On the nonnegative rank of distance matrices

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Abstract

For real numbers $a_1, \ldots, a_n$, let $Q(a_1, \ldots, a_n)$ be the $n \times n$ matrix whose i, j-th entry is $(a_i - a_j)^2$. We show that $Q(1, \ldots, n)$ has nonnegative rank at most $2\log_2 n+2$. This refutes a conjecture from [1] (and contradicts a "theorem" from [5]). We give other examples of sequences $a_1, \ldots, a_n$ for which $Q(a_1, \ldots, a_n)$ has logarithmic nonnegative rank, and pose the problem whether this is always the case. We also discuss examples of matrices based on hamming distances between inputs of a Boolean function, and note that a lower bound on their nonnegative rank implies lower bounds on Boolean formula size.

1. Introduction

The *nonnegative rank*[2] of an $n \times m$ matrix M of nonnegative real numbers is the smallest k so that M can be written as $M = A \cdot B$, where A and B are nonnegative of dimensions $n \times k$ and $k \times m$. The notion of nonnegative rank was introduced in [12], where some of its computational applications were presented. Perhaps the most intriguing question is how much can the rank and the nonnegative rank of M differ. If M is a matrix of zeros and ones, a separation between nonnegative rank and rank is closely related to the so-called log-rank conjecture [12, 6]. For a general nonnegative matrix M, a separation between rank and nonnegative rank can potentially be used to separate non-commutative monotone and general branching programs [7]. In this paper we mention a connection with lower bounds on Boolean formula size.

Let us denote the rank and the nonnegative rank by rk and rk_+. One can easily construct a nonnegative 4×4 matrix with $\mathrm{rk}(M) < \mathrm{rk}_+(M)$. A stronger separation can be obtained using the *distance matrix* Q. For $a_1, \ldots, a_n \in \mathbb{R}$, the matrix $Q(a_1, \ldots, a_n)$ is an $n \times n$ matrix defined by

$$Q(a_1, \ldots, a_n)_{i,j} = (a_i - a_j)^2 \text{ for } i, j \in \{1, \ldots, n\}\,.$$

In [1] it was shown that whenever $a_1, \ldots, a_n$ are distinct and $n \geq 3$ then

$$\mathrm{rk}(Q(a_1, \ldots, a_n)) = 3 \text{ but } \mathrm{rk}_+ Q(a_1, \ldots, a_n) = \Omega(\log n)\,.$$

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[2]A.k.a. "monotone rank", "positive rank", or "rank over the semiring of nonnegative real numbers".

Comparing just rk and rk_+, this is the ultimate separation: we see that rk_+ cannot be upper-bounded by any function of rk. On the other hand, one would like to have an $n \times n$ matrix such that the gap between rank and nonnegative rank is large in terms of n.

Question 1: *Is there a nonnegative $n \times n$ matrix M with $rk_+(M) = n$ and $rk(M) = const$? Or at least $rk_+(M)/rk(M) = n^{1-o(1)}$?*

Some such estimate is indeed required in the branching program separation, mentioned above. In [1], it was conjectured that the distance matrix is again the right candidate: that $Q(a_1, \ldots, a_n)$ has nonnegative rank equal to n whenever $a_1, \ldots, a_n$ are distinct. This result was later announced in [5]. However, here we show that this is not the case. The matrix $Q(1, 2, 3, \ldots, n)$ has nonnegative rank $O(\log n)$. We do not know whether $Q(a_1, \ldots, a_n)$ can have large nonnegative rank for some other choice of the points $a_1, \ldots, a_n$. However, we note that if $Q(a_1, \ldots, a_n)$ has linear nonnegative rank with integer $a_1, \ldots, a_n$ then $\max |a_i|$ must be exponential. Moreover, one can construct fast growing $a_1, \ldots, a_n$ such that the nonnegative rank of the distance matrix is still logarithmic.

In sum, distance matrices are not the right candidate to answer Question 1, unless the points $a_1, \ldots, a_n$ satisfy some extra property. In Section 3, we suggest a different candidate, based on hamming distances between inputs of a Boolean function. We make a simple observation that for such matrices, a separation between $\mathrm{rk}(M)$ and $\mathrm{rk}_+(M)$ implies lower bounds on Boolean formula size. While finalising this manuscript, the author has learnt about recent results of Rothvoß [10] and Fiorini et al. [3]. They present much stronger gap between rank and nonnegative rank, and hence a more satisfactory answer to Question 1. For example, in Theorem 12 of [3] , an explicit $2^n \times 2^n$ matrix M is constructed with $\mathrm{rk}(M) = O(n^2)$ and $\mathrm{rk}_+(M) = 2^{\Omega(n)}$.

2. Nonnegative rank and Euclidean distance matrices

We consider matrices over the field of real numbers.[3] A real matrix M will be called *nonnegative*, if every entry of M is nonnegative. The *nonnegative rank* of an $n \times m$ nonnegative matrix M is the smallest k such that there exist nonnegative matrices A and B of dimensions $n \times k$, $k \times m$ respectively and $M = A \cdot B$. We denote nonnegative rank by rk_+.

Let us state some elementary properties of the nonnegative rank. If M_1, M_2 are nonnegative matrices then:

- (i). Permuting rows or columns of M_1, or multiplying a row or column by $r > 0$, does not change $\mathrm{rk}_+(M_1)$.
- (ii). $\mathrm{rk}_+(M_1+M_2) \leq \mathrm{rk}_+(M_1)+\mathrm{rk}_+(M_2)$, if M_1, M_2 have the same dimensions.
- (iii). $\mathrm{rk}_+(M_1)$ is the smallest k such that M can be written as a sum of k nonnegative rank one matrices.

[3]However, one can note that the presented upper bounds work also over integers.

(iv). If $\mathrm{rk}(M_1) \leq 2$ then $\mathrm{rk}_+(M_1) = \mathrm{rk}(M_1)$, see [2].

For a sequence of real numbers $a_1, \ldots, a_n$, let $Q(a_1, \ldots, a_n)$ be the $n \times n$ matrix

$$Q(a_1, \ldots, a_n) = \begin{pmatrix} 0 & (a_1 - a_2)^2 & (a_1 - a_3)^2 & \ldots & (a_1 - a_n)^2 \\ (a_2 - a_1)^2 & 0 & (a_2 - a_3)^2 & \ldots & (a_2 - a_n)^2 \\ \vdots & & & & \\ (a_n - a_1)^2 & (a_n - a_2)^2 & (a_1 - a_3)^2 & \ldots & 0 \end{pmatrix}.$$

That is, the i,j-th entry is the square of the Euclidean distance between a_i and a_j. A well-known fact, see [1], is that the rank of $Q(a_1, \ldots, a_n)$ is at most three (and equals to three if $|\{a_1, \ldots, a_n\}| \geq 3$). In [1], it was also shown that whenever $a_1, \ldots, a_n$ are distinct then the nonnegative rank of $Q(a_1, \ldots, a_n)$ is at least $\Omega(\log n)$.

In Section 4, will show the following:

Theorem 1. *$rk_+ Q(1, \ldots, n) \leq 2\log_2 n + 2$.*

If one wonders whether such an upper bound can be achieved for $a_1, \ldots, a_n$ not containing any long arithmetic progression, let us note that the relevant property is not that of *containing* an arithmetic progression, but rather *being contained in* an arithmetic progression. This observation gives:

Corollary 2. *Let $a_1 \leq \cdots \leq a_n$ be integers. Then*

$$rk_+ Q(a_1, \ldots, a_n) = O(\log(a_n - a_1)).$$

This implies that if there exist natural numbers $a_1 \leq \cdots \leq a_n$ such that $\mathrm{rk}_+ Q(a_1, \ldots, a_n)$ is linear in n then a_n must be exponential. In Section 5, we give an example of a distance matrix with small nonnegative rank for "exponentially growing" $a_1, \ldots, a_n$. We are left with the following question:

Question 2. *Are there real numbers $a_1, \ldots, a_n$ with $rk_+ Q(a_1, \ldots, a_n) = \Omega(n)$?*

3. Nonnegative rank and hamming distance

We now discuss examples of matrices based on hamming distances between Boolean vectors. For $x, y \in \{0,1\}^n$, let $h(x, y)$ be the number of bits where x, y differ. That is, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have

$$h(x, y) = \sum_{i \in \{1, \ldots, n\}} x_i(1 - y_i) + \sum_{i \in \{1, \ldots, n\}} (1 - x_i) y_i .$$

For two sets $A, B \subseteq \{0,1\}^n$, consider the matrix $H(A, B)$ whose columns are labelled with elements of A and rows with elements of B such that

$$H(A, B)_{x,y} = h(x, y) \text{ for } x \in A,\ y \in B.$$

First, we observe that the nonnegative rank of $H(A, B)$ – and hence also the rank – is always small:

Lemma 3. *Let $A, B \subseteq \{0,1\}^n$. Then $rk_+ H(A,B) \leq 2n$.*

Proof. Let $H := H(A,B)$. It is sufficient to consider the case $A = B = \{0,1\}^n$. For vectors $u, v \in \mathbb{R}^k$, let (u,v) be their inner product. First, note that a $p \times q$ matrix M has nonnegative rank $\leq k$ iff there exist nonnegative vectors $u_1, \dots, u_p, v_1, \dots, v_q \in \mathbb{R}^k$ such that $M_{i,j} = (u_i, v_j)$ for every i, j. For $x = (x_1, \dots, x_n)$, define $x^\star := (1 - x_1, \dots, 1 - x_n)$. Then

$$h(x,y) = (x, y^\star) + (x^\star, y) = (xx^\star, y^\star y)\,,$$

where the last inner product is in $\mathbb{R}^{2n}$ and $xx^\star$ is the concatenation of x and $x^\star$. Hence $H_{x,y} = (xx^\star, y^\star y)$ for every $x, y \in \{0,1\}^n$ and so $\mathrm{rk}_+ H \leq 2n$. $\square$

Second, assume that A, B are disjoint. This means that for every $x \in A$ and $y \in B$, $h(x,y) \geq 1$. Hence if we consider the $|A| \times |B|$ matrix $J(A,B)$ whose every entry is equal to one, the matrix

$$H(A,B) - J(A,B)$$

is nonnegative. The rank of $H(A,B) - J(A,B)$ is at most $2n+1$, but for its nonnegative rank, no upper bound is apparent.

Question 3. *Let $A, B \subseteq \{0,1\}^n$ be disjoint. How large can be the nonnegative rank of $H(A,B) - J(A,B)$?*

Let us point out one connection between this question and Boolean complexity. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and $e \in \{0,1\}$, let $f^{-1}(e) := \{x : f(x) = e\}$. The sets $f^{-1}(1)$ and $f^{-1}(0)$ are disjoint, and let us define

$$H(f) := H(f^{-1}(1), f^{-1}(0))\,,\ J(f) := J(f^{-1}(1), f^{-1}(0))\,.$$

The following proposition shows that a lower bound on $\mathrm{rk}_+(H(f) - J(f))$ gives a lower bound on the formula complexity of f. Hence a strong enough separation between rank and nonnegative rank, as anticipated in Question 3, implies that nonnegative rank can give nontrivial Boolean lower bounds.

For a Boolean function f, let $L(f)$ denote the size of a smallest Boolean formula computing f (in the de Morgan basis). We refer the reader to, e.g., [11] for background.

Proposition 4. *Let $f : \{0,1\}^n \to \{0,1\}$. Then*

$$L(f) \geq \frac{rk_+(H(f) - J(f))}{2n}\,.$$

Proof. This is an adaptation of a theorem in [8], the reader may also use the expository paper [4].

A set $A \times B \subseteq f^{-1}(1) \times f^{-1}(0)$, will be called a *rectangle*. A rectangle is *monochromatic*, if there exists $i \in \{1, \dots, n\}$ and $e \in \{0,1\}$ such that for every $x \in A, y \in B$ we have $x_i = e$ and $y_i = 1 - e$ (where x_i is the i-th bit in x). Denote $\mathcal{R}$ the set of all rectangles and $\mathcal{R}_m$ the set of all monochromatic rectangles. Let $\mu : \mathcal{R} \to \mathbb{R}$ be a nonnegative function. We call it a *subadditive measure*, if for every rectangles $R_1, R_2, R \in \mathcal{R}$ such that R is a disjoint union of R_1 and R_2, we have

$$\mu(R) \leq \mu(R_1) + \mu(R_2)\,.$$

A general fact is that every subadditive measure μ gives a lower bound on $L(f)$:

$$L(f) \geq \frac{\mu(f^{-1}(1) \times f^{-1}(0))}{\max_{R \in \mathcal{R}_m} \mu(R)}\,.$$

For a rectangle $A \times B$, define

$$\mu(A \times B) := \mathrm{rk}_+(H(A,B) - J(A,B))\,.$$

Clearly, μ is a subadditive measure. Since $\mu(f^{-1}(1) \times f^{-1}(0)) = \mathrm{rk}_+(H(f) - J(f))$, it is sufficient to show that for every monochromatic rectangle R, $\mu(R)$ is at most $2n$.

Let $R = A \times B$ be such a rectangle. Without loss of generality, we can assume that $x_1 = 1$ and $y_1 = 0$ for every $x \in A$ and $y \in B$. For $z = (z_1, \ldots, z_n) \in \{0,1\}^n$, let $z' := (z_2, \ldots, z_n) \in \{0,1\}^{n-1}$. Let $A' := \{x' : x \in A\}$ and $B' := \{y' : y \in B\}$. For every $x \in A$ and $y \in B$ we have $h(x,y) = 1 + h(x', y')$, and therefore

$$H(A,B) = H(A', B') + J(A,B)\,.$$

This implies

$$\mu(A \times B) = \mathrm{rk}_+(H(A,B) - J(A,B)) = \mathrm{rk}_+ H(A', B')\,.$$

By Lemma 3, we have $\mathrm{rk}_+ H(A', B') \leq 2(n-1)$, which completes the proof. □

Let us add few remarks:

(i). In order to answer Question 3, we would need the converse of Proposition 4. That is: can we lower bound $\mathrm{rk}_+(H(f) - J(f))$ in terms of $L(f)$?

(ii). In view of [9], one should note that rk_+ is not a submodular function in the sense of matroid theory. Identifying a matrix with the set of its rows, we can find nonnegative 4×4 matrices M_1, M_2 such that $\mathrm{rk}_+(M_1 \cap M_2) + \mathrm{rk}_+(M_1 \cup M_2) > \mathrm{rk}_+(M_1) + \mathrm{rk}_+(M_2)$. Namely,

$$\begin{aligned} M_1 &= \{(0,0,1,1), (0,1,1,0), (1,1,1,1), (1,1,0,0)\}\,, \\ M_2 &= \{(0,0,1,1), (0,1,1,0), (1,1,1,1), (1,0,0,1)\}, \end{aligned}$$

for then $\mathrm{rk}_+(M_1) = \mathrm{rk}_+(M_2) = \mathrm{rk}_+(M_1 \cap M_2) = 3$ but $\mathrm{rk}_+(M_1 \cup M_2) = 4$.

(iii). In the case of a monotone Boolean function f, one can replace $h(x,y)$ by the quantity $h^1(x,y) := \sum_i x_i(1 - y_i)$, and consider the matrix $H^1(f)$ defined by $H^1(f)_{x,y} := h^1(x,y)$. Proposition 4 remains valid if we replace $H(f)$ by $H^1(f)$ and $L(f)$ by monotone formula size. Again, one can ask how large can $\mathrm{rk}_+(H^1(f) - J(f))$ be, hoping to gain some insight from the known lower bounds on monotone formula size.

4. Proof of Theorem 1

Lemma 5. *Let M_1 and M_2 be nonnegative matrices and $M_1 \times M_2$ their Kronecker product. Then $rk_+(M_1 \times M_2) \leq rk_+(M_1) \cdot rk_+(M_2)$.*

Proof. Assume $\mathrm{rk}_+(M_1) = p$ and $\mathrm{rk}_+(M_2) = q$. Then $M_1 = U_1 + \cdots + U_p$ and $M_2 = V_1 + \cdots + V_q$, where U_i, V_j are nonnegative matrices of rank one. (Note that for rank one matrices, there is no difference between rank and nonnegative rank). Hence

$$M_1 \times M_2 = \sum_{(i,j)\in\{1,\ldots,p\}\times\{1,\ldots,q\}} U_i \times V_j \,.$$

Every $U_i \times V_j$ is a nonnegative matrix of rank one, and hence of nonnegative rank one. Hence $\mathrm{rk}_+(M_1 \times M_2) \leq pq$. □

For real numbers $a_1, \ldots, a_n, b_1, \ldots b_n$, let $Q(a_1, \ldots, a_n || b_1, \ldots, b_n)$ be the $n \times n$ matrix with

$$Q(a_1, \ldots, a_n || b_1, \ldots b_n)_{i,j} = (a_i - b_j)^2 \text{ for } i, j \in \{1, \ldots, n\} \,.$$

Lemma 6. *$rk_+Q(1, 2, \ldots, 2n) \leq rk_+Q(1, 2, \ldots, n) + 2$.*

Proof. The nonnegative rank of $Q(a_1, a_2, \ldots, a_n)$ does not change if we permute $a_1, \ldots, a_n$. So let us consider the matrix $Q(1, \ldots, n, 2n, 2n-1, \ldots, n+1)$. We can divide this matrix into four $n \times n$ blocks in the following manner:

$$Q(1, \ldots, n, 2n, \ldots, n+1) = \\ = \begin{pmatrix} Q(1, \ldots, n) & Q(1, \ldots, n || 2n, \ldots, n+1) \\ Q(2n, \ldots, n+1 || 1, \ldots, n) & Q(2n, \ldots, n+1) \end{pmatrix} .$$

For $i, j \in \{1, \ldots, n\}$ we have

$$\begin{aligned} Q(1, .., n)_{i,j} &= (i-j)^2 \,, & (1) \\ Q(2n, \ldots, n+1)_{i,j} &= ((2n-i+1) - (2n-j+1))^2 \\ &= (i-j)^2 \,, \\ Q(1, \ldots, n || 2n, \ldots, n+1)_{i,j} &= (i - (2n-j+1))^2 \\ &= (2n-i-j+1)^2 \,, & (2) \\ Q(2n, \ldots, n+1 || 1, \ldots, n)_{i,j} &= ((2n-i+1) - j)^2 \\ &= (2n-i-j+1)^2 \,. \end{aligned}$$

In particular, we see that

$$\begin{aligned} Q(1, \ldots, n) &= Q(2n, \ldots, n+1) \,, \\ Q(1, \ldots, n || 2n, \ldots, n+1) &= Q(2n, \ldots, n+1 || 1, \ldots, n) \,. \end{aligned}$$

Next, we want to show that we can write

$$Q(1, \ldots, n || 2n, \ldots, n+1) = Q(1, \ldots, n) + V \,,$$

where V is a matrix of nonnegative rank one. Let $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, where $u_i = (2n - 2i + 1)$ and $v_j = (2n - 2j + 1)$. Then

$$(2n - i - j + 1)^2 = (i - j)^2 + u_i v_j \,. \tag{3}$$

This follows from the identity

$$(\frac{1}{2}(u_i + v_j))^2 = (\frac{1}{2}(u_i - v_j))^2 + u_i v_j \,,$$

noting that $u_i + v_j = 2(2n - i - j + 1)$ and $u_i - v_j = 2(j - i)$. (1),(2) and (3) give

$$Q(1, \ldots, n || 2n, \ldots, n + 1) = Q(1, \ldots, n) + u^t \cdot v \,.$$

The vectors u, v are nonnegative, and so $V := u^t v$ is a nonnegative rank one matrix. Altogether, we have shown that

$$\begin{aligned} Q(1, \ldots, n, 2n, \ldots, n + 1) &= \begin{pmatrix} Q(1, \ldots, n) & Q(1, \ldots, n) + V \\ Q(1, \ldots, n) + V & Q(1, \ldots, n) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \times Q(1, \ldots, n) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times V \,. \end{aligned}$$

Hence, by Lemma 5, $\mathrm{rk}_+ Q(1, \ldots, n, 2n, \ldots, n + 1) \leq \mathrm{rk}_+ Q(1, \ldots, n) + 2$. □

Theorem 1 follows from the last lemma. We have $\mathrm{rk}_+ Q(1) = 0$. If $n > 1$ is a power of two then $\mathrm{rk}_+ Q(1, \ldots, n) \leq \mathrm{rk}_+ Q(1, \ldots, n/2) + 2$, which implies

$$\mathrm{rk}_+ Q(1, \ldots, n) \leq 2 \log_2 n \,.$$

If n is not a power of two, $Q(1, \ldots, n)$ is a submatrix of $Q(1, \ldots, 2^{\lceil \log_2 n \rceil})$ and so $\mathrm{rk}_+ Q(1, \ldots, n) \leq 2 \lceil \log_2 n \rceil \leq 2(\log_2(n) + 1)$.

The Corollary follows by noting that for any k, $Q(a_1, \ldots, a_n) = Q(a_1 + k, \ldots, a_n + k)$. Hence we can without loss of generality assume that $a_1, \ldots, a_n$ are nonnegative and that $a_1, \ldots, a_n \in \{1, \ldots, m\}$ with $m = a_n - a_1 + 1$. Hence $Q(a_1, \ldots, a_n)$ is a submatrix of the matrix $Q(1, \ldots, m)$, where the latter has rank $O(\log m)$.

5. Other rank deficient distance matrices

One can construct several other examples of points $a_1, \ldots, a_n$ for which the distance matrix has logarithmic nonnegative rank. The innocuous Corollary 2 already applies to slowly growing sequences, such as $1^2, 2^2, \ldots, n^2$, but one can give more imaginative examples. However, they would hardly illuminate the question whether there exist some points for which the nonnegative rank is linear. We therefore give only one additional example. Lemma 8 is a general construction which is used in Proposition 9 to give a fast growing sequence whose distance matrix has a logarithmic nonnegative rank.

Lemma 7. *Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be nonnegative real numbers such that $\min_{i \in \{1,\dots,n\}} a_i \geq \max_{j \in \{1,\dots,n\}} b_j$. Then the matrix $M_{i,j} = a_i - b_j, i, j \in \{1, \dots, n\}$ has nonnegative rank at most two.*

Proof. Let $m := \min_{i \in \{1,\dots,n\}} a_i$. Then $M_{i,j} = (a_i - m) + (m - b_j)$. This means that M can be written as the sum of two nonnegative rank one matrices. □

Lemma 8. *Let $a_1 \leq \dots \leq a_n$ be positive real numbers and let r be such that $r \geq (a_n/a_1)^2$. Then*

$$rk_+ Q(a_1, \dots, a_n, ra_1, \dots, ra_n) \leq rk_+ Q(a_1, \dots, a_n) + 4\,.$$

Proof. Write

$$Q(a_1, \dots, a_n, ra_1, \dots, ra_n) = \\ = \begin{pmatrix} Q(a_1, \dots, a_n) & Q(a_1, \dots, a_n || ra_1, \dots, ra_n) \\ Q(ra_1, \dots, ra_n || a_1, \dots, a_n) & Q(ra_1, \dots, ra_n) \end{pmatrix}.$$

One can see that

$$Q(ra_1, \dots, ra_n) = r^2 Q(a_1, \dots, a_n)\,.$$

We want to show that for some nonnegative matrix V with $\mathrm{rk}_+ V \leq 2$,

$$Q(ra_1, \dots, ra_n || a_1, \dots, a_n) = rQ(a_1, \dots, a_n) + V\,. \quad (4)$$

We have

$$\begin{aligned} Q(ra_1, \dots, ra_n || a_1, \dots, a_n)_{i,j} &= (ra_i - a_j)^2\,, \\ Q(a_1, \dots, a_n)_{i,j} &= (a_i - a_j)^2\,. \end{aligned}$$

Note that

$$\begin{aligned} (ra_i - a_j)^2 &= r(a_i - a_j)^2 + (r^2 - r)a_i^2 - a_j^2(r - 1) \\ &= r(a_i - a_j)^2 + (r - 1)(ra_i^2 - a_j^2)\,. \end{aligned} \quad (5)$$

Let V be the matrix $V_{i,j} = (r-1)(ra_i^2 - a_j^2), i, j \in \{1, \dots, n\}$. By the assumption on r, we have $r \geq 1$ and $\min_{i \in \{1,\dots,n\}} ra_i^2 = ra_1^2 \geq a_n^2 = \max_{j \in \{1,\dots,n\}} a_j^2$. Hence by the previous lemma, V is a nonnegative matrix with $\mathrm{rk}_+ V \leq 2$ and by (5), it is such that (4) holds. Similarly, we can construct a V' of nonnegative rank ≤ 2 such that

$$Q(a_1, \dots, a_n || ra_1, \dots, ra_n) = rQ(a_1, \dots, a_n) + V'\,.$$

Altogether, we have shown that

$$\begin{aligned} Q(a_1, \dots, a_n, ra_1, \dots, ra_n) &= \\ &= \begin{pmatrix} Q(a_1, \dots, a_n) & rQ(a_1, \dots, a_n) + V' \\ rQ(a_1, \dots, a_n) + V & r^2 Q(a_1, \dots, a_n) \end{pmatrix} \\ &= \begin{pmatrix} 1 & r \\ r & r^2 \end{pmatrix} \times Q(a_1, \dots a_n) + \begin{pmatrix} 0 & V' \\ V & 0 \end{pmatrix}. \end{aligned}$$

Hence, by Lemma 5, we have $\mathrm{rk}_+ Q(a_1, \dots, a_n, ra_1, \dots, ra_n) \leq \mathrm{rk}_+ Q(a_1, \dots, a_n) + 4$. □

Proposition 9. *There exists an infinite sequence of natural numbers $b_1 < b_2 < \dots$ such that for every n, $rk_+Q(2^{b_1}, \dots, 2^{b_n}) \leq 4\log_2 n + 2$.*

Proof. We are supposed to give an increasing sequence of natural numbers $a_1, a_2, \dots$ such that every a_i is a power of two and $Q(a_1, \dots, a_n)$ has a logarithmic nonnegative rank. Construct the sequence recursively by means of the previous lemma. Let $a_1 := 1$ and $a_2 := 2$. Assume that $n \geq 2$ is a power of two and that we have already constructed $a_1, \dots, a_n$. Let $r := (a_n/a_1)^2 = a_n^2$ and for $i \in \{1, \dots, n\}$, define $a_{n+i} := ra_i$.

By construction, every a_i is a power of two and the sequence $a_1, a_2, \dots$ is increasing. Since $\mathrm{rk}_+Q(a_1, a_2) = 2$, Lemma 8 implies that whenever $n \geq 2$ is a power of two then $\mathrm{rk}_+Q(a_1, \dots, a_n) \leq 4\log_2(n) - 2$. Hence $\mathrm{rk}_+Q(a_1, \dots, a_n) \leq 4\log_2(n) + 2$ for any n. □

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