ a family of finite subsets of $\Lambda$. Then there exists a subfamily $\Omega \subset \Lambda$ of cardinality $\Lambda$ that is a $\Delta$-system.*

By a more or less standard argument, we obtain the next mild strengthening of the previous result.

**Corollary 3.** *Let $\Lambda > \omega$ be a regular cardinal, $X$ a Banach space with an M-basis $\{x_\gamma; f_\gamma\}_{\gamma \in \Gamma}$, $\{v_\lambda\}_{\lambda \in \Lambda}$ a long sequence of finitely supported vectors in $X$ with supports $\{A_\lambda\}_{\lambda \in \Lambda}$ and only rational coefficients $\langle v_\lambda, f_\gamma \rangle$. Then there exists a subset*

*$\Omega \subset \Lambda$ of cardinality $\Lambda$ and a finite set $B \subset \Lambda$ such that $A_\lambda \cap A_\alpha = B$ for all $\lambda, \alpha \in \Omega$, and the coefficients of $v_\lambda$ on $B$ are independent of $\lambda \in \Omega$.*

*Proof.* Apply Lemma 2 to the family $\{A_\lambda\}_{\lambda\in\Lambda}$ to obtain a $\Delta$-system $\{A_\lambda\}_{\lambda\in\Lambda'}$ with root $B$ such that $|\Lambda'| = |\Lambda|$. The set $\mathbb{Q}^B$ is countable. We define a mapping $r : \Lambda' \to \mathbb{Q}^B$ by $r(v_\lambda)(b) = v_\lambda(b)$ for all $b \in B$ and $\lambda \in \Lambda'$. Assume that $|\{\lambda \in \Lambda' : r(\lambda) = \mathbf{q}\}| < |\Lambda|$ for all $\mathbf{q} \in \mathbb{Q}^B$. We have $\Lambda' = \bigcup_{\mathbf{q}\in\mathbb{Q}^B}\{\lambda \in \Lambda' : \ r(\lambda) = \mathbf{q}\}$. Since $\mathbb{Q}^B$ is countable and $|\Lambda'|$ ($> \omega$) is regular we obtain a contradiction, hence there exists $\mathbf{q} \in \mathbb{Q}^B$ such that $|\Omega| = |\Lambda|$, where $\Omega := \{\lambda \in \Lambda' : \ r(\lambda) = \mathbf{q}\}$. For all $\lambda \in \Omega$ and $b \in B$ we get $v_\lambda(b) = \mathbf{q}(b)$. □

Recall (see, e.g., [J78]) that every non-limit cardinal is regular, and thus in particular every cardinal is a limit of a transfinite increasing sequence of regular cardinals. We rely on orthonormal matrices with special properties, described below.

**Lemma 4.** *Given $n \in \mathbb{N}$, there exists an orthonormal matrix $W := (a^n_{k,j})_{0\le k,j<2^n}$ with real coefficients, such that*

$$a^n_{k,0} = 2^{-\frac{n}{2}} \qquad \text{for } \ 0 \le k < 2^n, \tag{1}$$

$$\sum_{j=1}^{2^n-1} |a^n_{k,j}| < 1 + \sqrt{2} \qquad \text{for } \ 0 \le k < 2^n. \tag{2}$$

Such matrices were used by Ovsepian and Pełczyński in [OP75]. For a concrete example of Walsh matrices see, e.g., [HMVZ, Lemma 5.17] or [LT77, Lemma 1.f.5].

The following is the main result of this note.

**Theorem 5.** *Let $X$ be a Banach space with an M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$, and let $\varepsilon > 0$. Then $X$ admits an M-basis $\{x'_\gamma; f'_\gamma\}_{\gamma\in\Gamma}$ such that $\|x'_\gamma\|.\|f'_\gamma\| \le 2(1+\sqrt{2}) + \varepsilon$ for every $\gamma \in \Gamma$. Moreover,* $\operatorname{span}\{x_\gamma : \ \gamma \in \Gamma\} = \operatorname{span}\{x'_\gamma : \ \gamma \in \Gamma\}$ *and* $\operatorname{span}\{f_\gamma : \ \gamma \in \Gamma\} = \operatorname{span}\{f'_\gamma : \ \gamma \in \Gamma\}$.

*Proof.* For convenience, we may assume without loss of generality that $\Gamma$ is an ordinal of cardinality $|\Gamma|$. We are going to find a system consisting of a splitting $\Gamma = \bigcup_{\lambda\in\Gamma} A_\lambda$, where all $A_\lambda$ are countable and pairwise disjoint, together with biorthogonal systems $\{x'_\gamma; f'_\gamma\}_{\gamma\in A_\lambda}$, so that

A. $\operatorname{span}\{x'_\gamma : \gamma \in A_\lambda\} = \operatorname{span}\{x_\gamma : \gamma \in A_\lambda\}$

B. $\operatorname{span}\{f'_\gamma : \gamma \in A_\lambda\} = \operatorname{span}\{f_\gamma : \gamma \in A_\lambda\}$

C. $\|x'_\gamma\|\|f'_\gamma\| \le 2(1+\sqrt{2}) + \varepsilon$, for all $\gamma \in A_\lambda$, $\lambda \in \Gamma$

The existence of such a system clearly implies the statement of the theorem. We construct the $A_\lambda$'s and the biorthogonal system associated to each of them by using induction in $\lambda \in \Gamma$.

We start by putting $A_1 := \{0\}$ (the first element in $\Gamma$), and letting $\{x'_0; f'_0\}$ be a (single-element) biorthogonal system in $\operatorname{span}\{x_0\}\times\operatorname{span}\{f_0\}$ with $\|x'_0\| = \|f'_0\| = 1$. Suppose we achieved this for all $\lambda < \beta \in \Gamma$. It remains to obtain the objects $A_\beta$ and $\{x'_\gamma; f'_\gamma\}_{\gamma\in A_\beta}$. To this end we are going to construct an increasing sequence $\{A^j\}_{j=1}^\infty$, $A^j \subset A^{j+1}$, of finite subsets of $\Gamma$, so that $A_\beta = \bigcup_{j=1}^\infty A^j$. We are simultaneously going to build finite biorthogonal systems $\{x^j_\gamma; f^j_\gamma\}_{\gamma\in A^j}$, $j \in \mathbb{N}$, and a sequence of finite sets $\{C^j\}_{j=1}^\infty$ satisfying the following conditions for all $j \in \mathbb{N}$.

1. $\operatorname{span}\{x_\alpha : \alpha \in A^j\} = \operatorname{span}\{x^j_\alpha : \alpha \in A^j\}$.
2. $\operatorname{span}\{f_\alpha : \alpha \in A^j\} = \operatorname{span}\{f^j_\alpha : \alpha \in A^j\}$.
3. $C^j = \{\alpha \in A^j : \|x^j_\alpha\|\|f^j_\alpha\| \le 2(1+\sqrt{2}) + \varepsilon\}$.
4. $A^j \subset C^{j+1}$.
5. $x^{j+1}_\gamma = x^j_\gamma$ whenever $\gamma \in C^j$.
6. $f^{j+1}_\gamma = f^j_\gamma$ whenever $\gamma \in C^j$.

7. $\operatorname{span}\{x^j_\alpha:\ \alpha\in A^j\}\subset\operatorname{span}\{x^{j+1}_\alpha:\alpha\in C^{j+1}\}$.
8. $\operatorname{span}\{f^j_\alpha:\ \alpha\in A^j\}\subset\operatorname{span}\{f^{j+1}_\alpha:\alpha\in C^{j+1}\}$.

The existence of such systems now implies the inductive step in the proof of the main theorem.

Indeed, we put $A_\beta:=\bigcup_{j=1}^\infty A^j$ $(=\bigcup_{j=1}^\infty C^j)$. If $\gamma\in C^j$ for some $j\in\mathbb{N}$ then, by 5., $x^j_\gamma=x^{j+l}_\gamma$ for all $l\in\mathbb{N}$, and so we can put $x'_\gamma:=x^j_\gamma$. Similarly, by 6., $f^j_\gamma=f^{j+l}_\gamma$ for all $l\in\mathbb{N}$, and we put $f'_\gamma=f^j_\gamma$. The biorthogonality of $\{x'_\gamma;f'_\gamma\}_{\gamma\in A_\beta}$ follows from the fact that $\{x^j_\gamma;f^j_\gamma\}_{\gamma\in A^j}$ is biorthogonal for every $j\in\mathbb{N}$. Conditions A. and B. are checked easily. On one hand, if $\gamma\in A_\beta$, then $\gamma\in C^j$ for some $j\in\mathbb{N}$, so by 1.,

$$x'_\gamma=x^j_\gamma\in\operatorname{span}\{x_\alpha:\ \alpha\in A^j\}\subset\operatorname{span}\{x_\alpha:\ \alpha\in A_\beta\},$$

and, since $\gamma\in A^j$, by 7.,

$$\begin{aligned}x_\gamma\in\operatorname{span}\{x^j_\alpha:\ \alpha\in A^j\}&\subset\operatorname{span}\{x^{j+1}_\alpha:\ \alpha\in C^{j+1}\}\\&=\operatorname{span}\{x'_\alpha:\ \alpha\in C^{j+1}\}\subset\operatorname{span}\{x'_\alpha:\ \alpha\in A_\beta\}.\end{aligned}$$

We obtain similar results for $f'_\gamma$ and $f_\gamma$. Note that conditions 3. and 4. imply C.. It remains to check that $\bigcup_{\lambda\in\Gamma}A_\lambda=\Gamma$. This follows from the fact (see below) that in the construction of $\{A^j\}_{j\in\mathbb{N}}$ we start by taking $A^1:=\{\gamma_0\}$, where $\gamma_0$ is the first element in $\Gamma\setminus\bigcup_{\lambda<\beta}A_\lambda$, so $A_\beta\neq\emptyset$ while $\bigcup_{\lambda<\beta}A_\lambda\neq\Gamma$.

To start, put $A^1=\{\gamma_0\}$, where $\gamma_0$ is the first element in $\Gamma\setminus\bigcup_{\lambda<\beta}A_\lambda$, $x^1_{\gamma_0}:=x_{\gamma_0}$, and $f^1_{\gamma_0}:=f_{\gamma_0}$. Put $C^1:=\{\gamma_0\}$ if $\|x_{\gamma_0}\|\|f_{\gamma_0}\|\le 2(1+\sqrt{2})+\varepsilon$, $C^1=\emptyset$ otherwise. Let us describe the inductive step from $j$ to $j+1$. Suppose that $A^p$, $x^p_\gamma,f^p_\gamma$ for $p\le j$ have been constructed, such that 1.-8. are satisfied whenever the indices exist. Put $L=\{\lambda_1,\dots,\lambda_k\}=A^j\setminus C^j$, and find $C>0$ such that $\sup\{\|x_\lambda\|,\|f_\lambda\|:\ \lambda\in L\}<C$. Put $N=2^n-1$, with $n\in\mathbb{N}$ large enough to have $2^{-n/2}C<\varepsilon$. Use Theorem 1 to find $K:=K(N,\varepsilon)$. We are going to build a family $\{S_\lambda:\ \lambda\in L\}$ of finite pairwise disjoint subsets of $\Gamma$, disjoint also from $\bigcup_{\lambda<\beta}A_\lambda\cup\bigcup_{i\le j}A^i$, together with finite biorthogonal systems $\{y_\gamma;g_\gamma\}_{\gamma\in S_\lambda}$, $\lambda\in L$, such that, for all $\lambda\in L$,

a. $S_\lambda=S^1_\lambda\cup S^2_\lambda$, $|S^1_\lambda|=N$, $S^1_\lambda\cap S^2_\lambda=\emptyset$.
b. $\operatorname{span}\{x_\gamma:\gamma\in S_\lambda\}=\operatorname{span}\{y_\gamma:\gamma\in S_\lambda\}$.
c. $\operatorname{span}\{f_\gamma:\gamma\in S_\lambda\}=\operatorname{span}\{g_\gamma:\gamma\in S_\lambda\}$.
d. $\{g_\gamma:\gamma\in S^1_\lambda\}$ is $(1+\varepsilon)$-equivalent to the unit basis of $\ell_2^N$.
e. $\|y_\gamma\|\le 2+2\varepsilon$, for $\gamma\in\bigcup_{\lambda\in L}S_\lambda$.

Finding the above system is the main step of our construction. We have $|\beta|<\Gamma$ and so there exists a regular cardinal $R$, $\beta<R\le\Gamma$. Denote $\{B_{\lambda,\alpha}\}_{\lambda\in L,\alpha<R}$ a system of pairwise disjoint subsets of $\Gamma\setminus\bigcup_{\delta<\beta}A_\delta$, each of them of cardinality $K$. By Theorem 1, we have that every $\operatorname{span}\{f_\gamma:\ \gamma\in B_{\lambda,\alpha}\}$ contains a $(1+\varepsilon)$-isometric copy $G_{\lambda,\alpha}$ of $\ell_2^N$. Since the pair of finite dimensional spaces

$$(\operatorname{span}\{f_\gamma:\ \gamma\in B_{\lambda,\alpha}\},\operatorname{span}\{x_\gamma:\ \gamma\in B_{\lambda,\alpha}\})$$

is a dual pair, it follows by standard linear algebra that there exist a splitting $B_{\lambda,\alpha}=D_{\lambda,\alpha}\cup E_{\lambda,\alpha}$, $D_{\lambda,\alpha}:=\{\gamma^{\lambda,\alpha}_1,\dots,\gamma^{\lambda,\alpha}_N\}$, $D_{\lambda,\alpha}\cap E_{\lambda,\alpha}=\emptyset$, and a finite biorthogonal system $\{h_\gamma;z_\gamma\}_{\gamma\in B_{\lambda,\alpha}}$, with properties

$$\operatorname{span}\{h_\gamma:\ \gamma\in B_{\lambda,\alpha}\}=\operatorname{span}\{f_\gamma:\ \gamma\in B_{\lambda,\alpha}\},$$

$$\operatorname{span}\{z_\gamma:\ \gamma\in B_{\lambda,\alpha}\}=\operatorname{span}\{x_\gamma:\ \gamma\in B_{\lambda,\alpha}\},$$

$$\{h_\gamma\}_{\gamma\in D_{\lambda,\alpha}}\text{ is }(1+\varepsilon)\text{-equivalent to the unit basis of }\ell_2^N.$$

Let $G_{\lambda,\alpha}:=\operatorname{span}\{h_\gamma:\ \gamma\in D_{\lambda,\alpha}\}$.

Fix $\lambda\in L$, $\alpha<R$, and $\gamma\in D_{\lambda,\alpha}$. Put $X_\gamma:=z_\gamma\restriction_{G_{\lambda,\alpha}}$.

Clearly, $1\le\|X_\gamma\|\le 1+\varepsilon$. Denote $X_\gamma$ again the Hahn-Banach norm-preserving extension of $X_\gamma$ from $G_{\lambda,\alpha}\hookrightarrow X^*$ to the whole $X^*$, so $X_\gamma\in X^{**}$. Since obviously $\overline{\operatorname{span}}_{\mathbb{Q}}\{x_\zeta\}_{\zeta\in\Gamma}=X$, a standard application of Helly's theorem (see, e.g., [F$^{\sim}$, Exercise 3.36]) provides an element $\tilde{x}_\gamma \in \mathrm{span}_{\mathbb{Q}}\{x_\zeta : \zeta \in \Gamma\}$ such that $\|\tilde{x}_\gamma\| < \|X_\gamma\| + \varepsilon$ $(< 1+2\varepsilon)$ and $\tilde{x}_\gamma \restriction_{G_{\lambda,\alpha}} = X_\gamma$. Denote by $F_m^{\lambda,\alpha} \subset \Gamma$ the finite support sets of $\tilde{x}_{\gamma_m^{\lambda,\alpha}}$, $m \in \{1,\ldots,N\}$. Apply Corollary 3 to the given M-basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ and to each system $\{\tilde{x}_{\gamma_m^{\lambda,\alpha}}\}_{\alpha<R}$, $m \in \{1,\ldots,N\}$, $\lambda \in L$, to obtain a (single) subset $R' \subset R$ of cardinality $|R|$, such that the following conditions hold. There exists finite sets $\Delta_{\lambda,m} \subset \Gamma$, such that for all $\alpha < \xi \in R', m \in \{1,\ldots,N\}$, $\lambda \in L$,

1. $F_m^{\lambda,\alpha} \cap F_m^{\lambda,\xi} = \Delta_{\lambda,m}$ (so $\mathrm{supp}\,(\tilde{x}_{\gamma_m^{\lambda,\alpha}} - \tilde{x}_{\gamma_m^{\lambda,\xi}}) \cap \Delta_{\lambda,m} = \emptyset$).
2. $F_m^{\lambda,\xi} \setminus \Delta_{\lambda,m} \subset \Gamma \setminus (\bigcup_{\alpha<\xi} B_{\lambda,\alpha} \cup \bigcup_{i\le j} A^i \cup \bigcup_{\lambda<\beta} A_\lambda)$

It is also easy to see that by a suitable choice of $\alpha_\lambda, \xi_\lambda \in R'$, for $\lambda \in L$, we may, without loss of generality, assume that putting for $m \in \{1,2,\ldots,N\}$

$$\hat{x}_{\lambda,m} := \tilde{x}_{\gamma_m^{\lambda,\alpha_\lambda}} - \tilde{x}_{\gamma_m^{\lambda,\xi_\lambda}},$$
$$\hat{f}_{\lambda,m} = h_{\gamma_m^{\lambda,\alpha_\lambda}},$$

we have, in addition, that $\mathrm{supp}\,(\hat{x}_{\lambda,m}) \cap \mathrm{supp}\,(\hat{x}_{\lambda',m'}) = \emptyset$ unless $\lambda = \lambda', m = m'$. Thus we have that

$$\{\hat{x}_{\lambda,m}, \hat{f}_{\lambda,m}\}_{m\in\{1,\ldots,N\}}$$

is a biorthogonal $(2+2\varepsilon)$-bounded biorthogonal system such that vectors $\hat{x}_{\lambda,m}$, $m \in \{1,2,\ldots,N\}$, have disjoint supports with similar systems built previously in the inductive process. Next, we put

$$S_\lambda := B_{\lambda,\alpha_\lambda} \cup \bigcup_{m=1}^{N} \mathrm{supp}\,(\hat{x}_{\lambda,m}), \text{ for } \lambda \in L.$$

Again, we have $S_\lambda \cap S_{\lambda'} = \emptyset$, unless $\lambda = \lambda'$. Let $S_\lambda^1 = D_{\lambda,\alpha_\lambda} = \{\gamma_1^{\lambda,\alpha_\lambda}, \ldots, \gamma_N^{\lambda,\alpha_\lambda}\}$. For every $\gamma = \gamma_m^{\lambda,\alpha_\lambda} \in S_\lambda^1$, we put $g_\gamma := \hat{f}_{\lambda,m}$, $y_\gamma := \hat{x}_{\lambda,m}$. This choice guarantees that conditions a., d., and e. are satisfied. It remains to use standard linear algebra in order to add elements $g_\gamma, y_\gamma$ for $\gamma \in S_\lambda^2$, so that b. and c. will be satisfied.
To finish the inductive step, put $A^{j+1} := A^j \cup \bigcup_{\lambda\in L} S_\lambda$. For $\gamma \in C^j$, we let $x_\gamma^{j+1} := x_\gamma^j, f_\gamma^{j+1} := f_\gamma^j$. For $\lambda \in L$ put $\hat{x}_{\lambda,0} := x_\lambda$, $\hat{f}_{\lambda,0} := f_\lambda$. We have that $\{\hat{x}_{\lambda,m}; \hat{f}_{\lambda,m}\}_{m\in\{0,\ldots,N\}}$ is a biorthogonal system. Let $W := (a_{i,j})_{i,j=0,\ldots,N}$ be a matrix from Lemma 4. Put, for $k = 0,1,2,\ldots,N$,

$$u_k^\lambda := \sum_{m=0}^{N} a_{k,m}\hat{x}_{\lambda,m}, \qquad v_k^\lambda := \sum_{m=0}^{N} a_{k,m}\hat{f}_{\lambda,m}.$$

Finally, define $x_\gamma^{j+1}$ and $f_\gamma^{j+1}$ for $\gamma \in A^{j+1}$ in the following way:

$$x_\gamma^{j+1} := \begin{cases} u_0^\lambda, & \text{if } \gamma = \lambda \in L, \\ u_m^\lambda, & \text{if } \gamma \in S_\lambda^1 \ (= D_{\lambda,\alpha_\lambda}),\ \gamma = \gamma_m^{\lambda,\alpha_\lambda}, \\ y_\gamma, & \text{if } \gamma \in S_\lambda^2. \end{cases}$$

$$f_\gamma^{j+1} := \begin{cases} v_0^\lambda, & \text{if } \gamma = \lambda \in L, \\ v_m^\lambda, & \text{if } \gamma \in S_\lambda^1 \ (= D_{\lambda,\alpha_\lambda}),\ \gamma = \gamma_m^{\lambda,\alpha_\lambda}, \\ g_\gamma, & \text{if } \gamma \in S_\lambda^2. \end{cases}$$

Since $W$ is an orthonormal matrix, we obtain that $\{x_\gamma^{j+1}; f_\gamma^{j+1}\}_{\gamma\in\{\lambda\}\cup S_\lambda^1}$ is again a biorthogonal system, for every $\lambda \in L$.
It remains to estimate the norms of the new vectors and functionals. By using the condition d., (1), and the orthonormality of $W$, we get the following estimate,

whenever $\gamma \in \{\lambda\} \cup S^1_\lambda$:

$$\|f^{j+1}_\gamma\| < 2^{-n/2}\|f_\lambda\| + (1+\sqrt{2}) \max_{1\le m\le N} \|\hat{f}_{\lambda,m}\|$$
$$\le 2^{-n/2}C + (1+\sqrt{2})(1+\varepsilon) < (1+\sqrt{2}) + 4\varepsilon.$$

Similarly, using (2) instead,

$$\|x^{j+1}_\gamma\| < 2^{-n/2}\|x_\lambda\| + (1+\sqrt{2}) \max_{1\le m\le N} \|\hat{x}_{\lambda,m}\|$$
$$\le 2^{-n/2}C + (1+\sqrt{2})2(1+2\varepsilon) < 2(1+\sqrt{2}) + 13\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, these estimates imply conditions 4., 7., 8.. The remaining conditions follow from our construction by standard arguments.

□

Let us recall that Plichko in [Pl86] ([HMVZ], Example 5.19) has constructed an example of a WCG space which has no $C$-bounded M-basis, for every $C < 2$. On the other hand, in [Pl79] there is a generalization of the construction of $(1+\varepsilon)$-bounded M-basis in a separable space, to the case of WCG spaces, where one obtains $(2+\varepsilon)$-bounded M-bases. This result can be generalized to spaces with "many projections". In particular, one gets the following result.

**Proposition 6.** *Every Banach space belonging to a $\mathcal{P}$-class of nonseparable Banach space admits a $(2+\varepsilon)$-bounded M-basis for every $\varepsilon > 0$.*

*Proof.* Only formal changes in the proof in [Pl79] are needed. Let $\{P_\alpha\}_{\alpha\in\Gamma}$ be a projectional resolution of the identity in $X$, such that $P_\alpha(X)$ belong to $\mathcal{P}$ for all $\alpha$. Each space $X_\alpha = (P_{\alpha+1} - P_\alpha)(X)$ contains a 1-complemented separable space $Y_\alpha$, which is 2-complemented in the whole $X$. In each of $Y_\alpha$, we can build an M-basis, $\{x^\alpha_i; f^\alpha_i\}_{i\in\mathbb{N}}$, such that $\{x^\alpha_i\}_{i=1,\dots,N}$ is almost isometric to the unit basis of $\ell_2^N$, for suitable values of $N$. Using complementability, it is possible to extend $f^\alpha_i$, $i = 1,\dots,N$, onto the whole $X$ keeping the norm below $2+\varepsilon$. Using a standard device (see, e.g., [Fa97, Proposition 6.2.4]), we can glue all those partial biorthogonal systems into a full M-basis for $X$. This is the key ingredient in the proof, and the rest follows along the lines of [Pl79].

□

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