Deterministic approximation algorithms for partition functions and zeros of graph polynomials

with applications to graph limits

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Workshop Graph limits in Bohemian Switzerland 2018

March 25-30, 2018

Counting problems

- number of matchings, perfect matchings, independent sets, k-colorings, the permanent, the weight enumerator of a code, partition function of Ising/Potts model, ...

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- Two approaches: Monte Carlo Markov Chains (MCMC), Correlation decay
- This lecture: recent new approach due to Barvinok. Based on location of complex zeros.

Two concrete examples

The permanent of a matrix $A \in \mathbb{C}^{n\times n}$,

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The permanent of a matrix $A \in \mathbb{C}^{n \times n}$,

$$\mathrm{per}(A) := \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{i,\pi(i)},$$

and the independence polynomial of a graph $G = (V, E)$,

$$Z_G(z) := \sum_{\substack{I \subseteq V \\ I \text{ ind.}}} z^{|I|} = \sum_{k=0}^{\alpha(G)} i_k z^k.$$

Overview

- Lecture 1-2.
  - approximating the permanent: what is known?
  - Barvinok's approach for approximating the permanent
  - Taylor approximations
  - Zero-free regions for the permanent
- Lecture 2-3
  - approximating evaluations of the independence polynomial: what is known?
  - Applying Barvinok's approach
  - Zeros-free regions for the independence polynomial
  - connections with complex dynamics
- Lecture 3-4
  - computing the coefficients of the independence polynomial faster
  - connections with sparse graph limits

Approximating the permanent: what do we want?

$$\text{per}(A) := \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{i,\pi(i)}$$

If $\text{per}(A) > 0$, $A \geq 0$ and $\alpha > 1$, an α-approximation to $\text{per}(A)$ is a number $\xi > 0$ such that

$$\alpha^{-1} \leq \frac{\text{per(A)}}{\xi} \leq \alpha$$

We want, given $\alpha > 1$, an efficient algorithm that computes an α-approximation to per(A).

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- If A has values in $\{0, 1\}$ and the underlying graph is an expander, there exists a (deterministic) algorithm that computes an $(1+\varepsilon)^n$-approximation for each $\varepsilon > 0$ in polynomial time, due to Gamarnik and Katz, 2010 (using correlation decay approach).

Barvinok's results

Theorem (Barvinok)

Fix any positive δ such that $\delta < 1/2$. Then there exists a constant $\gamma > 0$ such that for any $\varepsilon > 0$ and any matrix A satisfying

$$|A_{i,j} - 1| < \delta \quad \textit{for all } i, j$$

one can compute in time $n^{\gamma(\log n - \log \varepsilon)}$ a number ξ such that

$$|\log(per(A)) - \xi| \leq \varepsilon.$$

Barvinok's results

Theorem (Barvinok)

Fix any $0 \leq \delta < 1$. Then there exists a constant $\gamma > 0$ such that for any $\varepsilon > 0$ and any real matrix A satisfying

$$|1 - A_{i,j}| \leq \delta \quad \text{for all } i, j$$

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Barvinok's approach

- Define a univariate polynomial

$$p(z) := \mathrm{per}(J + z(A - J)).$$

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- Compute Taylor coefficients and analyze running time.

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(Taylor)

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- Write $p(z) = \sum_{i=0}^{d} a_i z^i = a_d \prod_{i=1}^{d} (z - \zeta_i)$. Then

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- and consequently,

$$f'(z) = \sum_{i=1}^{d} \frac{1}{z - \zeta_i} \text{ and thus } f^{(k)}(0) = -(k-1)! \sum_{i=1}^{d} \frac{1}{\zeta_i^k}$$

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(Newton identities:)

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Coefficient of z^k:

$$\sum_{\pi \in S_n} \prod_{i=1}^{n} (1 + z(A_{i,\pi(i)} - 1)) = \sum_{\pi \in S_n} \sum_{j=0}^{n} \sum_{\substack{U \subseteq [n] \\ |U| = j}} z^j \prod_{i \in U} (A_{i,\pi(i)} - 1).$$

$$= \sum_{j=0}^{n} z^j \sum_{\substack{U \subseteq [n] \\ |U| = j}} \sum_{\pi : U \hookrightarrow [n]} (n-j)! \prod_{i \in U} (A_{i,\pi(i)} - 1).$$

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So coefficient of z^k can be computed in time $n^{O(k)}$.

Running time

Taylor series

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- If $|z| < |\zeta_i|$ for each i, then for $m = O(\log(d/\varepsilon))$ we have ε-approximation.

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- If $|z| < |\zeta_i|$ for each i, then for $m = O(\log(d/\varepsilon))$ we have ε-approximation.
- Since $1 < |\zeta_i|$ for all i and since $d \leq n$ we see that algorithm runs in time $n^{O(\log(n/\varepsilon))}$.

Summary of Barvinok's approach

Goal: approximate a polynomial p at some $\lambda \in \mathbb{C}$.

Recipe

- Show that $p(z)$ is nonzero in some open (simply connected) region Ω of $\mathbb{C}$ that contains λ.
- approximate the logarithm of p on Ω by a low order Taylor polynomial T.
- Compute the coefficients of T from the first coefficients of p.

Result: a quasi-polynomial time algorithm.

Zero-free region for the permanent

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Lemma (Barvinok)

Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be non-zero vectors. Suppose the angle between any pair does not exceed α for some $0 \leq \alpha < 2\pi/3$.

(i) *Let $u = \sum_{i=1}^n u_i$. Then*

$$\|u\| \geq \cos(\alpha/2) \sum_{i=1}^{n} \|u_i\|.$$

(ii) *Let $0 \leq \delta < \cos(\alpha/2)$ and let a_i, b_i $(i \in [n])$ be complex numbers such that*

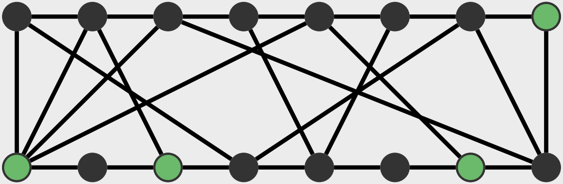
$$|1 - a_i| < \delta \text{ and } |1 - b_i| < \delta.$$

Then

$$v = \sum_{i=1}^{n} a_i u_i \text{ and } w = \sum_{i=1}^{n} b_i u_i \text{ are nonzero,}$$

and their angle does not exceed $2 \arcsin\left(\frac{\delta}{\cos(\alpha/2)}\right)$.

Independence polynomial



For a graph $G = (V, E)$, the independence polynomial is defined as

$$Z_G(\lambda) = \sum_{\substack{I \subseteq V \\ I \text{ independent}}} = \sum_{k=0}^{\alpha(G)} i_k \lambda^k.$$

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- $Z_G(1)$ equal the number of independent sets of G.
- Z_G is know as the *hardcore model* in statistical physics.
- for $p \in (0, 1)$, $Z_G(-p)$ is related to the Lovász Local Lemma.

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- Weitz (2006) breakthrough result: a deterministic efficient approximation algorithm for $Z_G(\lambda)$ whenever

$$0 \leq \lambda < \lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$$

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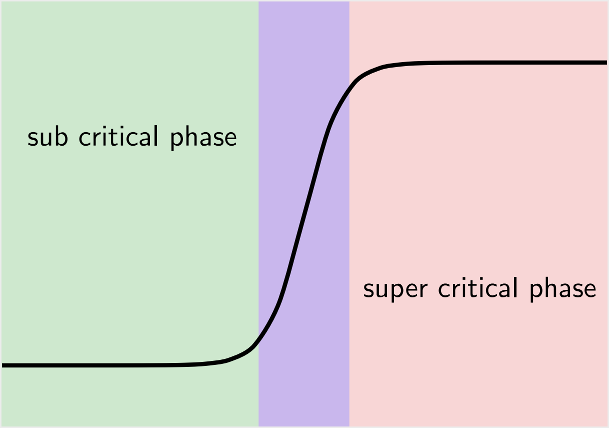
- Sly and Sun (2012): **NP**-hard to approximate $Z_G(\lambda)$ when $\lambda > \lambda_c(\Delta)$ and $\Delta(G) = \Delta$.

Phase transition in approximate counting



$0 \leq \lambda < \lambda_c$ $\qquad$ λ_c $\qquad$ $\lambda_c < \lambda$

Weitz '06 $\qquad$ Sly and Sun '12

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Theorem (Patel, R. 17)

Can compute the number of independent sets of size k in a graph G of max. degree Δ in time $\Delta^{O(k)}n$.

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Corollary

Polynomial time algorithms for bounded degree graphs.

Zero-free regions for Z_G

For $\Delta \in \mathbb{N}_{\geq 3}$ Let

$$\lambda^*(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta} \quad \text{and} \quad \lambda_c(\Delta) := \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}.$$

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For any graph G with max. degree at most Δ and any $\lambda \in \mathbb{C}$ with $|\lambda| \leq \lambda^*(\Delta)$,

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New algorithmic results for Z_G

Theorem (Patel, R. 2017 and Harvey, Srivastava and Vondrák 2018)

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for all graphs G *of max. degree at most* Δ.

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for all graphs G of max. degree at most Δ.

Theorem (Bezáková, Galanis, Goldberg and Šefankovič, 2018)

#P-*hard to approximate $Z_G(\lambda)$ when $\lambda < -\lambda^*(\Delta)$ and G of max. degree Δ.*

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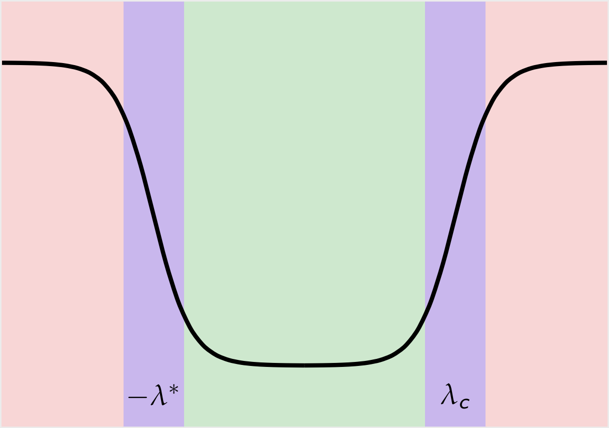

A useful reformulation

Fundamental recurrence for Z_G: for a fixed vertex v

$$Z_G(\lambda) = \lambda Z_{G\setminus N[v]}(\lambda) + Z_{G-v}(\lambda).$$

Definition

Let us define, assuming $Z_{G-v}(\lambda) \neq 0$,

$$R_{G,v} := \frac{\lambda Z_{G\setminus N[v]}(\lambda)}{Z_{G-v}(\lambda)}.$$

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A useful observation:

$$R_{G,v} \neq -1 \text{ if and only if } Z_G(\lambda) \neq 0.$$

A recurrence relation

Definition

Let H be a graph with fixed vertex u_0. Let $u_1, \ldots, u_d$ be the neighbors of u_0 in H (in any order). Set $H_0 = H - v_0$ and define for $i = 1, \ldots, d$, $H_i := H_{i-1} - u_i$. Then $H_d = H \setminus N[u_0]$.

Lemma

Suppose $Z_{H_i}(\lambda) \neq 0$ *for all* $i = 0, \ldots, d$. *Then*

$$R_{H,u_0} = \frac{\lambda}{\prod_{i=1}^{d}(1 + R_{H_{i-1},u_i})}.$$

Proof sketch of Shearer's bound

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Proof.

Idea: assume G connected. Use the identity

$$R_{H,u_0} = \frac{\lambda}{\prod_{i=1}^{d}(1 + R_{H_{i-1},u_i})},$$

to prove inductively that the following holds for all $U \subseteq V \setminus \{v_0\}$ (for some fixed v_0):

(i) $Z_{G[U]}(\lambda) \neq 0$,

(ii) if $u_0 \in U$ has a neighbour in $V \setminus U$, then $|R_{G[U],u_0}| < 1/\Delta$.

□

Proof of Sokal's conjecture: ideas

Theorem (Peters, R. 2017)

There exists an open region D_Δ *in* $\mathbb{C}$ *containing* $[0, \lambda_c(\Delta))$ *such that for any graph* G *of max. degree at most* Δ *and* $\lambda \in D_\Delta$, $Z_G(\lambda) \neq 0$.

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1 Let

$$F(x_1, \dots, x_d) = \lambda / \prod_{i=1}^{d} (1 + x_i).$$

Find a 'trapping region' for F.

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2 Find conjugation $g = \varphi \circ f \circ \varphi^{-1}$ with $|g'| < 1$ on $\mathbb{R}_{\geq 0}$.

3 Find open sets D_Δ in the parameter space and 'trapping region' $\mathcal{U}$ such that for all $z \in \mathcal{U}$ and $\lambda \in D_\Delta$, $g(z) \in \mathcal{U}$.

Proof of Sokal's conjecture: ideas

Theorem (Peters, R. 2017)

There exists an open region D_Δ in $\mathbb{C}$ containing $[0, \lambda_c(\Delta))$ such that for any graph G of max. degree at most Δ and $\lambda \in D_\Delta$, $Z_G(\lambda) \neq 0$.

1 Let

$$F(x_1, \dots, x_d) = \lambda / \prod_{i=1}^{d} (1 + x_i).$$

Find a 'trapping region' for F. Not straightforward, univariate version $f(x) = \lambda/(1+x)^d$ is not a contraction on $\mathbb{R}_{\geq 0}$.

2 Find conjugation $g = \varphi \circ f \circ \varphi^{-1}$ with $|g'| < 1$ on $\mathbb{R}_{\geq 0}$.

3 Find open sets D_Δ in the parameter space and 'trapping region' $\mathcal{U}$ such that for all $z \in \mathcal{U}$ and $\lambda \in D_\Delta$, $g(z) \in \mathcal{U}$.

4 Show that $\mathcal{U}$ also works for $\varphi \circ F \circ \varphi^{-1}$.

Regular trees and complex dynamics

Regular trees and complex dynamics

Fix $\Delta \geq 3$, Let T_k be the tree with k levels, in which each non-leaf vertex has $\Delta - 1$ descendants. Then

$$R_{T_k,v} = \frac{\lambda}{(1 + R_{T_{k-1},v})^{\Delta-1}} = f^{\circ k}(\lambda)$$

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Theorem (Peters, R. 2017)

(i) *Let $\lambda \in U$. Then $Z_{T_k}(\lambda) \neq 0$ for all k.*

(ii) *For any $\lambda \in \partial U$, there exists λ' arbitrarily close to λ and $k \in \mathbb{N}$ such that $Z_{T_k}(\lambda') = 0$.*

Regular trees and complex dynamics II

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Theorem (Bezáková, Galanis, Goldberg, Štefankovič 2018)

Let $\lambda \notin \overline{U}$ and suppose λ is not positive. Then it is $\#\mathbf{P}$-hard to approximate $Z_G(\lambda)$ on graphs G of max. degree Δ.

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Question

Is it true that for any $\lambda \in U$ and any graph G of max. degree at most Δ, $Z_G(\lambda) \neq 0$?

Summary of Barvinok's approach

Recipe

1. Show that $p(z)$ is nonzero in some open (simply connected) region Ω of $\mathbb{C}$ that contains λ.
2. approximate the logarithm of p on Ω by a low order Taylor polynomial T.
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In many cases, item 3 can be done in polynomial time!

Polynomial time algorithms from Barvinok's approach

- The matching polynomial (Patel, R. 2017; zero-free region by Heilman and Lieb 1972)
- Partition functions of edge-coloring models/Holants (Patel, R. 2017; zero-free region by R. 2016+)
- Tutte polynomial (Patel, R. 2017; zero-free region by Jackson, Procacci and Sokal 2013)
- Partition functions of spin models (Patel, R. 2017; zero-free region by Barvinok and Soberón 2017)
- Hypergraph Ising model (Liu, Sinclair, Srivastava 2017)
- Weight enumerator for linear codes (Barvinok and R. 2017+)

The chromatic polynomial

Let χ_G denote the chromatic polynomial of a graph G.

Theorem (Jackson, Procacci and Sokal 2013)

For any graph G of max. degree at most Δ and any z such that $|z| \geq 6.91\Delta$,

$$\chi_G(z) \neq 0.$$

Question

Is it true that for any $\Delta > 2$ there exists $\varepsilon > 0$ such that for any graph G of max. degree at most Δ,

$$\chi_G(z) \neq 0$$

whenever $\Re(z) > \Delta$ and $|\Im(z)|a \leq \varepsilon$?

Counting induced subgraphs

Counting induced subgraphs

Definition

Let H and G be graphs. Then $\text{ind}(H, G)$ denotes the number of subsets $S \subseteq V(G)$ such that $H = G[S]$.

Example

Let H be the graph consisting of k isolated vertices. Then $\text{ind}(H, G)$ is $\#$ independent sets of size k.

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Theorem (Patel, R. 2017+)

Let $\Delta \in \mathbb{N}$. Then there is a $\Delta^{O(k)}n$-time algorithm that computes $ind(H, G)$ for graphs H and G of order k and n respectively and max degree at most Δ.

Corollary

Let $\Delta \in \mathbb{N}$. We can compute the number of independent sets of size k in a graph G of maximum degree at most Δ and order n in $\Delta^{O(k)} n$ time.

Application to the independence polynomial and questions

Corollary

Let $\Delta \in \mathbb{N}$. We can compute the number of independent sets of size k in a graph G of maximum degree at most Δ and order n in $\Delta^{O(k)} n$ time.

Theorem (Nederlof, Patel, R. 2018+)

There exists a constant $c >$ such that we can compute the number of independent sets of size k in a planar graph G of order n in $c^{O(k)} n$ time

Questions

- Does there exist $c > 0$ and an algorithm, which given a graph H of order k and a planar graph G of order n that computes $\text{ind}(H, G)$ in time $c^{O(k)}\text{poly}(n)$?
- Recall $p(z) = \text{per}(J + z(A - J))$. Does there exist $c > 0$ and an algorithm that computes the coefficient of z^k of p in time $c^{O(k)}\text{poly}(n)$ for matrices A of order n?

Computing # ind. sets of size k: Tools

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Definition

Call a graph parameter f additive if $f(G_1 \cup G_2) = f(G_1) + f(G_2)$.

Lemma (Csikvári and Frenkel 2016)

Let $f = \sum_{i=1}^{n} a_i ind(H_i, \cdot)$ be a graph parameter. Then f is additive if and only if $a_i = 0$ whenever H_i is not connected.

Computing # ind. sets of size k: Tools II

Let $\zeta_1, \ldots \zeta_\alpha$ be the roots of $Z_G(\lambda) = \sum_{k=0}^{\alpha} i_k \lambda^k$ and let $p_k = \sum_{i=1}^{\alpha} \zeta_i^{-k}$.

Lemma

The p_k are additive graph parameters.

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(Newton identities)

$$k \cdot i_k = -\sum_{m=0}^{k-1} i_m \cdot p_{k-m} \iff p_k = -\sum_{m=1}^{k-1} i_m \cdot p_{k-m} - k \cdot i_k$$

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Corollary

We can write

$$p_k(\cdot) = \sum_{i=1}^{m} a_{k,i} \cdot ind(H_i, \cdot)$$

for certain coefficients a_i and connected graphs H_i of order at most k.

Use this to iteratively compute the coefficients a_i for each p_k.

Connections with sparse graph limits

Definition

A sequence of bounded degree graphs (G_n) is called *Benjamini-Schramm convergent* if $|V(G_n)| \to \infty$ and for each connected graph H the sequence

$$\frac{\mathrm{ind}(H, G_n)}{|V(G_n)|}$$

is convergent.

(Observation)

If (G_n) is convergent, $|\Delta(G_n)| \leq \Delta$ and $|\lambda| < \frac{(\Delta-1)^{\Delta-1}}{\Delta^{\Delta}}$, then the sequence

$$\frac{\log(Z_{G_n}(\lambda))}{|V(G_n)|}$$

is convergent.

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Thank you for your attention!

Some references

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