Extremal Graph Theory

Jan Hladký
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An alternative definition: substructures in graphs

In this talk:
- Turán’s Theorem
- Erdős–Sós Conjecture
- Szemerédi Regularity Lemma
Mantel 1907/Turán 1941  

If $G$ has $n$ vertices

If $G$ has more than $n^2/4$ edges then it contains a triangle.

- optimal $\implies$ extremal graph
- starting point of extremal graph theory
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- optimal \( \Rightarrow \) extremal graph
- starting point of extremal graph theory

Extensions:
- other graphs than the triangle (Turán, Erdős-Stone 1964)
- 3-uniform hypergraphs (still open!!!)
- “triangle density problem”
  Alexander Razborov, 2013 Robbins Prize (AMS)
Erdős–Sós Conjecture

Setting

$G$ . . . graph on $n$ vertices
$T_\ell$ . . . all trees on $\ell$ vertices
Erdős–Sós Conjecture

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$G$ ... graph on $n$ vertices
$\mathcal{T}_\ell$ ... all trees on $\ell$ vertices

Embedding trees: motivation $\delta(G) \geq k$, then $\mathcal{T}_{k+1} \subset G$. 
Erdős–Sós Conjecture

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Can this be weakened?
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Loebl-Komlós-Sós Conjecture ’95 If at least $n/2$ of the vertices of $G$ have degrees at least $k$, then $T_{k+1} \subset G$.
approximate solution by H., Piguet, Komlós, Simonovits, Stein, Szemerédi
Szemerédi Regularity Lemma

Szemerédi 1975: Dense subsets of integers contain arithmetic progressions of arbitrary length
If \( A \subset \mathbb{N} \) such that \( \limsup_n \frac{|A \cap \{1, \ldots, n\}|}{n} > 0 \) then
\[ \forall k \text{ there exists } a_0, d \in \mathbb{N} \text{ such that } a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (k - 1)d \in A. \]

History: 1953 Roth \( k = 3 \); 1977 Furstenberg (ergodic theory)

Szemerédi 1978: Regularity lemma Every graph can be decomposed into a bounded number of quasirandom pieces

Ruzsa and Szemerédi 1976: Removal lemma
easy consequence of the Regularity lemma (next slide)

2012: Abel Prize to Szemerédi

2002-2007: Hypergraph regularity lemma
Rödl, Schacht, Skokan, \ldots; Gowers
2012 Pólya Prize to Rödl and Schacht
Removal Lemma

Ruzsa and Szemerédi 1976: (Triangle) Removal lemma:
If a graph contains few triangles then it can be made triangle-free by removing few edges.
For every $\epsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.
If $G$ is an $n$-vertex graph ($n > n_0$) which has at most $\delta n^3$ triangles then there is a set of at most $\epsilon n^2$ edges deletion of which makes $G$ triangle-free.

Regularity-lemma free proof: Fox (Annals Math 2012)
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Application I: Property testing
Removal Lemma

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Application I: Property testing

Application II: Roth’s Theorem: Dense sets contain 3-AP’s
(Version I) If $A \subset \mathbb{N}$ such that $\lim \sup_n \frac{|A \cap \{1, \ldots, n\}|}{n} > 0$ then
there exists $a_0, d \in \mathbb{N}$ such that $a_0, a_0 + d, a_0 + 2d \in A$.
(Version II) For every $\alpha > 0$ there exists $n_0$ such that the
following holds. $A \subset \{1, \ldots, n\}$ (for some $n > n_0$) $|A| > \alpha n$ then
there exists $a_0, d \in \mathbb{N}$ such that $a_0, a_0 + d, a_0 + 2d \in A$. 