Handbook of Set Theory
The Covering Lemma

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I. The Covering Lemma

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1. The statement

Jensen's discovery of the covering lemma arose out of work on the singular cardinals problem. Cohen published his proof of the independence of the continuum hypothesis [3, 4] in 1963, and one year later Easton's thesis [12, 13] completely settled the question of the size of the continuum for regular cardinals. The continuum problem for singular cardinals remained open, and the *Singular Cardinal Hypothesis* (SCH), stating that $2^\lambda = \lambda^+$ for every singular strong limit cardinal, became one of the most important problems in set theory. It was ten years before Silver made the first significant advance on the problem: In a sharp contrast to Easton's result, which stated that the only constraints on the size of the continuum for regular cardinals are the obvious ones, Silver [44] proved that the SCH cannot fail at a singular cardinal of uncountable cofinality unless it already fails at all but a nonstationary set of smaller cardinals. Silver's proof, which depended heavily on the use of the filter of closed unbounded subsets of λ, fails badly at cardinals of cofinality ω, and attention turned immediately to understanding this case. A year later, in 1974, Jensen distributed a series of handwritten notes titled *Marginalia on a Theorem of Silver*.[1] These notes, later revised by Devlin and Jensen and published [6] under the same title, stated and proved the basic covering lemma for L:

1.1 Theorem. *If $0^\sharp$ does not exist then for any set x of ordinals there is a set $y \in L$ such that $x \subseteq y$ and $|y| = |x| + \omega_1$.*

It is an immediate corollary that $\neg 0^\sharp$ implies the SCH, since if λ is a singular strong limit cardinal then theorem 1.1 implies that $2^\lambda = \lambda^{\mathrm{cf}(\lambda)} = (\lambda^\tau)^L\, 2^\tau \leq \lambda^+\, \lambda = \lambda^+$ where $\tau = \max\{\omega_1, \mathrm{cf}(\lambda)\}$.

[1] It is worth pointing out that this same problem also led to one of the major advances in set theory during the 1990's, Shelah's pcf theory ([41], see $\boxed{\text{pcf}}$).

The most obvious direction in which to extend the covering lemma is by weakening the assumption $\neg 0^\sharp$ to allow larger cardinals in the universe. The first step in this direction was due to Dodd and Jensen, who constructed a core model K^{DJ} under the assumption that there is no inner model with a measurable cardinal [8, 9, 7]. The Dodd-Jensen core model is, in many ways, similar to L: it satisfies the GCH along with most of the combinatorial properties of L, and it satisfies the same covering lemma:

1.2 Theorem. *Assume that there is no inner model with a measurable cardinal, and let K^{DJ} be the Dodd-Jensen core model. Then for any set x of ordinals there is a set $y \in K^{DJ}$ such that $y \supseteq x$ and $|y| = |x| + \omega_1$.*

The statement cannot be extended directly to larger cardinals, since Prikry forcing [38] gives a counterexample. However, Dodd and Jensen generalize theorem 1.2 to show that Prikry forcing is the only possible counterexample [10]:

1.3 Theorem. *Assume that $0^\dagger$ does not exist, but there is an inner model with a measurable cardinal, and that the model $L[U]$ is chosen so that $\kappa = \operatorname{crit}(U)$ is as small as possible. Then one of the following two statements holds:*

1. *For every set x of ordinals there is a set $y \in L[U]$ with $y \supseteq x$ and $|y| = |x| + \omega_1$.*
2. *There is a sequence $C \subseteq \kappa$, which is Prikry generic over $L[U]$, such that for all sets x of ordinals there is a set $y \in L[U, C]$ such that $y \supseteq x$ and $|y| = |x| + \omega_1$.*

Furthermore, the sequence C of clause (2) is unique up to finite initial segments.

Theorem 1.3 can easily be generalized to models with no inaccessible limit of measurable cardinals, but two problems have to be surmounted to extend it to larger cardinals: (i) it is necessary to construct a core model which is consistent with the larger cardinals and for which the basic argument of the proof of the covering lemma can be made to work, and (ii) it is necessary to find a useful statement of the covering lemma for this core model which can be proved from the basic argument. The construction and basic properties of the core model is given in chapter $\boxed{\text{Steel}}$. In addition this chapter includes, in section 4, an outline of the theory of the core model for sequences of measures and for non-overlapping extenders.

A statement of the full covering lemma for these models will be deferred to section 4 of this paper, but we state here a simplified version which generalizes the result of Dodd and Jensen by showing that a singular cardinal which is regular in K is made singular by a set which approximates a Prikry-Magidor generic set (see chapter $\boxed{\text{introduction/§ 2.2}}$).

Say that a cardinal κ is *μ-measurable* if there is an embedding $i\colon V \to M$ such that the measure $\{\, x \subseteq \kappa : \kappa \in i(x) \,\}$ associated with i is a member of M. This is the weakest cardinal property which requires the existence of something more than normal ultrafilters.

For the rest of this subsection we assume that there is no inner model with a μ-measurable cardinal, and we assume that K is the core model. If κ is a cardinal of κ then use $\mathcal{U}(\kappa, \beta)$, for $\beta < o(\kappa)$, to denote the measures on κ in K. First we define what appears to be a rather weak notion of indiscernablity:

1.4 Definition. Assume that κ is a singular cardinal which is regular in K. A closed unbounded subset C of κ is a weak Prikry-Magidor set for K if (i) $|C| < \kappa$, and (ii) if x is any closed unbounded subset of κ with $x \in K$ then $C - x$ is bounded in κ.

Any Prikry-Magidor generic subset of κ is a weak Prikry-Magidor set.

1.5 Theorem. *If there is no model with a μ-measurable cardinal then any weak Prikry-Magidor set $C \subseteq \kappa$ for K satisfies the following two conditions:*

1. *C is eventually contained in any set $a \in K$ such that $a \in \mathcal{U}(\kappa, \beta)$ for all $\beta < o(\kappa)$.*
2. *$C \cap \lambda$ is a weak Prikry-Magidor set over K for every sufficiently large limit point λ of C.*

1.6 Definition. A function σ is an *assignment function in K* for C if

1. There is $h \in K$ such that $\sigma(\nu) = h(\nu)$ for all sufficiently large $\nu \in C$.
2. C is a set of indiscernibles for $\mathcal{U}(\kappa, \sigma(\nu))$ in the sense that for any sequence $(\, a_\xi : \xi < \kappa \,) \in K \cap {}^{\kappa}\mathcal{P}(\kappa)$ of subsets of κ, and for all sufficiently large $\nu \in C$, we have $\forall \xi < \nu \left(\nu \in a_\xi \iff a_\xi \in \mathcal{U}(\kappa, \sigma(\nu))\right)$.

If $o(\kappa) < \kappa$, as in Prikry-Magidor forcing, then we can always take σ to be the function $\sigma(\nu) = o(\nu)$. If $o(\kappa) \geq \kappa^+$ then Radin forcing can be used to add a set C which satisfies the definition of a weak Prikry-Magidor set except that κ remains regular, and hence $|C| = \kappa$. Clearly such a set does not have an assignment function, since any assignment function would be bounded in κ^+. Thus the following theorem requires the condition that $|C| < \kappa$ in definition 1.4.

1.7 Theorem. *Any weak Prikry-Magidor set set $C \subseteq \kappa$ for K has an assignment function in K. Furthermore*

1. *The assignment function σ is unique except for initial segments,*
2. *The assignment function is weakly increasing in the sense that $\sigma(\nu) \geq \limsup_{\xi \in C \cap \nu} \sigma(\xi) + 1$ for every sufficiently large limit point ν of C.*

If the semicontinuity property of clause (2) is strengthened by replacing the equality with an inequality, then any weak Prikry-Magidor set satisfying the modified statement is a Prikry-Magidor generic set.

We cannot hope to actually cover subsets of κ using indiscernibles for only a single cardinal κ, but the following theorem generalizes Dodd and Jensen's theorem 1.3 to say that any small subset of κ can be approximated by a weak Prikry-Magidor set:

1.8 Theorem. *If the singular cardinal κ is regular in K then for any set $x \in \mathcal{P}_\kappa(\kappa)$ there is a weak Prikry-Magidor set $D \subseteq \kappa$ for K and a function $g\colon \kappa \to \kappa$ in K such that $x - \bigcup_{\nu \in C}(g(\nu) - \nu)$ is bounded in κ.*

The weak covering lemma

No satisfactory statement of the full covering lemma is known for cardinals much larger than a single strong cardinal. What remains is known as the weak covering lemma, which is proved by using the proof of the full covering lemma, but applying it only to intervals in which there are no indiscernibles.

1.9 Definition. A class model M of set theory satisfies the *weak covering property* if ${\lambda^+}^M = \lambda^+$ for every singular strong limit cardinal λ of V.

The weak covering lemma, stating that K has the weak covering property, is among the most important consequences of the covering lemma. If K contains more than a few measurable cardinals then the weak covering property is needed to prove the basic properties of the core model, including the full covering lemma; indeed the weak covering property may be taken as part of the definition of what it means to be a "core model". The best results known to date are as follows:

1.10 Theorem.

1. *If the sharp for a model with a class of strong cardinals, does not exist then there is a core model K of the form $L[\mathcal{E}]$ which satisfies the weak covering property.*

2. *If there are no models with a Woodin cardinal and there is a measurable cardinal Θ, then the Steel core model $K_\theta = L_\theta[\mathcal{E}]$ below Θ exists, and satisfies the weak covering property for $\lambda < \Theta$ (see $\boxed{\text{Steel}}$, [36, 37, 39]).*

3. *If $AD^{L(\mathbb{R})}$ is false, then either the Steel core model of clause* (2) *exists and satisfies the weak covering property, or else there is a ordinal α such that the Woodin core model K_{V_α} exists and satisfies the weak covering property above α (see $\boxed{\text{Steel}}$,[39]).*

The proof of clause (1) of theorem 1.10 will be sketched in section 4. The proof of clause (2) is given in [37] and [36], and part of the proof of clause (3) is given in [39].

It might be noted that the core model K_{V_α} of clause (3) contains all of V_α and thus only gives useful information about the universe above α. It is not clear what, if anything, can be done in the actual vicinity of a Woodin cardinal. Mitchell [35] reports some unsatisfactory results from applying the standard proof at a Woodin cardinal, but the following result of Woodin may suggest a more useful direction. The theorem only applies below 2^ω (assuming AC in V) but that is the region where the large cardinals implied by AD exist. This result also goes beyond the large cardinal limit of $\neg$AD imposed by theorem 1.10.

1.11 Theorem (Woodin [50]). *Suppose that the nonstationary ideal on ω_1 is ω_2-saturated, and suppose that M is a transitive inner model of ZF + DC + AD containing all reals and ordinals such that every set of reals in M is, in V, weakly homogeneously Souslin. Let X be a bounded subset of θ^M such that $|X| = \omega_1$. Then there exists $Y \supseteq X$ in M such that $|Y|^M = \omega_1$.*

The Strong covering lemma

This concludes, until section 4, the discussion of cardinals larger than a measurable cardinal. We now return to the models L and $L[U]$ in order to look at another direction in which the original covering lemma has been extended. The *strong covering lemmas* use Jensen's proof but show that more can be extracted from it. Theorem 1.12, which is our version of the strong covering lemma for the Dodd-Jensen core model, is essentially taken from unpublished notes of Carlson, who proved it for L by using a variant, influenced by ideas of Silver, of Jensen's proof. The idea, as well as the name, comes from work of Shelah (see [41, theorem VII.0.1] and [42]) who obtains the strong covering property in a more general setting by assuming the ordinary covering property together with some extra combinatorial structure. We will describe his main application in the next section.

1.12 Theorem. *Assume that there is no inner model with a measurable cardinal, and let K be the Dodd-Jensen core model. Then there is a class $\mathbf{C} \subseteq K$, definable in K, such that the following statements hold:*

1. *If x is any uncountable set of ordinals then there is a set $X \in \mathbf{C}$ such that $x \subseteq X$ and $|x| = |X|$.*
2. *The class $\mathbf{C}$ is closed in V under increasing unions of uncountable cofinality; that is, if $(X_\nu : \nu < \eta)$ is an increasing set of members of $\mathbf{C}$ and $\mathrm{cf}(\eta) > \omega$ then $\bigcup_{\nu<\eta} \in \mathbf{C}$.*

The statement of theorem 1.12 remains valid if $L[U]$ exists but $0^\dagger$ does not, provided that K is replaced by the appropriate model $L[U]$ or $L[U, C]$ from theorem 1.3. The following statement, however, is stronger and is easier to generalize to larger core models (see section 4).

1.13 Theorem. *Assume that $0^\dagger$ does not exist, and that the measure U and Prikry sequence C are as in theorem 1.3(2). Then there is a class $\mathbf{C} \subseteq L[U, C]$ which satisfies clauses (1) and (2) of theorem 1.12, and in addition*

3. *For each set $X \in \mathbf{C}$ there is an ordinal $\rho < \kappa$ and a function $h \in L[U]$ such that $X = \mathcal{H}^h(\rho \cup C)$, the smallest set containing $\rho \cup C$ and closed under h.*

4. *The class $\mathbf{C}$ is definable in $L[U]$ in the sense that there is a formula φ such that a set X is in $\mathbf{C}$ if and only if there is a set $A \in U$, a function $h \in L[U]$ and an ordinal ρ such that $L[U] \models \varphi(A, h, \rho)$ and $X = \mathcal{H}^h(\rho \cup (C \cap A))$.*

Clause (4) follows from the definability of forcing: the formula $\varphi(A, h, \rho)$ asserts that $(\varnothing, A) \Vdash \mathcal{H}^h(\check{\rho} \cup \dot{C}) \in \dot{\mathbf{C}}$, where the forcing is Prikry forcing for the measure U, $\dot{C}$ is a name for the resulting Prikry sequence, and $\dot{\mathbf{C}}$ is a name, derived from the proof of the covering lemma, for the class $\mathbf{C}$.

The following proposition gives a very useful property of the function h. It is also true for L, for the Dodd-Jensen core model, and for the core model for sequences of measures.

1.14 Proposition. *Let h be as in theorem 1.13 for $X \in \mathbf{C}$. Then h can be written as $h = \bigcup_{\nu<\alpha} h_\alpha$ for some functions $h_\nu \in X$ such that $h_\nu \subseteq h_{\nu'}$ whenever $\nu < \nu' < \alpha$.*

The Covering lemma without second order closure

The strong covering lemma can be viewed as asserting that if $0^\sharp$ does not exist then every sufficiently closed set is a member of L. The precise statement of the requirement that X be sufficiently closed has both first order and second order components. Magidor's covering lemma [25] for L weakens the conclusion of the covering lemma in order to eliminate the second order components:

1.15 Theorem (Magidor [25]). *If $0^\sharp$ does not exist and x is a set of ordinals which is closed under the primitive recursive set functions then there are sets $y_n \in L$ for $n < \omega$ such that $x = \bigcup_{n<\omega} y_n$.*

Magidor also extends theorem 1.15 to the Dodd-Jensen core model, by requiring closure under a larger set of functions in K^{DJ} and assuming that there is no inner model with a ω_1-Erdős cardinal. He points out that this assumption is necessary, since if there is a ω_1-Erdős cardinal in K then there is a generic extension M of K such that for any countable set $\mathcal{F}$ of functions in K there is a set $X \in M$ which is closed under the functions in $\mathcal{F}$, but is not a countable union of sets in K.

The following theorem was proved independently of theorem 1.15, but the same idea lies behind both theorems.

1.16 Theorem (Mitchell [26, 34], Jensen [11]). *If there is no model with a Woodin cardinal then every regular Jónsson cardinal is Ramsey in the core model K.*

Furthermore, if κ is δ-Jónsson for some uncountable ordinal $\delta < \kappa$ then κ is δ-Erdős in K.

A cardinal κ is said to be δ-Jónsson if any structure with universe κ and countably many predicates has an elementary substructure with order type δ; and κ is said to be δ-Erdős [2] if for any such structure and any closed unbounded subset C of κ there is a normal set of indiscernibles contained in C.

A similar proof shows that Chang's conjecture implies that ω_2 is ω_1-Erdős in K, and together with a result of Silver (1967, unpublished) proves the equiconsistency of the two notions.

This concludes our discussion of the various statements of the covering lemma. In section 2 we will briefly describe some of the basic applications of the core model, and in section 3 we will outline the basic proof of the covering lemma and its variants under the hypothesis that $0^\dagger$ does not exist. The final section looks at larger cardinals, giving the statement and an outline of the proof of the covering lemma for sequences of ultrafilters or extenders. The basic proof is taken almost unchanged from section 3, but the analysis of the resulting system of indiscernibles is much more difficult.

2. Basic applications

We pointed out earlier that the source of the covering lemma, as well as its first application, is the Singular Cardinal Hypothesis:

2.1 Theorem (Jensen [6]). *If $0^\sharp$ does not exist then $2^\lambda = \lambda^+$ for every singular strong limit cardinal λ.*

Jensen's proof can be generalized to larger cardinals, but the full strength of the failure of the SCH was not discovered until Gitik combined the covering lemma with Shelah's pcf theory:

2.2 Theorem (Gitik [15, 18]). *The failure of the singular cardinal hypothesis is equiconsistent with $o(\kappa) = \kappa^{++}$.*

See chapter $\boxed{\text{gitik}}$ for the proof of this result and a full introduction to the extensive theory, due almost entirely to Gitik, which has grown from it.

The following theorem is Shelah's main application of the strong covering lemma 1.12. The sufficiency of slightly stronger large cardinal assumptions is proved in [43].

2.3 Theorem. *If M is a model containing K such that $M \models GCH$, and r is a real such that $M[r] \models \neg CH$, then there is an inner model with an inaccessible cardinal. If, in addition, the cardinals of $M[r]$ are the same as those of M then there is a model with a measurable cardinal.*

The weak covering lemma

By far the most important consequence of the covering lemma is the weak covering property, definition 1.9. Indeed it is arguably more accurate to turn the statement around: the covering lemma is an application, and not necessarily the most important application, of the weak covering lemma. Below a strong cardinal the proof of the weak covering lemma is a special case of the proof of the full covering lemma, so that the importance of the weak covering lemma is not immediately apparent. Beyond a strong cardinal, in Steel's core model, it is not possible to even begin the proof of the covering lemma without first proving, by an entirely different method requiring a weak large cardinal hypothesis, a slightly weaker version of the weak covering lemma.

Among the most important properties of the core model K (stated under the assumption that $0^{\P}$ does not exist) which follow from the the weak covering lemma are the following:

- The construction of K from the countably complete core model K^{c}.
- If $i\colon K \to M$ is an elementary embedding, where M is well founded, then i is an iterated ultrapower of K.
- If U is a normal K-ultrafilter on κ and $\mathrm{ult}(K, U)$ is well founded then $U \in K$. If $\mathrm{crit}(U) > \omega_2$ then the hypothesis that $\mathrm{ult}(K, U)$ is well founded can be omitted.

Many results which are usually regarded as consequences of the covering lemma in fact use only these basic properties of the core model. Among such results are the lower bounds in the following theorem:

2.4 Theorem. *1. The failure of the GCH at a measurable cardinal κ is equiconsistent with $o(\kappa) = \kappa^{++}$.*

2. If κ is weakly compact and $o^K(\kappa) < \kappa^{++}$ then $\kappa^{+K} = \kappa^+$.

3. The consistency of a Woodin cardinal implies that of the existence of a saturated ideal on ω_1. If the Steel core model exists then the existence of such an ideal implies in turn that there is a Woodin cardinal in an inner model [45].

Sketch of proof. We prove, as an example, the lower bound for clause 1. The upper bound is proved in Gitik [14]. Suppose that U is a measure on

κ and $2^\kappa \geq \kappa^{++}$, but that $o(\kappa) < \kappa^{++}$ in K. Let $i^U : V \to M = \mathrm{ult}(V, U)$, and consider $i = i^U \restriction K : K \to K^M$. Then i is an iterated ultrapower of K, so let $i = i_{0,\theta}$ where $i_{\nu,\nu'} : N_\nu \to N_{\nu'}$. If $\nu < \theta$ is a limit ordinal there there is $\xi_\nu < \nu$ and $U_\nu \in N_{\xi_\nu}$ such that $N_{\nu+1} = \mathrm{ult}(N_\nu, i_{\xi_\nu,\nu}(U_\nu))$. Since $o(\kappa) < \kappa^{++} \leq \theta$, there is a stationary class $S \subseteq \kappa^{++}$ of ordinals of cofinality ω such that $\xi_\nu = \bar{\xi}$ and $U_\nu = \bar{U}$ are constant for $\nu \in S$. Now fix $\nu \in S \cap \lim(S)$. If $\vec{\kappa} = (\,\nu_n : n < \omega\,)$ is a cofinal sequence in $S \cap \nu$ and $\kappa_n = \mathrm{crit}(i_{\nu_n,\nu})$ then the sequence $\vec{\kappa}$ generates the measure $i_{\bar{\xi},\nu}(\bar{U})$. Since ${}^\omega M \subseteq M$, the sequence $\vec{\kappa}$ and hence the measure $i_{\bar{\xi},\nu}(\bar{U})$ is a member of M. It follows that $i_{\bar{\xi},\nu}(\bar{U}) \in K^M$; but this is impossible since $i_{\bar{\xi},\nu}(\bar{U})$ is not in $N_{\nu+1}$ and hence not in $N_\theta = K^M$. This contradiction completes the proof that $o(\kappa) \not< \kappa^{++}$ in K. $\dashv$

The naive proof of clause 2 uses the fact that κ is inaccessible; however Schindler [40] has used $\square_{\kappa,\kappa}$ in K to adapt it to successor cardinals, showing that if κ and κ^+ both have the tree property then κ is strong in K.

The reason why the weak covering property is so important is that it makes it possible to apply to K techniques which Kunen (see [23, §21]) originally used in proving that $0^\sharp$ follows from the existence of a nontrivial elementary embedding from L into L. As applied to L these techniques make use of the fact that any proper class $\Gamma \prec L$ is isomorphic to L. The corresponding fact for K is that any class $\Gamma \prec K$ is isomorphic to K, provided that

$$\left\{\, \lambda : \mathrm{otp}(\Gamma \cap \lambda^{+K}) = \lambda^{+K} \,\right\} \tag{I.1}$$

is a proper class. Cardinal calculations show that the classes Γ used in Kunen's arguments satisfy that

$$\left\{\, \lambda : 2^{<\lambda} = \lambda \wedge \mathrm{cf}(\lambda) < \lambda \wedge \left|\Gamma \cap \lambda^+\right| = \lambda^+ \,\right\} \tag{I.2}$$

is stationary. The weak covering lemma implies that the class (I.2) is contained in the class I.1 and hence implies that $\Gamma \cong K$.

The full covering lemma

The singular cardinal hypothesis has already been mentioned as a result which requires the full covering lemma. We now look at other such results.

2.5 Theorem (Dodd-Jensen [10], Mitchell [30]). *Let κ be a singular cardinal of cofinality λ which is regular in K. Then κ is measurable in K, and if $\lambda > \omega$ then $o(\kappa) \geq \lambda$ in K.*

The proof, using theorem 1.7, is easy, and a more careful analysis yields a classification of singular cardinals [31]:

2.6 Theorem ($\neg\exists\kappa\, o(\kappa) = \kappa^{++}$). *Assume that κ is a singular cardinal and that κ is regular in K. Then there is a weak Prikry-Magidor set C (with assignment function σ) witnessing the singularity of κ such that*

1. *If $\operatorname{cf}(\kappa) = \omega$ then let $\beta \leq o(\kappa)$ be the least ordinal such that $o(\nu) < \beta$ for all but boundedly many members of C. Then*
   (a) *If β is a successor ordinal then C is Prikry generic over K.*
   (b) *If $\operatorname{cf}(\beta) < \kappa$ then C is a weak Prikry-Magidor sequence.*
   (c) *If $\operatorname{cf}^K(\beta) = \kappa$ and $\tau \colon \kappa \to \beta$ is a cofinal function in K then there is a weak Prikry-Magidor sequence D with assignment function σ' such that $C = (\, c_n : n < \omega \,)$ where c_{n+1} is the least member c of D such that $\sigma'(c) \geq \tau(c_n)$.*
   (d) *If $\operatorname{cf}^K(\beta) = \kappa^+$ then C is a sequence of accumulation points for κ (the definition of an accumulation point is given in definition 4.14).*
2. *The set C is maximal in a sense which makes it definable (up to initial segment) except in case (1b), where any two such sequences eventually alternate.*

2.7 Theorem (Mitchell [27]). *If the closed, unbounded filter on ω_1 is an ultrafilter then there is a weak repeat point in K.*

A measure U on κ is a *weak repeat point* if for every set $A \in U$ there is $U' \lhd U$ with $A \in U'$. Analogous results for the nonstationary ideal have been proved by Gitik [17, 20].

2.8 Theorem (Sureson [47], Mitchell[29]). *The following four statements are equiconsistent, where $\delta < \kappa$ is a regular cardinal.*

1. *There is a κ-complete ultrafilter U on κ extending the closed, unbounded filter such that $\{\, \alpha : \operatorname{cf}(\alpha) = \delta \,\} \in U$.*
2. *There is a κ-complete ultrafilter U on κ with δ skies; that is, there is an increasing sequence $(\, \alpha_\nu : \nu < \delta \,)$ of ordinals between κ and $i^U(\kappa)$ such that if $\nu < \nu' < \delta$ and $f \colon \kappa \to \kappa$ then $i^U(f)(\alpha_\nu) < \alpha_{\nu'}$.*
3. *There is a κ^+-saturated normal filter F with $\{\, \alpha : \operatorname{cf}(\alpha) = \delta \,\} \in F$.*
4. *$o(\kappa) = \delta + 1$ if $\delta > \omega$, and $o(\kappa) = 2$ if $\delta = \omega$.*

The covering lemma is used to prove that each of clauses (1–3) imply that clause 4 holds in K. The forcing used in [29] to prove the other direction direction has been simplified and extensively generalized by Gitik; in particular it is used to give the upper bounds for the consistency strength of the failure of the SCH.

The Σ^1_3 absoluteness theorem, theorem 2.9 below, was originally proved by Jensen assuming $\neg 0^\dagger$; Magidor (unpublished, see [46, §4]) has given a simpler proof but one which gives slightly less information. Clause (1) was proved under the assumption that $\neg\exists\kappa\, o(\kappa = \kappa^{++})$ by Mitchell [32] using Jensen's method. Steel and Welch [46] later proved clause (1) using Magidor's method, and Steel, using results of Hjorth, extended it [45, theorem 7.9] to prove clause (2).

2.9 Theorem. *1. If there is no model of $o(\kappa) = \kappa^{++}$, and $r^\sharp$ exists for every real r, then K is Σ^1_3-correct.*

2. Assume that that there are two measurable cardinals and no inner model with a Woodin cardinal. Then K is Σ^1_3-correct.

3. The proof

This section outlines the proof of the Jensen and Dodd-Jensen covering lemmas up through a single measurable cardinal. Section 4 will continue, using the same basic ideas, to describe the covering lemmas for larger cardinals.

Subsection 3.1 briefly describes the basic tools, including fine structure, needed for the proof. Subsection 3.2 gives the proof of Jensen's covering lemma for L, theorem 1.1, (including the proof of strong variant, theorem 1.12, for L). Subsection 3.3 extends this proof to the Dodd-Jensen covering lemma, theorems 1.2 and 1.3. Finally section 3.4 looks at the two major variants on the covering lemma: Magidor's theorem 1.15, and the theorem 1.16 stating that Jónsson cardinals are Ramsey in K.

The proofs given in this section are not complete, but enough details are given that a reader with some understanding of fine structure should be able to fill in the rest. Complete proofs may be found in the original sources, [6, 7, 8, 9, 10, 22, 25], or in later references such as Devlin [5] and [23].

3.1. Fine Structure and Other Tools

This chapter has two incompatible aims: the first is to be accessible to a reader without a sophisticated knowledge of fine structure, and the second is to present a proof which is sufficiently complete that a reader with a understanding of fine structure can fill in the details.

One very interesting approach to this dilemma was invented by Silver (see [23, 25]), who gave a proof of the Jensen covering lemma which essentially eliminates any need for fine structure. He has extended this method to yield the Dodd-Jensen covering lemma, and it has been further extended and publicized by Magidor. In unpublished work, Magidor and Silver have used this approach at least up a model with a cardinal α such that $o(\alpha)$ is measurable. It is not known whether this approach works up to $o(\alpha) = \alpha^{++}$, and it seems unlikely that it will work for the newer models containing cardinals up to a Woodin cardinal. This rules out its use here, since this section is intended to serve as an introduction to covering lemmas for larger models.

The approach we have used is to work in a generic presentation of fine structure; one which has never actually been worked out in detail, especially for larger core models, but one which could (and possibly should) be. The hope is that this presentation will be sufficiently generic that a knowledgeable reader will be readily able to translate it to his preferred version, while at the same time it is sufficiently specific (without being too detailed) that it is understandable to a naive reader. Our presentation of fine structure, like Jensen's original papers, is based directly on master code structures. We use the L_α hierarchy, rather than the newer J_α hierarchy, since the former is more familiar to the naive reader and the substantial advantages of the later would not be apparent at this level of detail. At some points in the arguments, primarily those involving the downward extension lemma, it did not seem possible to give the full proof without being more specific about the fine structure; In these cases we restrict ourselves to giving the proof in the simplest case, which is Σ_1 definability over L_α for a limit ordinal α. This case may seem very special, but as we will argue that it essentially contains the general case.

As stated earlier, we take the basic models of our fine structure to be the sets L_α. We will call these models *mice* in anticipation of larger core models.

Two concepts are basic to the fine structure of a mouse $M = L_\alpha$: the Σ_n-projectum ρ_n^M and the Σ_n-Skolem function h_n^M. A third concept which is central to the proof of the covering lemma is the Σ_n-ultrafilter $\mathrm{ult}_n(M, \pi, \kappa)$ of M, obtained by using the embedding π as an extender. A fourth concept, the use of substructures of mice, is used in the definition of fine structure and is central to the proof of $\square_\kappa$ and other combinatorial applications of fine structure; however it is more peripheral to the proof of the covering lemma, where it is only needed for the non-countably closed case.

We discuss these four concepts further before beginning the actual proof. We begin with the definition of the fine structure of L_α in the special case when $n = 1$ and α is a limit ordinal.

3.1 Definition. Assume that α is a limit ordinal, and that $M = (L_\alpha, A)$ is *amenable*, that is, $A \cap x \in L_\alpha$ for all $x \in L_\alpha$.

1. The Σ_1 *projectum* ρ_1^M of an amenable structure $M = (L_\alpha, A)$ is the least ordinal ρ such that there is Σ_1 subset x of ρ which is not a member of L_α, but is Σ_1 definable in M using a finite set $p \subseteq \alpha$ as a parameter.

2. The Σ_1 *standard parameter* p_1^M of M is the least finite sequence $p \in$

$[\alpha]^{<\omega}$ of ordinals such that there is some set $x \subseteq \rho_1^M$ so that $x \notin L_\alpha$, but x is Σ_1 definable in M from parameters in $\rho_1^M \cup p$.

The ordering of the parameters is lexicographical on descending sequences of ordinals; that is, $p < p'$ if $\max(p \triangle p') \in p'$.

3. The Σ_1 *standard master code* is the set A_1^M of pairs $(\ulcorner\varphi\urcorner, \xi)$ such that $\xi < \rho_1^M$ and $\ulcorner\varphi\urcorner$ is the Gödel number of a Σ_1 formula φ over M, with parameter p_1^M, such that $M \models \varphi(\xi)$.
4. The Σ_1 *Skolem function* h_1^M of M is defined as follows: fix an enumeration $(\, \exists z\varphi_n : n < \omega \,)$ of the Σ_1 formulas of set theory. Then $h_1^M(\langle n, x\rangle)$ is defined if and only if there are z and y such that $M \models \varphi_n(x, y, z, p_1^M)$. In this case $h_1^M(\langle n, x\rangle) = y$ where (α', z, y) is the lexicographically least triple such that $(L_{\alpha'}, A \cap \alpha') \models \varphi_n(x, y, z, p_1^M)$.
5. The Σ_1-code $\mathfrak{C}_1(M)$ of M is the structure $(L_{\rho_1^M}, A_1^M)$.

The Σ_1-Skolem function is a function of one variable; however we will frequently abuse notation by writing it as a function with a variable number of arguments. Thus $h_1^M(x_1, x_2, x_3)$ should be understood to mean $h_1^M(\langle x_1, x_2, x_3\rangle)$ where $\langle . \rangle$ is an appropriate coding of finite sequences. In addition, we will abuse notation by writing h_1“x to mean h_1“$[{}^{<\omega}x]$, the closure of x under the Skolem function function h_1.

In addition, we will seldom consider the Skolem functions in connection with a specific formula φ_n; hence we will usually not mention the parameter n explicitly, but instead regard it as coded in the stated parameters.

If α is a successor ordinal, $\alpha = \gamma + 1$, then the definitions are the same, except that the hierarchy $(\, L_{\alpha'} : \alpha' < \alpha \,)$ used for the definition of h_1^M is replaced by a hierarchy, with length ω, of the sets in $L_\alpha - L_{\alpha+1}$. The hierarchy depends on the specific fine structure being used. Jensen originally used the Levy hierarchy on L_γ, the k^{th} level of which contains the subsets of L_γ which are Σ_k definable in (L_γ, A). Later he invented the rudimentary functions and the J_α-hierarchy in order to avoid technical complications caused by the use of the Levy hierarchy. One major advantage of the J_α-hierarchy over the L_α-hierarchy is that $[J_\gamma]^{<\omega} \subseteq J_\gamma$ for all ordinals γ.

We will not say more about the the successor case, except to mention that arguments involving fine structure generally treat the case of successor α as simpler special cases of the arguments for limit ordinals α. This characterization of the case of successor α as “simpler” assumes, of course, an understanding of the detailed definition of the fine structure.

We now turn to consider the fine structure for $n > 1$. The central theme of fine structure is that it is never necessary to deal directly with Σ_{n+1} definability for any n greater than zero; instead a Σ_{n+1} formula is reduced to an equivalent Σ_1 formula over the Σ_n code of L_α. The definition of the Σ_n code $\mathfrak{C}_n(L_\alpha)$ is itself a good example of this theme.

3.2 Definition. We define the Σ_n-codes of L_α by recursion on $n < \omega$. We set $\mathfrak{C}_0(L_\alpha) = (L_\alpha, \varnothing)$, and for $n \geq 0$

$$\rho^M_{n+1} = \rho_1^{\mathfrak{C}_n(L_\alpha)} \qquad p^{L_\alpha}_{n+1} = p_1^{\mathfrak{C}_n(L_\alpha)} \qquad h^{L_\alpha}_{n+1} = h_1^{\mathfrak{C}_n(L_\alpha)}$$
$$A^{L_\alpha}_{n+1} = A_1^{\mathfrak{C}_n(L_\alpha)} \qquad \mathfrak{C}_{n+1}(L_\alpha) = \mathfrak{C}_1(\mathfrak{C}_n(L_\alpha))$$

Finally, the projectum of L_α is defined to be $\mathrm{proj}(L_\alpha) = \rho^{L_\alpha} = \inf_n \rho^{L_\alpha}_n$. Since the projecti $\rho^{L_\alpha}_n$ are nonincreasing, $\rho^{L_\alpha}_n = \mathrm{proj}(L_\alpha)$ for all sufficiently large $n < \omega$.

Note that if $n > 1$ then the Σ_n Skolem function h^M_n need not be Σ_n definable over M. Jensen's Σ_n-uniformization theorem states that this construction can be used to define a Skolem function for Σ_n formulas over L_α which is Σ_n-definable in L_α (though not uniformly so). This Σ_n-uniformization theorem was an important part of the motivation for Jensen's invention of fine structure, but it has turned out to have little direct importance because it is simpler and more useful to work directly with the fine structure.

We now consider the problem of recovering the original structure $M = (L_\alpha, A)$ from its code $M_1 = \mathfrak{C}_1(M) = (L_{\rho^M_1}, A^M_1)$. In order to do so, recall that the Skolem function h^M_1 is Σ_1 definable in M from the parameter p^M_1, and that A^M_1 is the set of Gödel numbers of the Σ_1 theory of M with parameters from $\rho^M_1 \cup p^M_1$. Now define $X = \left\{ [\xi]_\sim : \xi \in \rho^M_1 \cap \mathrm{dom}(h^M_1) \right\}$, where $[\xi]_\sim$ is the equivalence class of ξ under the equivalence relation $\xi \sim \xi'$ iff $\ulcorner h^M_1(\xi) = h^M_1(\xi') \urcorner \in A^M_1$. The membership relation E on X is defined by $[\xi] \mathrel{\mathrm{E}} [\xi']$ iff $\left\ulcorner h^M_1(\xi) \in h^M_\xi(\xi') \right\urcorner \in A^M_1$; and the subset $\bar{A} \subset X$ is defined by setting $[\xi] \in \bar{A}$ iff $\ulcorner h^M_1(\xi) \in A \urcorner \in A^M_1$.

It is straightforward to verify that we can define a Σ_1-elementary embedding $i\colon (X, E, \bar{A}) \to (L_\alpha, \in, A)$ by setting $i([\xi]) = h^M_1(\xi)$. This embedding i is an isomorphism if and only if $M = h^{M_1}``(\rho^M_1)$, in which case we can say that this construction recovers M from its Σ_1-code $\mathfrak{C}_1(M)$.

3.3 Definition. The structure $M = (L_\alpha, A)$ is said to be *1-sound* if $L_\alpha = h^M_1``(\rho^M_1)$. Further, M is said to be *n-sound* if it is $n-1$-sound and $\mathfrak{C}_{n-1}(M)$ is 1-sound; and M is said to be *sound* if M is n-sound for all n.

Notice that if M is sound then one can repeat the process described above n times in order to recover M from any of its codes $\mathfrak{C}_n(M)$. Thus the following lemma is the basic fact of fine structure:

3.4 Lemma. *If α is any ordinal then the structure L_α is sound.*

Sketch of proof. We will only consider the case when α is a limit ordinal. First we show that L_α is 1-sound.

Let $Z = h^M_1``(\rho^M_1) \prec_1 L_\alpha$, and let i be the collapse map $i\colon \bar{M} \cong Z \prec_1 L_\alpha$. Since $\bar{M} \models$ "V = L", we must have $\bar{M} = L_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$. Since i is

Σ_1 elementary and $\rho_1^M \cup p_1^M \subseteq Y$, the set $A_1^{L_\alpha}$ is Σ_1 definable in $L_{\bar\alpha}$. Since $A_1^{L_\alpha} \notin L_\alpha$ it follows that $\bar\alpha = \alpha$.

Similarly, p_1^M is the least parameter which can be used to define A_1^M in L_α, and $i^{-1}(p_1^M) \leq p_1^M$, so $p_1^M = i(p_1^M)$. But every member of $\mathrm{dom}(i)$ is Σ_1 definable in $L_{\bar\alpha}$ from parameters in $\rho_1^M \cup p_1^M$, and it follows that i is the identity. Thus h_1^M "$(\rho_1^M) = Z = \mathrm{ran}(i) = L_\alpha$. This completes the proof that L_α is 1-sound.

Lemma 3.4 is proved for arbitrary n by a joint induction, which proves lemma 3.4 along with lemma 3.5 below. Assume as an induction hypothesis that L_α is n-sound, and let $\mathfrak{C}_n(M) = (L_{\rho_n^M}, A_n^M)$. Assume, in addition, that lemma 3.5 holds for $\mathfrak{C}_n(M)$. Now, in order to show that $\mathfrak{C}_n(M)$ is 1-sound, apply the argument above to $\mathfrak{C}_n(M)$ so that the isomorphism $i\colon L_{\bar\alpha} \cong Z \prec_1 L_\alpha$ becomes $i\colon (L_{\bar\rho}, \bar A) \cong Z \prec_1 (L_{\rho_n^M}, A_1^M)$. In order to continue the argument we need to know that $(L_{\bar\rho}, \bar A) = \mathfrak{C}_n(L_{\bar\alpha})$ for some $\bar\alpha \leq \alpha$, and this is exactly what lemma 3.5, assumed by the induction hypothesis to hold for $\mathfrak{C}_n(M)$, asserts. ⊣

The downward extension property, also known as condensation, is central to many applications of fine structure. This is one place where the advantages of the J_α hierarchy appear: the assumption $i\colon (L_{\rho'}, A') \prec_1 \mathfrak{C}_n(L_\alpha)$ can be weakened to $i\colon (L_{\rho'}, A') \prec_0 \mathfrak{C}_n(L_\alpha)$, provided that "$\prec_0$" is interpreted in the sense of the J_α-hierarchy to mean that $L_{\rho'}$ is A'-rudimentarily closed.

3.5 Lemma (Downward Extension). *Suppose that $i\colon (L_{\rho'}, A') \prec_1 \mathfrak{C}_n(L_\alpha)$. Then there is $\alpha' \leq \alpha$ such that $(L_{\rho'}, A') = \mathfrak{C}_n(L_{\alpha'})$, and i extends to a Σ_n-embedding $\tilde\imath\colon L_{\alpha'} \to L_\alpha$. Furthermore $\tilde\imath$ preserves the first n stages of the fine structure, so that $\tilde\imath\, h_k^{L_{\alpha'}} = h_k^{L_\alpha}\, \tilde\imath$ for all $k \leq n$.*

Sketch of proof. The lemma is proved by a joint induction on n, along with lemma 3.4 above. Apply to the structure $(L_{\rho'}, A')$ the construction described before definition 3.3 which recovers a structure (L_α, A) from its Σ_1-code $\mathfrak{C}_1(L_\alpha, A)$. The assumption that i is Σ_0-elementary implies that the construction succeeds at least to the extent of defining the model $(X, E, \bar A)$ and the embedding $i'\colon (X, E, \bar A) \to \mathfrak{C}_{n-1}(L_\alpha)$, and the embedding i' ensures that (X, E) is well founded, so that $X \cong L_{\rho''}$ for some ordinal ρ''. If A'' is the image of $\bar A$ under this isomorphism then i' induces an embedding $\tilde\imath_{n-1}\colon (L_{\rho''}, A'') \to \mathfrak{C}_{n-1}(L_\alpha)$.

By the construction, the set A' encodes the Σ_1 theory of $(L_{\rho''}, A'')$, and since i is σ_0 elementary and A encodes the Σ_1 theory of $\mathfrak{C}_{n-1}(L_\alpha)$ it follows that $\tilde\imath_{n-1}$ is a Σ_1-elementary embedding. By the induction hypothesis it follows that there is an ordinal α' such that $(L_{\rho''}, A'') = \mathfrak{C}_{n-1}(L_{\alpha'})$, and an embedding $\tilde\imath_{n-1}\colon L_{\alpha'} \to L_\alpha$ which extends $\tilde\imath_{n-1}$ and which preserves the first $n-1$ stages of the fine structure, as far as $\mathfrak{C}_{n-1}(L_{\alpha'})$.

Now it only remains to verify that $(L_{\rho'}, A') = \mathfrak{C}_1(L_{\rho''}, A'')$, which entails verifying that $\rho_1^{(L_{\rho''},A'')} = \rho'$ and $p_1^{(L_{\rho''},A'')} = \tilde{\imath}_{n-1}^{-1}(p_1^{\mathfrak{C}_{n-1}(L_\alpha)})$. The inequalities $\rho_1^{(L_{\alpha''},A'')} \leq \rho'$ and $p_1^{(L_{\rho''},A'')} \leq \tilde{\imath}_{n-1}^{-1}(p_1^{\mathfrak{C}_{n-1}(L_\alpha)})$ follow from the fact that $A' \notin L_{\rho''}$, which is proved by the argument of the Russell paradox. The inequality $\rho' \leq \rho_1^{(L_{\alpha''},A'')}$ follows from the fact that $A' \cap \xi \in L_{\rho'}$ for $\xi < \rho'$, which follows from the assumption that $(L_{\rho'}, A') \prec_1 \mathfrak{C}_n(L_\alpha)$. The final inequality $p_1^{(L_{\rho''},A'')} \geq \tilde{\imath}_{n-1}^{-1}(p_1^{\mathfrak{C}_{n-1}(L_\alpha)})$ is the point in the proof which uses the inclusion of lemma 3.4 in the induction hypothesis: assume for the sake of contradiction that $\rho_n^{L_{\bar{\alpha}}} < \tilde{\imath}_{n-1}^{-1}(p_n)^{L_\alpha}$ and apply lemma 3.4 to $L_{\bar{\alpha}}$. This implies that $\tilde{\imath}_{n-1}^{-1}(p_n^{L_\alpha})$ is Σ_1 definable in $(L_{\alpha'}, A')$ from $p_n^{L_{\bar{\alpha}}}$. It follows that $p_n^{L_\alpha}$ is Σ_1 definable in $\mathfrak{C}_{n-1}(L_\alpha)$ from $\tilde{\imath}_{n-1}(p_n^{L_{\bar{\alpha}}}) < p_n^{L_\alpha}$, but this contradicts the definition of $p_n^{L_\alpha}$.

This completes the proof of lemma 3.5, except for the claim that $\tilde{\imath}$ is Σ_n-elementary. To see this, notice that the embedding i'' constructed in the induction step is one quantifier stronger than i'. The map $\tilde{\imath}$ is obtained by repeating this process n times, and hence the original Σ_0 embedding is strengthened to a Σ_n-elementary embedding $\tilde{i}\colon L_{\alpha'} \to L_\alpha$. ⊣

It should be noted that the statement that the embedding $\tilde{\imath}$ preserves the fine structure is stronger—and usually more useful—than the statement that $\tilde{i}$ is Σ_n-elementary.

If we define $\bar{h}_n^{L_\alpha} = h_1^{L_\alpha} \dots h_n^{L_\alpha}$ then $\bar{h}^{L_\alpha}\colon \rho_n^{L_\alpha} \to L_\alpha$, and an induction using lemma 3.4 shows that $L_\alpha = \bar{h}_n^{L_\alpha}$“$\rho_n^{L_\alpha}$. In order to avoid considering detailed fine structure as much as possible, we make the following convention:

Notation. Unless stated otherwise, we abuse notation by using $h_n^{L_\alpha}$ to denote the function $\bar{h}_n^{L_\alpha}$ described above, and we call it the Σ_n-Skolem function of L_α.

We end the discussion of lemma 3.5 by using it to prove lemma 1.14 from the introduction, which is frequently useful in applications of the covering lemma.

3.6 Lemma. *The Σ_n-Skolem function $h_n^{L_\alpha}$ of L_α can be written as an increasing union $h_n^{L_\alpha} = \bigcup_{\nu<\eta} g_\nu$ of functions $g_\nu \in L_\alpha$, with $\eta \leq \rho_n^{L_\alpha}$.*

Sketch of proof. First consider the case when $n = 1$ and α is a limit ordinal. Pick a sequence of ordinals α_ν cofinal in α, and define g_ν to be the function defined in L_{α_ν} by the same Σ_1 formula (with the same parameter $p_1^{L_\alpha}$) as was used to define $h_1^{L_\alpha}$ in L_α. Thus $g_\nu(x) = y$ whenever $h_1^{L_\alpha}(x) = y$, with $x, y \in L_{\alpha_\nu}$, and in addition the witness to the Σ_1 fact “$h_1^{L_\alpha}(x) = y$” is also a member of L_{α_ν}.

For $n > 1$, apply the construction above to $\mathfrak{C}_{n-1}(L_\alpha)$, noting that $\rho_n^{L_\alpha}$ is always a limit ordinal for $n > 0$. ⊣

The importance of lemma 3.5 to fine structure theory extends far beyond the arguments above; however its importance in the proof of the covering lemma is secondary to the that of the upward extension property, lemma 3.8, described below.

Embeddings of mice

In this subsection we define a generalized ultrapower which is central to the proof of the covering lemma. This ultrapower, which is used to extend a given embedding $\pi\colon L_{\bar\kappa} \to L_\kappa$ to an embedding $\tilde\pi\colon L_{\bar\alpha} \to L_{\alpha'}$ with a larger domain, can be described in modern terms, as the ultrapower by the extender $E_{\pi,\beta}$ of length β which is associated with the embedding π. It should be noted, however, that this construction of Jensen is older, and in fact is ancestral to, the modern notion of an extender. Extenders are more completely described in $\boxed{\text{steel}}$.

We first explain the extender construction by defining the Σ_0-ultrapower $\mathrm{ult}(M,\pi,\beta)$ of a model M.

3.7 Definition. Assume that $\pi\colon N \to N'$ is a Σ_0-elementary embedding, and that $\mathcal{P}(\nu)\cap M \subseteq N$ for all $\nu \le \bar\kappa$ such that $\sup \pi``\nu < \beta$. Then

$$\mathrm{ult}(M,\pi,\beta) = \big\{\, [a,f]_\pi : f \in M \text{ and } \mathrm{dom}(f) \in \mathrm{dom}(\pi) \text{ and } a \in [\beta]^{<\omega} \cap \pi(\mathrm{dom}(f)) \,\big\} \qquad \text{(I.3)}$$

where $[a,f]_\pi$ is the equivalence class of the pair (a,f) under the relation

$$(a,f) \sim_\pi (a',f') \iff (a,a') \in \pi\big(\{\, (\vec\nu,\vec\nu') : f(\vec\nu) = f'(\vec\nu') \,\}\big). \qquad \text{(I.4)}$$

The membership relation E_π and any other predicates of $\widetilde{M}$ are defined similarly, and the embedding $i\colon M \to \mathrm{ult}(M,\pi,\beta)$ is defined as usual by $i(x) = [a, C_x]_\pi$ where a is arbitrary and C_x is the constant function, $\forall z\, C_x(z) = x$.

The embedding $i\colon M \to \mathrm{ult}(M,\pi,\beta)$ satisfies the Łos theorem for Σ_0 formulas. If $M = L_\alpha$ for some ordinal α then $\mathrm{ran}(i)$ is cofinal in $\mathrm{ult}(M,\pi,\beta)$, and it follows that i is Σ_1-elementary. In particular, i preserves the Σ_1-Skolem function of L_α.

We will need to define Σ_n-ultrapowers, for arbitrary $n \in \omega$, so that they preserve Σ_{n+1}-Skolem functions. The obvious way to define such an ultrapower is to modify definition 3.7 by replacing the condition "$f \in M$" of (I.3) with "f is Σ_n-definable in L_α"; however doing so requires first proving Jensen's uniformization theorem, which states that there is a Σ_n-definable Skolem function for Σ_n formulas on L_α. A second possible approach is that of Silver, who showed that it is possible to define $\mathrm{ult}_n(L_\alpha,\pi,\beta)$ by using compositions of the naive Σ_n-Skolem function, and that the naive Skolem function is preserved by the resulting embedding even though it is not defined by a Σ_n-formula. This is the simplest approach, as it avoids the use of fine structure, but it appears to have difficulties with models for larger cardinals.

Our approach will be closer to the first one, but will use the fine structure directly. The notion of Σ_n ultrapower which we use can be defined in two different, but equivalent, ways. One way is to define $\mathrm{ult}_n(L_\alpha, \pi, \beta)$ directly, using definition 3.7, but allowing any function f of the form $f(x) = h_n(x, q)$ where h_n is the Σ_n-Skolem function mapping a subset of $\rho_n^{L_\alpha}$ onto L_α, and $q \in L_{\rho_n^{L_\alpha}}$ is an arbitrary parameter. The other way is indirect, by taking the ordinary Σ_0 ultrapower $i\colon \mathfrak{C}_n(L_\alpha) \to \mathrm{ult}\big(\mathfrak{C}_n(L_\alpha), \pi, \beta\big)$ of the Σ_n code of L_α, and then extending this to a map $\tilde{i}\colon L_\alpha \to L_{\tilde{\alpha}}$. This approach has the advantage that most arguments can be carried out at the level of the Σ_n-code of L_α, which involves the easily understandable Σ_0-ultrapower and Σ_1-Skolem function.

The extension of π to an embedding $\tilde{\pi}$ with the larger domain L_α depends on lemma 3.8 below, which is the counterpart of the downward extension lemma 3.5 given earlier. One major difference between the upward and downward extension lemmas concerns the well foundedness of the new structure. In the downward extension lemma, this structure is a substructure of a given well founded structure and hence is automatically well founded. In the upward extension lemma the well foundedness of $\mathrm{ult}_n(L_\alpha, \pi, \beta)$ must be explicitly assumed.

3.8 Lemma (Upward Extension). *Suppose that $\pi\colon L_{\bar{\kappa}} \to L_\kappa$, that $\beta \le \kappa$, and either $\rho_n^{L_\alpha} > \min(\{\, \nu : \pi(\nu) \ge \beta \,\})$ or π is cofinal in β and $\pi(\rho_n^{L_\alpha}) \ge \beta$. Set $M_n = \mathfrak{C}_n(L_\alpha)$ and $\widetilde{M}_n = \mathrm{ult}(M_n, \pi, \kappa)$.*

1. *There is a structure $\widetilde{M}_0$ such that $\widetilde{M}_n$ is, formally, equal to $\mathfrak{C}_n(\widetilde{M}_0)$. If this structure $\widetilde{M}_0$ is well founded then $\widetilde{M}_0 = L_{\tilde{\alpha}}$ for some ordinal $\tilde{\alpha}$.*

2. *There is an embedding $\tilde{\pi}\colon L_\alpha \to \widetilde{M}_0$ such that $\pi{\restriction}L_{\bar{\beta}} = \tilde{\pi}{\restriction}L_{\bar{\beta}}$, where $\bar{\beta}$ is the least ordinal such that $\pi(\bar{\beta}) \ge \beta$ if $\beta < \kappa$, or $\bar{\beta} = \bar{\kappa}$ if $\beta = \kappa$.*

3. *The embedding $\tilde{\pi}$ preserves the Σ_k codes for $k \le n$: in particular, $\tilde{\pi}\, h_k^{L_\alpha}(x) = h_k^{\widetilde{M}_0}\, \tilde{\pi}(x)$ for all x for which either side is defined.*

4. *The embedding $\tilde{\pi}$ preserves the Σ_1-Skolem function of M_n in the sense that there is a function $\tilde{h}$, which is Σ_1 definable over $\widetilde{M}_n$, such that such that $\tilde{\pi}\, h_{n+1}^{M}(x) = \tilde{h}\, \tilde{\pi}(x)$ for all $x \in M$ such that either side is defined.*

The proof of the upward extension lemma is nearly the same as that of the downward extension lemma 3.5. The models $\widetilde{M}_{n-k}$, and embeddings

$\tilde{\pi}_{n-k}\colon \mathfrak{C}_{n-k}(L_\alpha) \to \widetilde{M}_{n-k}$ are defined by recursion on $k \leq n$: The embedding $\tilde{\pi}_0\colon \mathfrak{C}_n(L_\alpha) \to \widetilde{M}_n$ is the Σ_0 ultrapower, and $\widetilde{M}_{n-(k+1)}$ is constructed from $\widetilde{M}_{n-k}$ by the same recovery process as was used for the downward extension lemma. This process uses the fact that $\widetilde{M}_{n-k}$ has the form $(\widetilde{M}_{n-k}, \mathrm{E}, \tilde{A}_{n-k})$ and satisfies the first order sentences asserting that $(\widetilde{M}_{n-k}, \mathrm{E})$ is a model of $\mathrm{V} = \mathrm{L}$, and that $\tilde{A}_{n-k}$ is the Σ_n theory of a larger model $\widetilde{M}_{n-(k+1)}$ of which $\widetilde{M}_{n-k}$ is the Σ_1-code.

The embedding $\tilde{\pi}$ does not, in general, preserve fine structure below ρ_n^M; for example if β is a cardinal in L then $\rho^{\widetilde{M}} \geq \beta$, since every bounded subset of β in L is a member of L_β, but it may happen that $\rho_{n+1}^M < \bar{\beta}$. In this case $\tilde{\pi}(\rho_{n+1}^M) = \pi(\rho_{n+1}^M) < \beta$. It follows that the function $\tilde{h}$ will not, in general, be the Σ_1-Skolem function $h_1^{\widetilde{M}_n}$. It is defined by the same formula as $h_1^{\widetilde{M}_n}$, but using the image $\tilde{\pi}_0(p_1^{L_{M_n}})$ of the standard parameter of M_n instead of the standard parameter $p_1^{\widetilde{M}_n}$ of $\widetilde{M}_n$.

In order for the ultrapower $\mathrm{ult}_n(M, \pi, \beta) = \mathrm{ult}_{\Sigma_M}(M, \pi, \beta)$ described above to be defined, the set $\{\, (\vec{\nu}, \vec{\nu}') : f(\vec{\nu}) = f'(\vec{\nu}') \,\}$ must be in the domain of π for each pair (f, f') of functions in $\Sigma_n(M)$. This yields the following condition for the existence of $\mathrm{ult}_n(M, \pi, \beta)$:

3.9 Proposition. *Let M, π and β be as above, and let $\bar{\beta}$ be the least ordinal such that $\pi(\bar{\beta}) \geq \beta$ (or $\bar{\beta} = \mathrm{Ord}(N)$ if $\beta = \sup(\mathrm{ran}(\pi))$). Then the ultrapower $\mathrm{ult}_n(M, \pi, \beta)$ is defined if and only if either $\rho_n^M > \bar{\beta}$, or else $\rho_n^M \geq \bar{\beta}$ and $\mathrm{ran}(\pi)$ is cofinal in β.*

Equivalently, the ultrapower is defined if and only if either

1. *Every subset of $\bar{\kappa}$ which is Σ_n definable in M is a member of $L_{\bar{\kappa}}$, or*
2. *$\mathrm{ran}(\pi)$ is cofinal in β and every bounded subset of $\bar{\kappa}$ which is Σ_n definable in M is a member of $L_{\bar{\kappa}}$.*

3.2. Proof of the covering lemma for L

The main part of the proof of the covering lemma for L is a construction which shows that any suitable set $X \prec_1 L_\kappa$ (in a sense to be made precise in definition 3.12) is a member of L. The concluding part of the proof is an analysis of the notion of suitability showing that every uncountable set of ordinals is contained in a suitable set of the same cardinality.

The construction, together with the definition 3.12 of suitability, will prove the following lemma:

3.10 Lemma.
1. *If $X \prec_1 L_\kappa$ is suitable then there is a cardinal $\rho < \kappa$ of L and a function $h \in L$ such that $X = h``(\rho \cap X)$.*
2. *If $X \prec_1 L_\kappa$ is suitable and $\rho < \kappa$ is a cardinal of L then $X \cap L_\rho$ is also suitable.*

3.11 Corollary. *Any suitable set X is a member of L.*

Proof. The proof is by induction on κ. Let $X \subseteq \kappa$ be suitable, and let h and ρ be as in clause (1). Then $X \cap L_\rho$ is suitable by clause (2) and hence is in L by the induction hypothesis, but then $X = h``(X \cap \rho) \in L$. $\dashv$

In order to describe the basic construction, we fix a cardinal κ of L and a set $X \prec_1 L_\kappa$ with $\sup(X) = \kappa \not\subseteq X$. Let $\pi\colon N \to X$ be the collapse map, so that $N = L_{\bar\kappa}$ for some ordinal $\bar\kappa$, and let (α, n) be the lexicographically largest pair such that $\mathrm{ult}_n(L_\alpha, \pi, \kappa)$ is defined. There are two cases:

1. If $\mathcal{P}(\delta) \cap L \subseteq N$ for all $\delta < \bar\kappa$ then $L_\alpha = L$ and $n = 0$.
2. Otherwise α is the least ordinal such that there is a bounded subset of $\bar\kappa$ in $L_{\alpha+1} - L_{\bar\kappa}$, and n is the least integer such there is such a subset which is Σ_{n+1}-definable in L_α. Thus, if $M = L_\alpha$ then $\rho^M_{n+1} < \bar\kappa \le \rho^M_n$, and $\rho^{L_{\alpha'}}_m \ge \bar\kappa$ whenever $\bar\kappa \le \alpha' < \alpha$ and $m < \omega$.

The basic construction will succeed whenever $\widetilde{M} = \mathrm{ult}_n(L_\alpha, \pi, \kappa)$ is well founded; the definition 3.12 of suitability, given in the next subsection, is a generalization of this requirement. If case (1) occurs for some suitable set X then $\tilde\pi\colon L \to \mathrm{ult}(L, \pi, \kappa) = L$ is a nontrivial embedding from L into L, which implies by Kunen's theorem (see introduction/theorem 1.13) that $0^\sharp$ exists, contrary to the hypothesis of theorem 1.12. Thus we can assume that case (2) occurs for all suitable sets X. Then by lemma 3.8, $\mathrm{ult}_n(L_\alpha, \pi, \kappa) = L_{\tilde\alpha}$ for some ordinal $\tilde\alpha$, and the following diagram commutes:

$$\begin{array}{ccc} L_\alpha & \xrightarrow{\tilde\pi} & \widetilde{\mathcal{M}} = \mathrm{ult}_n(L_\alpha, \pi, \kappa) = L_{\tilde\alpha} \\ \uparrow & & \uparrow \\ L_{\bar\kappa} & \xrightarrow{\pi} & X \prec_1 L_\kappa \end{array} \tag{I.5}$$

Now let $\bar\rho = \rho^{L_\alpha}_{n+1}$. Then $\bar\rho < \bar\kappa$ and if $\bar h = h^{L_\alpha}_{n+1}$ then $L_\alpha = \bar h``\bar\rho$, so that

$$X = \pi``\bar\kappa = \pi``(\bar\kappa \cap \bar h``(\bar\rho)), \tag{I.6}$$

and furthermore

$$\tilde h \circ \tilde\pi = \tilde\pi \circ \bar h \tag{I.7}$$

where $\tilde h$ is the function given by lemma 3.8(4). Putting equations (I.6) and (I.7) together, we get

$$X = \kappa \cap (\widetilde{\pi} \circ \bar h``\bar\rho) = \kappa \cap (\tilde h \circ \widetilde{\pi}``\bar\rho) = \kappa \cap \tilde h``(X \cap \rho) \tag{I.8}$$

where $\rho = \pi(\bar\rho) < \kappa$. Since $\tilde h \in L$, this completes the basic construction.

Suitable sets

Here is the formal definition of suitability:

3.12 Definition. Suppose $X \subseteq L$ and let $\pi\colon N \cong X$ be the inverse of the transitive collapse. Then X is *suitable* if $X \prec_1 L_\kappa$ for some ordinal κ and $\mathrm{ult}_n(L_\alpha, \pi, \beta)$ is well founded for all triples (α, n, β) such that the ultrapower is defined.

We write $\mathbf{C}$ for the class of suitable sets.

Proof of lemma 3.10. If $X \prec_1 L_\kappa$ is any set in $\mathbf{C}$ then the basic construction succeeds for X, and hence clause (1) of lemma 3.10 holds for X. Clause (2) of that lemma is clear. $\dashv$

It follows by corollary 3.11 that every suitable set is in L, so Jensen's covering lemma 1.1 for L will follow if we can show that every uncountable set is contained in a suitable set of the same cardinality. For the strong covering lemma 1.12 we additionally need to show that the class $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Notice that definition 3.12 is absolute, so that the class $\mathbf{C}$ is definable in L.

The countably closed sets give a easy, but useful, special case:

3.13 Definition. We will call a set $X \prec_1 L_\kappa$ *countably closed* if there is a set $Y \prec H_\lambda$, for some $\lambda \geq \kappa$, such that ${}^\omega Y \subseteq Y$ and $X = Y \cap L_\kappa$.

If $|x|^\omega < \kappa$ then it is always possible to find a countably closed $X \supseteq x$ with $|X| = |x|^\omega$, so the following easily proved observation is often all that is needed.

3.14 Proposition. *Every countably closed set $X \prec_1 L_\kappa$ is suitable, and hence is a member of L.* $\dashv$

It follows that if $0^\sharp$ does not exist then every set x is contained in a set $y \in L$ such that $|y| \leq |x|^\omega$. This result gives much of the strength of the covering lemma, and moreover its proof highlights the most important ideas of the proof of the full covering lemma while omitting the most delicate part of the argument. This is sufficient reason to consider the countably closed case specially, but for core models involving measurable cardinals or non-overlapping extenders the countably closed case of the covering lemma is a necessary step on the proof of the full lemma: it is used to prove that the weak covering lemma, definition 1.9, holds in a variant K^{c} of the core model. The weak covering lemma for K^{c} is then used to prove the existence and essential properties of the true core model K, and only after this can the full covering lemma be proved for K.

The following lemma will conclude the proof of theorems 1.1 and 1.12, the covering lemma and the strong covering lemma for L:

3.15 Lemma. *The class* $\mathbf{C}$ *is unbounded in* $[L_\kappa]^\delta$ *for every uncountable cardinal* δ*, and* $\bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$ *whenever* $(\, X_\nu : \nu < \eta \,)$ *is an increasing sequence of sets in* $\mathbf{C}$ *with* $\mathrm{cf}(\eta) > \omega$*.*

The proof of this lemma will fill up the remainder of section 3.2.

Fix, for the moment, a set X which is not suitable, and let α, n and β be such that $\widetilde{M} = \mathrm{ult}_n(L_\alpha, \pi, \beta)$ is defined but not not well founded. This ill-foundedness is witnessed by a descending E-chain, $\ldots \mathrel{E} z_2 \mathrel{E} z_1 \mathrel{E} z_0$, of members of $\widetilde{M}$, where E is the membership relation of $\widetilde{M}$. In order to prove lemma 3.15 we need to incorporate additional structure into such a witness:

3.16 Definition. A *witness w to the unsuitability* of $X \prec_1 L_\kappa$ is a ω-chain of Σ_0 elementary embeddings $i_k \colon m_k \to m_{k+1}$ such that

1. $i_k \in X$ and $m_k \in X$ for each $k < \omega$.
2. $\operatorname{dir\,lim}(\pi^{-1}[w]) = \mathfrak{C}_n(L_\alpha)$ for some ordinal α and some $n \in \omega$.
3. $\operatorname{dir\,lim}(w)$ is not the Σ_n-code of any well founded model $L_{\tilde{\alpha}}$.
4. Write β_k for the critical point of i_k. Then the sequence $(\, \beta_k : k < \omega \,)$ is nondecreasing.
5. For each k, $m_k \in m_{k+1}$, and there is a function $f \in m_{k+1}$ such that $f``\beta_k = i_k``m_k$.

We will call $\beta = \sup_k \beta_k$ the *support* of the witness w, and we will call the pair (α, n) the *height* of w in X. We will say that a witness w is *minimal* in X if it has minimal height in X among all witnesses with the same support β.

There may be more than one minimal witness for X with the same support β. It is possible, with some care, to modify the definition so that this minimal witness is unique; however we do not need to do so.

3.17 Lemma. *A set* $X \prec_1 L_\kappa$ *is unsuitable if and only if it has a witness to its unsuitability. Furthermore, if w is a witness to the unsuitability of X then*

1. *If* $w \subseteq X' \prec_1 X$ *then w is also a witness to the unsuitability of* X'*.*
2. *If, in clause 1, w is a minimal witness for X then it is also a minimal witness for* X'*, and furthermore any minimal witness for* X' *with the same support is a minimal witness for X.*
3. *If* $X = Y \cap L_\kappa$*, where* $Y \prec_1 \mathrm{H}_\tau$ *for some cardinal* $\tau > \kappa$*, then* $w \notin Y$*.*

First we give some consequences of lemma 3.17.

3.18 Corollary. *If* $(X_\nu : \nu < \eta)$ *is an increasing sequence of sets in* $\mathbf{C}$*, with* $\mathrm{cf}(\eta) > \omega$*, then* $X = \bigcup_{\nu<\eta} X_\nu \in \mathbf{C}$*.*

Proof. Otherwise there would be a witness w to the unsuitability of X; but since $\mathrm{cf}(\eta) > \omega$ this would imply that $w \subseteq X_\nu$ for some $\nu < \eta$ and hence w is a witness to the unsuitably of X_ν by clause 1. $\dashv$

We can use lemma 3.17 to give a proof of proposition 3.14, although a direct proof is somewhat simpler.

3.19 Corollary. *Every countably closed set* $X \prec_1 L_\kappa$ *is suitable, and hence is a member of* L*.*

Proof. By definition, X is countably closed iff $X = Y \cap L_\kappa$ for some $Y \prec_1 \mathrm{H}_\tau$ where ${}^\omega Y \subseteq Y$. Then any witness to the unsuitability of X would have to be a member of Y, contrary to clause 3 of lemma 3.17. $\dashv$

The following lemma will complete the proof of the covering lemma except for the proof of lemma 3.17.

3.20 Lemma. *The class* $\mathbf{C}$ *is unbounded in* $[L_\kappa]^\delta$ *for any cardinal* δ *with* $\omega < \delta < \kappa$*.*

Proof. Jensen's proof of this result begins by generically collapsing the cardinal κ onto δ^+. The proof given here is essentially the same, but the presentation is slightly different: instead of carrying out the generic collapse we work with the set $\mathrm{Col}(\delta^+, L_\kappa)$ of forcing conditions for the collapse. The members of $\mathrm{Col}(\delta^+, L_\kappa)$ are functions $\sigma\colon \xi \to L_\kappa$ with $\xi < \delta^+$. With the obvious notions of "closed" and "unbounded" this space satisfies Fodor's lemma: if $S \subseteq \mathrm{Col}(\kappa^+, L_\kappa)$ is a stationary set and F is a function with domain S such that $F(\sigma) \in \mathrm{ran}(\sigma)$ for all $\sigma \in S$ then F is constant on a stationary subset of S. The reason for using the space $\mathrm{Col}(\delta^+, L_\kappa)$ instead of $[L_\kappa]^\kappa$ is that $\mathrm{Col}(\delta^+, L_\kappa)$ also satisfies the following variant of Fodor's lemma:

3.21 Proposition. *Suppose that* $S \subseteq \mathrm{Col}(\delta^+, L_\kappa)$ *is a stationary set such that* $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ *for all* $\sigma \in S$*, and that* F *is a function defined on* S *such that* $F(\sigma)$ *is a countable subset of* $\mathrm{ran}(\sigma)$ *for all* $\sigma \in S$*. Then there is a stationary subset* S' *of* S *and a function* $\sigma_0 \in S'$ *such that*

$$\forall \sigma \in S' \big(\sigma_0 \subseteq \sigma \text{ and } F(\sigma) \subseteq \mathrm{ran}(\sigma_0)\big).$$

Proof. Let $f(\sigma) < \mathrm{dom}(\sigma)$ be the least ordinal η such that $F(\sigma) \subseteq \sigma``\eta$, and let $S'_0 \subseteq S$ be a stationary set on which $f(\sigma)$ is constant. Pick any $\sigma_0 \in S_0$ and let $S' = \{\, \sigma \in S_0 : \sigma_0 \subseteq \sigma \,\}$. Then S' and σ_0 are as required. $\dashv$

Let S_0 be the set of functions $\sigma \in \mathrm{Col}(\delta^+, L_\kappa)$ such that $\mathrm{ran}(\sigma) \notin \mathbf{C}$, $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma) \prec_1 L_\kappa$. We will prove that S_0 is nonstationary, which implies that $\mathbf{C}$ is unbounded in $[L_\kappa]^\delta$

Suppose to the contrary that S_0 is stationary, and for each $\sigma \in S_0$ let w^σ be a minimal witness to that unsuitability, with support β^σ. By the ordinary Fodor's lemma there is a stationary set $S_1 \subseteq S_0$ such that $\beta = \beta^\sigma$ is constant for $\sigma \in S_1$, and by proposition 3.21 there is a stationary set $S_2 \subseteq S_1$ and $\sigma_0 \in S_2$ such that $\sigma_0 \subseteq \sigma$ and $w^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_2$. It follows that w^{σ_0} is a minimal witness to the unsuitability of $\mathrm{ran}(\sigma)$ for each $\sigma \in S_2$. Now consider the class $\mathcal{Y}$ of sets $Y \prec_{\Sigma_1} \mathrm{H}_{\kappa^+}$ such that $w^{\sigma_0} \in Y$. Then

$$\mathcal{X} = \{\, \sigma \in S_2 : \exists Y \in \mathcal{Y}\ \mathrm{ran}(\sigma) = Y \cap L_\kappa \,\}$$

contains a closed unbounded subset of $\mathrm{Col}(\delta^+, L_\kappa)$, and hence $S_2 \cap \mathcal{X} \neq \varnothing$. However, this contradicts lemma 3.17(3), and this contradiction completes the proof of lemma 3.20. $\dashv$

This completes the proof of the covering lemma, except for the proof of lemma 3.17:

Proof of lemma 3.17. First, notice that if w is a witness with support β to the unsuitability of X then clauses 4 and 5 imply that $\mathrm{dir\,lim}(w) = \mathrm{ult}\big(\mathrm{dir\,lim}(\pi^{-1}[w]), \pi, \beta\big)$, and hence clause 3.16(3) implies that X is in fact unsuitable.

Now suppose that X is unsuitable, so that there are α, n and β such that $\mathrm{ult}_n(L_\alpha, \pi, \beta)$ is defined, but not well founded. If we write $M_n = \mathfrak{C}_n(L_\alpha)$ then this means that $\mathrm{ult}(M_n, \pi, \beta)$ is defined, but is not the Σ_n-code of any well founded structure $L_{\tilde{\alpha}}$. We will find a witness w to the unsuitability of X, such that w has height and support less than or equal to (α, n) and β, respectively.

If $\mathrm{ult}(M_n, \pi, \beta)$ is not well founded then there are $f_k \in M_n$ and $a_k \in \beta$ so that $z_{k+1} \mathrel{\mathrm{E}} z_k$, where $z_k = [a_k, f_k] = \tilde{\pi}(f_k)(a_k)$, and E is the membership relation of $\mathrm{ult}(M_n, \pi, \beta)$. If, on the other hand, $\mathrm{ult}(M_n, \pi, \beta)$ is well founded then, since $\tilde{\pi}\colon M_n \to \widetilde{M}_n = \mathrm{ult}(M_n, \pi, \beta)$ is Σ_1-elementary, there is a (nonwellfounded) structure $\widetilde{M}$ such that $\widetilde{M}_n = \mathfrak{C}_n(\widetilde{M})$, along with a map $\tilde{h}_n$, the Σ_n-Skolem function of $\widetilde{M}$, mapping $\widetilde{M}_n$ onto $\widetilde{M}$. Then we can find $z_k = [a_k, f_k]$ so that $\tilde{h}_n(z_{k+1}) \mathrel{\mathrm{E}} \tilde{h}_n(z_k)$ for each $k < \omega$.

Write $M_n = (L_{\rho_n}, A_n)$ (if $n = 0$ then $\rho_n = \alpha$ and $A_n = \varnothing$, in which case we assume that α is a limit ordinal). Let $\alpha_k < \rho_n$ be the least ordinal such that $\{\, f_1, \dots, f_k \,\} \subseteq L_{\alpha_k}$, and let β_k be the least member of X such that $\{\, a_0, \dots, a_k \,\} \subseteq \beta_k$. Finally let $\bar{\beta}_k = \pi^{-1}(\beta_k)$ and let $\bar{\jmath}_k\colon \bar{m}_k \cong \mathcal{H}_{\Sigma_1}^{(L_{\alpha_k}, A_n \cap L_{\alpha_k})}(\bar{\beta}_k \cup \{\, f_1, \dots, f_k \,\})$ be the transitive collapse of the Σ_1 hull of $\bar{\beta}_k \cap \{\, f_1, \dots, f_k \,\}$ in $(L_{\alpha_k}, A_n \cap L_{\alpha_k})$, with $\bar{\imath}_k = \bar{\jmath}_{k+1}^{-1}\bar{\jmath}_k\colon \bar{m}_k \to \bar{m}_{k+1}$.

Then $\bar{m}_k, \bar{i}_k \in L_{\bar\kappa}$ for each $k < \omega$. Set $w = (\,\pi(\bar{m}_k), \pi(\bar{i}_k) : k < \omega\,)$, with $m_k = \pi(\bar{m}_k)$ and $i_k = \pi(\bar{i}_k)$, and set $\beta' = \sup_k \beta_k \leq \beta$.

If $\bar{w} = (\,\bar{m}_k, \bar{i}_k : k < \omega\,) = \pi^{-1}[w]$ then $\operatorname{dir}\lim(\bar{w}) \prec_0 M_n = \mathfrak{C}_n(L_\alpha)$ and hence, by lemma 3.5, $\operatorname{dir}\lim(\bar{w})$ is the Σ_n-code of $L_{\alpha'}$ for some $\alpha' \leq \alpha$, but w was constructed so that $\operatorname{dir}\lim(w)$ is not the Σ_n code of any well founded model. Finally, since $\alpha_{k+1} > \alpha_k$, the Skolem function mapping β_k onto $j_k``m_k \prec_{\Sigma_1} L_{\alpha_k}$, with parameters $\{\, f_1, \ldots, f_k \,\}$, is a member of $L_{\alpha_{\kappa+1}}$. This gives the functions f required by clause 3.16(5).

Thus w is the desired witness to the unsuitabilty of X.

To prove clause 3.17(3), note that by the absoluteness of well foundedness we can find, working in Y, a sequence $a'_k < \beta'$ of ordinals and a sequence $f'_k \in m_{k+1}$ of functions so if f''_k is the image $j_k(f'_k)$ of f'_k in $\operatorname{dir}\lim(w)$ then the sets $z'_k = f''_k(a'_k)$ demonstrate, in the same way that $(\,z_k : k < \omega\,)$ above did for $\widetilde{M}_n$, that $\operatorname{dir}\lim(w)$ is not the Σ_n-code of a well founded structure. Then the sets a'_k and f'_k are members of $Y \cap L_\kappa = X$, so the sets $\bar{z}'_k = \bar{i}_k\, \pi^{-1}(f'_k)(\pi^{-1}(\alpha'_k))$ demonstrate that $\operatorname{dir}\lim(\pi^{-1}[w])$ is not the Σ_n-code of a well founded structure, contradicting clause 3.16(2).

Clause 3.17(1), stating that if $w \subseteq X' \prec_1 X$ then w is also a witness that X' is not suitable, is straightforward. Now suppose that w is minimal, and that w' is a minimal witness for X' having the same support β.

Let (α', n') and (α'', n) be the heights of w' and w, respectively, in X'. Then $(\alpha', n') \leq (\alpha'', n)$ since w' is minimal. Since $L_{\alpha'} \subseteq L_{\alpha''}$, there is an embedding from $\operatorname{ult}_{n'}(L_{\alpha'}, \bar\pi, \bar\beta) \to \operatorname{ult}_n(L_{\alpha''}, \bar\pi, \bar\beta) = L_\alpha$. Hence $\operatorname{ult}_{n'}(L_{\alpha'}, \bar\pi, \bar\beta)$ is well founded. It follows that w' is a witness to the unsuitability of X with support β, and by the minimality of w we must have $\operatorname{ult}_{n'}(L_{\alpha'}, \bar\pi, \bar\beta) = L_\alpha$ and $n' = n$. Hence the height of w' in X is (α, n), so that w' is also a minimal witness for X'. ⊣

This completes the proof of the covering lemma for L. In the rest of this section we consider two variations on this proof. The first, and most important, extends the argument to models with a measurable cardinal in order to obtain the Dodd-Jensen covering lemma; the second variation applies the argument to unsuitable sets X, obtaining Magidor's covering lemma 1.15 and the absoluteness theorem for Jónsson cardinals, theorem 1.16.

3.3. Measurable Cardinals

The primary aim of this subsection is to prove the Dodd-Jensen covering lemma, and an important secondary aim is to prepare the way for section 4 which describes the covering lemma for larger core models. In accordance with this secondary aim we do not assume $\neg 0^\dagger$ except when it is explicitly specified. For simplicity we do assume that there is no model of $\exists\kappa\, o(\kappa) =$

κ^{++}, but much of our discussion is true in general (though not in detail) for the larger core models described in $\boxed{\text{steel}}$.

In Dodd and Jensen's original papers [8, 9, 10], the minimal model $L[U]$ for a measurable cardinal is treated separately from the Dodd-Jensen core model K^{DJ}. The model $L[U]$ is the simplest natural analogue of L and was already well understood long before the core model was invented. While $L[U]$ has many of the properties of L, there is one vital difference: The existence of the model L is implied by the axioms of set theory, but the construction of the model $L[U]$ depends on being first given the filter U which will be the measure in the model $L[U]$.

If $L[U]$ does exist then K^{DJ} can easily be obtained by "iterating the measure U out of the universe":

$$K^{DJ} = \bigcap_{\nu\in\mathrm{Ord}} \mathrm{ult}^{\nu}(L[U],U) = \bigcup_{\nu\in\mathrm{Ord}} (\mathrm{ult}^{\nu}(L[U],U) \cap V_{i_\nu(\kappa)})$$

where $i_\nu \colon L[U] \to \mathrm{ult}^{\nu}(L[U],U)$ is the ν-fold iteration of the ultrapower by U. In order to define an inner model which would exist even in the absence of a model $L[U]$, Dodd and Jensen defined the core model K^{DJ} to be $L[\mathcal{M}]$, where $\mathcal{M}$ is a class of approximations, called *mice*, to models of the form $L[U]$. The mice are structures $M = L_\alpha[W]$ with the properties (i) $M \models$ "W is a measure", (ii) M is iterable (in the sense that every iterated ultrapower of M is well founded) and (iii) M is sound and has projectum smaller than $\mathrm{crit}(W)$. Note that Condition (iii) implies that $L_{\alpha+1}[W] \models |\alpha| < \mathrm{crit}(W)$, so that W is not a measure in any model larger than M.

In [28], the Dodd-Jensen core model was extended to obtain a core model for sequences of measures. This extended core model had the form $K[\mathcal{U}] = L[\mathcal{U},\mathcal{M}]$, where $\mathcal{U}$ was the sequence of measures in $K[\mathcal{U}]$ and $\mathcal{M}$ was a class of mice. The mice $M \in \mathcal{M}$ were models of the form $M = L_\alpha[\mathcal{U}']$ where the sequence $\mathcal{U}'$ was a concatenation $\mathcal{U}' = \mathcal{U}^\frown\mathcal{W}$ of the sequence of measures of $K[\mathcal{U}]$ with a sequence $\mathcal{W}$ of filters which are measures in M but not in $L_{\alpha+1}[\mathcal{U}']$. The sequence $\mathcal{W}$ corresponded to the measure W in a Dodd-Jensen mouse $L_\alpha[W]$.

The modern approach to mice, which we follow here, originated in attempts to extend the core model to cardinals approaching a supercompact cardinal. This program has many difficulties, some of which are still not solved, but a key to making a beginning was the observation that the original notion of a mouse was too simple. A key fact in the theory of the constructible sets is that all of the models L_α, as well as L itself, have the same structure, so that they differ only in length. It became clear in course of the investigation that the mice for an extended core model should similarly have the same structure as the full core model. That is, the mice are themselves constructed from smaller mice, like a well founded version of Swift's well known flea, which

hath smaller fleas that on him prey
And these have smaller still to bit 'em;
And so proceed *ad infinitum.*

This seemed prohibitably complicated, but a suggestion of S. Baldwin made it possible to realize the desired situation while simplifying, instead of complicating, the construction: The mice and the core model would be structures of the form $M = L_\alpha[\mathcal{E}]$ or $M = L[\mathcal{E}]$, respectively. The members of the sequence $\mathcal{E}$ would be extenders, but some would be only partial extenders, not measuring all of the sets in M. These partial extenders would be the full extenders of those mice which are members of M; thus the sequence $\mathcal{E}$ codes both the mice and the extenders in the structure M.

For the rest of this section we limit ourselves to sequences of measures, with no extenders, and we mark this restriction by using the letter $\mathcal{U}$ to denote the sequence instead of $\mathcal{E}$.

As hoped, this approach leads to a feasible fine structure for the extended core models, but surprisingly it also simplifies the fine structure for the previously existing core models. This is particularly surprising for the Dodd-Jensen core model, the mice of which have at most one measurable cardinal. It would seem at first glance that nothing could be simpler than a mouse of the form $L_\alpha[U]$, but the apparent simplicity of this model hides a complicated fine structure. For example, consider the key fact of the fine structure of L—and even of Gödel's proof of the continuum hypothesis—that every constructible subset of ω is a member of L_{ω_1}. This fact fails badly in the model $L[U]$: if $\kappa = \mathrm{crit}(U)$ then $L_\alpha[U] = L_\alpha$ for all $\alpha \leq \kappa + 1$. The first nonconstructible set to be constructed is $0^\sharp$, which is a subset of ω and is Δ_1 definable over $L_{\kappa+1}[U]$, so that $0^\sharp \in L_{\kappa+2}[U] - L_{\kappa+1}[U]$.

The newer fine structure avoids this problem because the subsets of ω in $L[\mathcal{U}]$ are all in $L_{\omega_1}[\mathcal{U}]$ and hence are constructed from the restriction $\mathcal{U}\restriction\omega_1$ of $\mathcal{U}$ to (partial) measures on countable ordinals. In fact, the first nontrivial member of $\mathcal{U}$ is the L-ultrafilter on the first Silver indiscernible c_0 which is induced by, and which constructs, the real $0^\sharp$. In the case $L[\mathcal{U}] = L[U]$, the sequence $\mathcal{U}$ has as its last nontrivial member the measure U, which is the only member of the sequence $\mathcal{U}$ which is a full measure on $L[\mathcal{U}]$.

The benefit of the new approach to the core model is suggested by the fact that the following two modifications are all that is necessary to adapt the definition 3.1 of fine structure for L to the core model.

1. An added predicate is needed to represent the sequence of measures, so that the Σ_1-code is a structure of the form $(L_\alpha[\mathcal{U}], \mathcal{U}\restriction\alpha, A)$ instead of (L_α, A).
2. For ordinals α such that $\mathcal{U}_\alpha \neq \varnothing$ it is necessary to begin the construction with a special *amenable code*, defined to be $\mathfrak{C}_0(L_\alpha[\mathcal{U}]) =$

$(L_{\rho_0}, \mathcal{U}{\restriction}\rho_0, \mathcal{U}_\alpha)$ where, if $\kappa = \mathrm{crit}(\mathcal{U}_\alpha)$, then ρ_0 is defined to be κ^+ of $L[\mathcal{U}{\restriction}\alpha]$.

The analogous structure in the fine structure of L was simply $\mathfrak{C}_0(L_\alpha) = (L_\alpha, \varnothing)$. The amenable code is needed because the obvious structure, $(L_\alpha[\mathcal{U}], \in, \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is not amenable.

The "Skolem function" $h_0^{L_\alpha[\mathcal{U}]}$ mapping $\rho_0 = \rho_0^{L_\alpha[\mathcal{U}]} = \kappa^{+L_\alpha[\mathcal{U}]}$ onto $L_\lambda[\mathcal{U}]$ is derived from the function which maps a function $f\colon \kappa \to L_\kappa[\mathcal{U}]$ in $L_{\rho_0}[\mathcal{U}]$ to $[f]_{\mathcal{U}_\alpha} \in L_\alpha[\mathcal{U}] = L_\alpha[\mathcal{U}{\restriction}\alpha] \subseteq \mathrm{ult}(L_\kappa[\mathcal{U}], \mathcal{U}_\alpha)$.

If $\mathcal{U}_\alpha = \varnothing$ then the amenable code is simply $(L_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \varnothing)$, as in L.

The amenable code is somewhat more complicated in the case of sequences $L[\mathcal{E}]$ involving extenders instead of only measures. See $\boxed{\text{Steel}}$ for details.

The proof that the fine structure given by this definition satisfies the necessary properties is, of course, more complicated that the proof in L. We begin with the definition of a mouse:

3.22 Definition. A *premouse* is a sound potential premouse, and a *mouse* is an iterable premouse.

We define the terms *potential premouse* and *sound* by a simultaneous recursion on α:

1. A *potential premouse* is a model $L_\alpha[\mathcal{U}]$ or $L[\mathcal{U}]$ which satisfies the following three conditions:
   (a) For each γ such that $\mathcal{U}_\gamma \neq \varnothing$, there is a cardinal κ of $L[\mathcal{U}{\restriction}\gamma]$ such that $L[\mathcal{U}{\restriction}\gamma] \models (\gamma = \kappa^{++}$ and $\mathcal{U}_\gamma$ is a normal $L[\mathcal{U}{\restriction}\gamma]$-measure on $\kappa)$.
   (b) (Coherence) If $\mathcal{U}_\gamma \neq \varnothing$ then $\left(i^{\mathcal{U}_\gamma}(\mathcal{U}{\restriction}\gamma)\right){\restriction}\gamma + 1 = \mathcal{U}{\restriction}\gamma$.
   (c) (Soundness) The structure $(L_{\alpha'}[\mathcal{U}], \mathcal{U}{\restriction}\alpha', \mathcal{U}_{\alpha'})$ is sound for every ordinal $\alpha' < \alpha$.

   We say that a sequence $\mathcal{U}$ is *good* if $L[\mathcal{U}]$ is a potential premouse.
2. A potential premouse $\mathcal{M} = (L_\alpha[\mathcal{U}], \mathcal{U}{\restriction}\alpha, \mathcal{U}_\alpha)$ is said to be *n-sound* if $h_m^{\mathcal{M}}``\rho_n = L_\alpha[\mathcal{U}]$ for each $m \leq n$, where $h_m^{\mathcal{M}}$ and $\rho_m^{\mathcal{M}}$ are the Σ_m Skolem functions and Σ_m projectum of $\mathcal{M}$, respectively. The model $\mathcal{M}$ is *sound* if $\mathcal{M}$ is n sound for all $n \in \omega$.

   We will say that $\mathcal{M}$ is *sound above* η if either $\mathcal{M}$ is sound or there is n such that $\rho_{n+1}^{\mathcal{M}} \leq \eta$, $\mathcal{M}$ is n-sound and $h_{n+1}^{\mathcal{M}}``\eta = L_\alpha[\mathcal{U}]$.
3. A *premouse* is a potential premouse which is sound.

4. (Iterability) A premouse $L_\alpha[\mathcal{U}]$ is *iterable* if every iterated ultrapower of $L_\alpha[\mathcal{U}]$ is well founded. In this case we call $L_\alpha[\mathcal{U}]$ a *mouse.*

   Note that this definition of the term *iterable* needs to be supplemented by definition 3.26, given later, of iterated ultrapowers of a premouse.

Again, see $\boxed{\text{steel}}$ for the somewhat more complicated conditions on the sequence $\mathcal{U}$ when it is allowed to contain extenders.

3.23 Lemma. *The formula asserting that a model M is a mouse is absolute for models N containing ω_1.*

Proof. All of the clauses of the definition of a mouse are first order over M except the last, iterability. If M is not iterable then there is a countable iterated ultrapower of M witnessing this failure. Working in N, we can define a tree of attempts to build such a countable witness, but then M is iterable if and only if this tree is well founded, which is absolute. $\dashv$

The ultrapower $\mathrm{ult}_n(L_\alpha[\mathcal{U}],\pi,\beta)$ of $M = L_\alpha[\mathcal{U}]$ by the extender derived from an embedding π is defined just like that for L, by taking the ultrapower $\mathrm{ult}_0(\mathfrak{C}_n(M),\pi,\beta)$ of the Σ_n-code using functions in $\mathfrak{C}_n(M)$ and then using the upward extension property (lemma 3.8) to extend the embedding to all of M. In particular the upward extension property, Lemma 3.8 (which we will not restate here) is still valid for these models. Since not every potential premouse $L_\alpha[\mathcal{U}]$ is sound, the proposition 3.9 giving the conditions for the existence of $\mathrm{ult}_n(L_\alpha[\mathcal{U}],\pi,\beta)$ needs to be supplemented with the requirement that $L_\alpha[\mathcal{U}]$ be n-sound.[2]

In addition, we now have the possibility of taking an ultrapower by one of the ultrafilters $U = \mathcal{U}_\gamma$ in $M = L_\alpha[\mathcal{U}]$. In this case we have a stronger preservation result:

3.24 Lemma. *Suppose that $M = L_\alpha[\mathcal{U}]$ is a potential premouse which is n-sound, that $U = \mathcal{U}_\gamma$ for some $\gamma \leq \alpha$, and that $\kappa = \mathrm{crit}(U) < \rho_n^M$. Let $i^U\colon M \to \mathrm{ult}(M,U) = M'$. Then M' is a $n-1$-sound potential premouse and i^U preserves the Skolem functions of M: we have $i^U h_m^M(x) = h_m^{M'} i^U(x)$ whenever either side is defined.*

The crux of the proof of lemma 3.24 is showing that the standard parameters are preserved. See $\boxed{\text{steel}}$ for more.

Not all of the fine structure of M is preserved by the ultrapower i^U. First, and most important, the ultrapower $M' = \mathrm{ult}_n(M,U)$ is never sound above $\kappa = \mathrm{crit}(U)$, even though M is, since $\kappa \notin (i^U h_{n+1}^M)``(\mathrm{crit}(U)) = h_{n+1}^{M'}``\rho_{n+1}$. The model M' is sound above $\kappa+1$. Second, the two projecti ρ_n^M and ρ_{n+1}^M

[2]The ultrapower can also be defined on any iterated ultrapower of a mouse, and hence by lemma 3.24 the extra condition is not needed for any a iterable potential premouse $\mathcal{M}$. However we will only need ultrapowers as described in the text.

need not be preserved: if $\operatorname{crit}(U) = \rho^M_{n+1}$ then $\rho^{M'}_{n+1} = \rho^M_{n+1} < i^U(\rho^M_{n+1})$; and in any case $\rho^{M'}_n = \sup i^U \text{“}\rho^M_n$, which may be smaller then $i^U(\rho^M_n)$.

The existence of unsound potential premice is an important difference between the fine structure of L and that of larger core models. The counterpart of lemma 3.4, which states that L_α is sound, is given by the following lemma:

3.25 Lemma. *Any iterable potential premouse $M = L[\mathcal{U}]$ is an iterated ultrapower of a mouse.*

Sketch of proof. Like lemma 3.4, lemma 3.25 is proved in a simultaneous induction along with the counterpart of the downward extension lemma 3.5. The Σ_n codes of M are defined as described above, ignoring the fact that M is not sound. If n is least such that $\rho^M_n = \rho^M$ then the counterpart of lemma 3.5 shows that $\mathfrak{C}_n(M) = \mathfrak{C}_n(M'_n)$ where M'_n is the sound structure obtained by decoding the structure $\mathfrak{C}_n(M)$. During the simultaneous induction on m, it is shown that M'_n is sound above ρ^M_m and that M is an iterated ultrapower M'_n.

Note that the iterability of M'_n follows from that of M, since any iterated ultrapower of N can be embedded into the corresponding iterated ultrapower of M. $\dashv$

Iterated ultrapowers were described in chapter $\boxed{\text{introduction}}$, but here they are complicated by the fact that they may involve ultrapowers of differing degrees and since they may involve filters $\mathcal{U}^{M_\nu}_\gamma$ which are not full ultrafilters on M_ν. Both of these situations result in the *drops* mentioned in the following definition:

3.26 Definition. An *iterated ultrapower of a premouse $M = L_\alpha[\mathcal{U}]$* is a sequence of models M_ν for $\nu \leq \theta$, together with a finite set $D \subseteq \theta + 1$, called the set of *drops*, and embeddings $i_{\nu,\nu'}\colon M_\nu \to M_{\nu'}$ defined for all pairs $\nu < \nu' \leq \theta$ such that $D \cap (\nu, \nu'] = \varnothing$.

All of these are determined by a sequence of filters $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} \in M_\nu$, with a strictly increasing sequence of critical points $\operatorname{crit}(U_\nu)$, as follows:

1. $M_0 = L_\alpha[\mathcal{U}]$.
2. If ν is a limit ordinal then $M_\nu = \operatorname{dir lim}_{\nu_0 \leq \nu' < \nu} M_{\nu'}$ where $\nu_0 = \sup D \cap \nu$. Note that this direct limit exists since the finiteness of D implies that $D \cap \nu$ is bounded in ν, so that the embeddings $i_{\nu'',\nu'}\colon M_{\nu''} \to M_{\nu'}$ exist for all sufficiently large $\nu' < \nu'' < \nu$.
3. If $\nu + 1 \leq \theta$ then $M_{\nu+1}$ is determined by the choice of the ultrafilter $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu}$, where $\gamma_\nu > \gamma_{\nu'}$ for all $\nu' < \nu$. There are three cases:

(a) If U_ν is a full ultrapower on M_ν then set $n_\nu = n$ where n is the largest number such that $\mathrm{ult}_n(M_\nu, \pi, \beta)$ is defined. If $n_\nu = n_{\nu'}$ for sufficiently large $\nu' < \nu$, then $M_{\nu+1} = \mathrm{ult}_{n_\nu}(M_\nu, U_\nu)$. In this case $i_{\nu,\nu+1}$ is the canonical embedding.

Note that since the critical points of the ultrafilters U_ν are increasing, M_ν is sound above $\mathrm{crit}(U_\nu)$ and hence the soundness hypothesis of lemma 3.24 is satisfied.

(b) If U_ν is a full ultrapower on M_ν, but $n_\nu > n_{\nu'}$ for all $\nu' \in \nu - \max(D \cap \nu)$, then $M_{\nu+1} = \mathrm{ult}_{n_\nu}(M_\nu, U_\nu)$. In this case we add $\nu + 1$ to D, so that $i_{\nu,\nu+1}$ is not defined.

Note that this happens when $\mathrm{crit}(U_\nu) \geq \rho^{M_\nu}_{n_\nu}$ but $\mathrm{crit}(U_{\nu'}) < \rho^{M_{\nu'}}_{n_\nu} = \rho^{M_\nu}_{n_\nu}$ for $\sup(D \cap \nu) < \nu' < \nu$. This case is known as a *drop in degree.*

(c) If U_ν is not a full ultrafilter on M_ν, then let $M^*_{\nu+1}$ be the largest initial segment of M_ν on which U_ν is an ultrafilter. Thus $M^*_{\nu+1} = L_{\alpha^*_\nu}[\mathcal{U}_\nu]$ where α^*_ν is the least ordinal $\beta < \alpha$ such that there is a subset x of $\mathrm{crit}(U_\nu)$ in $L_{\beta+1}[\mathcal{U}_\nu] - L_\beta[\mathcal{U}_\nu]$ which is not measured by U_ν.

In this case, which is known as a *normal drop*, we set $M_{\nu+1} = \mathrm{ult}(M^*_{\nu+1}, U_\nu)$, and we add $\nu{+}1$ to D so that $i_{\nu,\nu+1}$ is not defined.

We say that M is *iterable* if every model in any iterated ultrapower of M is well founded and no attempt to create an iterated ultrapower leads to infinitely many drops.

Here again the situation becomes more complicated in the case of extenders, where *iteration trees* are needed instead of the linear iterated ultrapowers described above. See $\boxed{\text{steel}}$.

3.27 Remark. In practice we will frequently make the trivial emendation that $U_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} = \varnothing$ is also allowed in an iterated ultrapower, and set $M_{\nu+1} = M_\nu$ in this case. Such trees are known as *padded iterated ultrapowers.*

Comparisons of mice In the case of L, the only mice are the structures L_α, and hence it is trivial that, given two mice M and N, one is an initial segment of the other. Under appropriate conditions the same crucial fact is true of the mice for higher core models, but the proof requires the use of iterated ultrapowers to compare the two mice. We describe this process below, using the previous notation for iterated ultrapowers from definition 3.26. Superscripts M and N are used to distinguish the iterated ultrapower on M from that on N.

3.28 Definition (Comparison for premice). We will say that two potential premice M and N *strongly agree up to* τ if $\operatorname{crit}(\mathcal{U}^M_{\gamma'}) \geq \tau$ and $\operatorname{crit}(\mathcal{U}^N_{\gamma'}) \geq \tau$ for all $\gamma' \geq \gamma$, where γ is the least ordinal such that $\mathcal{U}^M_\gamma \neq \mathcal{U}^N_{\gamma'}$. Assume that the mice M and N are sound above τ and strongly agree up to τ. Then the comparison of M and N is defined by the use of iterated ultrapowers on M and N, which we distinguish by means of superscripts M and N.

Start the comparison by setting $M_0 = M$ and $N_0 = N$. Now suppose that M_ν and N_ν have been defined. If either of the models N_ν or M_ν is an initial segment of the other then the comparison is complete, and we terminate the iterated ultrapower with $\theta = \nu$. Otherwise let γ_ν be the least ordinal γ such that $\mathcal{U}^{N_\nu}_\gamma \neq \mathcal{U}^{M_\nu}_\gamma$ and set $U^M_\nu = \mathcal{U}^{M_\nu}_\gamma$ and $U^N_\nu = \mathcal{U}^{N_\nu}_\gamma$. Now use these ultrafilters to define $M_{\nu+1}$ and $N_{\nu+1}$ as in the definition 3.26 of an iterated ultrapower.

Note that definition 3.28 uses padded iterated ultrapowers, since it may be that either $U^N_\nu = \varnothing$ or $U^M_\nu = \varnothing$.

The coherence property of premice ensures that both the indices γ_ν and the critical points of the ultrafilters U^M_ν and U^N_ν are strictly increasing. It can be shown that this comparison process always stops in fewer than τ^+ steps, with one of the final models M_θ and N_θ an initial segment of the other.

The next few results analyze some of the possible outcomes of this comparison. All results assume that M and N satisfy the requirements for definition 3.28: that is, they strongly agree up to τ and are sound above τ.

3.29 Lemma. *Suppose that M_θ is a proper initial segment of N_θ. Then M is sound and $D^M = \varnothing$, so that only full ultrafilters of M were used in the iterated ultrapower on M.*

Sketch of Proof. Suppose to the contrary that $D^M \neq \varnothing$, and let $\nu + 1 = \sup D^M$. By the remark following lemma 3.8, the models $M_{\nu'}$ are not sound above τ for any $\nu' > \nu$. In particular M_θ must not be sound but, since N_θ is a premouse, definition 3.22 implies that M_θ, as a proper initial segment of N_θ, is sound. ⊣

3.30 Lemma. *Suppose that $M_\theta = N_\theta$, that $D^M = D^N = \varnothing$, and that both $\rho^M \leq \tau$ and $\rho^N \leq \tau$. Then $M = N$.*

Sketch of Proof. Since $D^M = D^N = \varnothing$, both $i^M_{0,\theta}\colon M \to M_\theta$ and $i^N_{0,\theta}\colon N \to N_\theta$ are defined. Since $\tau \geq \rho^M$ we have $M = h^M``\tau$ By lemma 3.24, we have $i^M_{0,\theta} h^M = h^{M_\theta}$ so it follows that $M \cong h^{M_\theta}``\tau$. Similarly $N \cong h^{N_\theta}``\tau$, and since $N_\theta = M_\theta$ it follows that $M = N$. ⊣

3.31 Corollary. *At least one of D^M and D^N are empty.*

Sketch of Proof. Suppose to the contrary that both iterated ultrapowers drop. Then lemma 3.29 implies that neither of M_θ and N_θ is a proper initial segment of the other, so $M_\theta = N_\theta$. Now let $\nu + 1$ be the largest member of $D^M \cup D^N$, and suppose for example that $\nu + 1 \in D^M$. Then the remainder of the iterated ultrapower can be regarded as a comparison of $M^*_{\nu+1}$ with either N_ν or $N^*_{\nu+1}$, depending on whether $\nu + 1 \in D^N$. Furthermore, since D^N contains some ordinal $\nu' \leq \nu + 1$, this comparison satisfies the hypothesis of lemma 3.30, with $\tau = \mathrm{crit}(U^M_\nu)$, so $M^*_{\nu+1} = N_\nu$ or $M^*_{\nu+1} = N^*_{\nu+1}$. This is impossible since $U^M_\nu = \mathcal{U}^{M_\nu}_{\gamma_\nu} \neq \mathcal{U}^{N_\nu}_{\gamma_\nu}$. ⊣

3.32 Lemma. *Suppose that M and N are mice with $\max(\rho^N, \rho^M) \leq \tau$, and that M and N strongly agree up to τ. Then one of M and N is an initial segment of the other.*

Sketch of Proof. We prove the lemma by induction on the lengths of the mice M and N. If $M_\theta = N_\theta$ then $M = N$ by lemma 3.30. Thus we can assume without loss of generality that M_θ is a proper initial segment of N_θ; we will show that in this case M is an initial segment of N. Note that the assumption that $\rho^M \leq \tau$ implies that there is a subset A^M_n of τ which is definable in M, but not a member of M. Then A^M_n is definable in M_θ and hence is a member of N_θ. It follows that $A^M_n \in N$. Now let N' be the least initial segment of N such that A^M_n is definable over N'. Then $\rho^{N'} \leq \tau$, and by the induction hypothesis it follows that one of M and N' is an initial segment of the other. Clearly neither can be a proper initial segment of the other, so it follows that $M = N'$, an initial segment of N. ⊣

3.33 Corollary. *If neither of the mice M and N have a full ultrafilter, then one of M and N is an initial segment of the other.*

Proof. Assume, to the contrary, there is some γ such that $\mathcal{U}^M_\gamma \neq \mathcal{U}^N_\gamma$, then apply lemma 3.32 to the least initial segments M' of M and N' of N at which sets are constructed which are not measured by these ultrafilters. Then one of N' and M' is an initial segment of the other, so in particular $\mathcal{U}^{N'}_\gamma = \mathcal{U}^{M'}_\gamma$, contradicting the assumption that these are different. ⊣

The Dodd-Jensen Core Model. The discussion above is valid for the core model up through $o(\kappa) = \kappa^{++}$. We now turn our attention to the Dodd-Jensen core model, and hence we assume that $0^\dagger$ does not exist. Thus no mouse has any full measures except possibly for the final nontrivial measure on its sequence.

3.34 Definition. The Dodd-Jensen core K^{DJ} is defined to be the union of the class of mice M such that there is no measurable cardinal in M. If there is no inner model with a measurable cardinal, then the core model K is equal to K^{DJ}. If there is an inner model with a measurable cardinal, but

$0^\dagger$ does not exist, then $K = L[U]$, where the model $L[U]$ is chosen so that the critical point of the measure U of $L[U]$ is as small as possible.

Notice that if $M = L_\alpha[\mathcal{U}^M]$ and $M' = L_{\alpha'}[\mathcal{U}^{M'}]$ are mice with no full ultrafilter then lemma 3.32 implies that one of M and M' is an initial segment of the other, that is, either $\mathcal{U}^M = \mathcal{U}^{M'}\restriction\alpha$ or $\mathcal{U}^{M'} = \mathcal{U}^M\restriction\alpha'$. Thus $K^{DJ} = L[\mathcal{U}^{\mathrm{DJ}}]$, where $\mathcal{U}^{\mathrm{DJ}} = \bigcup\{\,\mathcal{U}^M : M$ is a mouse without a full ultrafilter$\,\}$.

The model K^{DJ} may also contain mice $L_\alpha[\mathcal{U}^{\mathrm{DJ}}]$ which do have a full ultrafilter $U = \mathcal{U}^{\mathrm{DJ}}_\gamma$; however for each such mouse there will be an ordinal $\alpha' > \alpha$ such that U is not a ultrafilter in $L_{\alpha'}[\mathcal{U}^{\mathrm{DJ}}]$.

Note that $K^{DJ} = L$ if $0^\sharp$ does not exist. If $K = L[U]$, with U a measure on κ, then we can write $L[U] = L[\mathcal{U}]$ where, setting $\gamma = \kappa^{++K^{DJ}}$, the sequence $\mathcal{U}$ has a largest member $\mathcal{U}_\gamma = U$ and $\mathcal{U}\restriction\gamma = \mathcal{U}^{\mathrm{DJ}}\restriction\gamma$.

The fact that a mouse is required to be iterable, while a level L_α of the L hierarchy is only required to be well founded, makes the definition of a mouse logically more complicated than that of the sets L_α. For example, the statement $\exists\alpha\,\big((\omega, E) \cong (L_\alpha, \in)\big)$ is a Π^1_1 statement about the set $E \subseteq \omega^2$, while the statement that (ω, E) is isomorphic to a mouse $L_\alpha[\mathcal{U}]$ is Π^1_2. However, the definition of a mouse is absolute for models containing ω_1, and hence $(K^{DJ})^M = K^{DJ} \cap M$ whenever M is a standard model of ZF containing ω_1.

However, it is not the case, as it is for L, that there is a sentence V = K such that K is the only well founded class model of the sentence V = K. To see this, suppose that M is a mouse in K^{DJ} with a measure U on κ, and let $\widetilde{M} = \mathrm{ult}_{\mathrm{Ord}}(M, U)$. Then $i^U_{\mathrm{Ord}}(\kappa) = \mathrm{Ord}$, and the initial segment $K' = V^{\widetilde{M}}_{\mathrm{Ord}}$ of $\widetilde{M}$ is a class model of ZFC + V = K which is not equal to K^{DJ}. Notice, for example, that the critical points of the iteration form a closed and unbounded class of indiscernibles for K' which is definable in $L[M]$, and hence in K^{DJ}.

Vickers and Welsh have observed [48] that the existence of a Ramsey cardinal implies that it is consistent that there is a proper class $X \prec K^{DJ}$ such that $X \not\cong K^{DJ}$.

The proof of Jensen's covering lemma for L relied on Kunen's result that the existence of a nontrivial embedding $i\colon L \to L$ implies that $0^\sharp$ exists. The proof of the covering lemma for K relies on an analogous result, lemma 3.39, stating that if $0^\dagger$ does not exist and there is a nontrivial elementary embedding $i\colon K \to M$ then then $K = L[U]$ for some measure U in K. In particular, if there is a nontrivial embedding $i\colon K^{DJ} \to K^{DJ}$ then $K \neq K^{DJ}$, so that (assuming $0^\dagger$ does not exist) $K = L[U]$ for some measure U in K. However Jensen has shown that it is not necessarily true, as one might expect, that U is the ultrafilter associated with i, or even that $\mathrm{crit}(U) = \mathrm{crit}(i)$:

3.35 Theorem. *Suppose that $L[U]$ satisfies that U is a measure on κ, and let G be a generic collapse of $\lambda = \kappa^{+(L[U])}$ onto ω. Then in $L[U,G]$ there is a fully iterable premouse N with measurable cardinal less that κ such that $L_\lambda(U)$ is an ultrapower of N. Hence there is an elementary embedding $i : K^{DJ} \to K^{DJ}$ with* $\mathrm{crit}(i) < \kappa$.

Proof. We will prove that the theorem is true in $M = \mathrm{ult}(L[U], U)$, so that by elementarity it is true in $L[U]$ as well. Let $j : L[U] \to M = \mathrm{ult}(L[U], U)$, let

$$\kappa' = j(\kappa), \quad \lambda' = j(\lambda), \quad \text{and} \quad U' = j(U),$$

and let G be generic over M collapsing λ' to ω.

Pick $\beta > \lambda'$ so that $L_\beta[U', G]$ is admissible, so that we can apply the Barwise compactness theorem to $L_\beta[U', G]$. Let $\mathcal{L}$ be the infinitary language with a constant $\mathbf{x}$ for each member x of $L_\beta[U', G]$ and one additional constant $\mathbf{W}$ which will denote a measure on κ. Let T be the theory with sentences

$$\forall z \left(z \in \mathbf{x} \iff \bigvee_{y \in x} (z = \mathbf{y}) \right)$$

for each $x \in L_\beta[U', G]$, together with a sentence asserting that $\mathbf{W}$ is an amenable measure in $L_\lambda[\mathbf{W}]$ such that $\mathrm{ult}(L_\lambda[\mathbf{W}], \mathbf{W}) \cong L_{\lambda'}[U']$. This theory is consistent because it is true in $L[U]$, so by the Barwise compactness theorem it has a model $\mathcal{A}$. Let $W = \mathbf{W}^{\mathcal{A}}$ and let N be the premouse $L_\lambda[W]$. Then $\mathrm{ult}(N, W) = L_{\lambda'}[U']$ by the construction of W, and it follows that N is fully iterable since $L_{\lambda'}[U']$ is fully iterable.

Now K^{DJ} is equal to the initial segment of $\mathrm{ult}_{\mathrm{Ord}}(N, W)$ below Ord, its measurable cardinal, and hence it follows easily that there is an embedding from K^{DJ} into K^{DJ} with critical point κ. $\dashv$

Notice that the model N of this proof gives an example of a mouse which is not a member of K, and of a mouse which is added by a set forcing.

A major difference between the proof of the covering lemma for L and the proof of the covering lemma for $L[U]$ is that the proof of the analog 3.39 of Kunen's result itself uses the covering lemma. In order to avoid circularity we prove the covering lemma in two steps: the first step proves enough of the strength of the covering lemma to prove lemma 3.39, using the following observation:

3.36 Lemma. *If U is a countably complete normal K^{DJ}-ultrafilter then U is a measure in $L[U]$.*

Proof. Suppose to the contrary that U is not a measure in $L[U]$. It follows that there is some subset x of $\kappa = \mathrm{crit}(U)$ such that $x \in L[U] - K$. If $\mathcal{U}$ is the good sequence such that $K^{DJ} = L[\mathcal{U}]$, and we set $\gamma = (\kappa^{++})^{K^{DJ}}$, then as pointed out earlier, $L[U]$ is equal to $L[\mathcal{U}']$, where $\mathcal{U}'$ is the sequence $\mathcal{U}'$

defined by $\mathcal{U}'\restriction\gamma = \mathcal{U}\restriction\gamma$, $\mathcal{U}'_\gamma = U$, and $\mathcal{U}' = \varnothing$ for $\gamma' > \gamma$. Let α be least such that there is some set $x \in \mathcal{P}(\kappa) \cap L_{\alpha+1}[\mathcal{U}] - K^{DJ}$. Now the model $L_\alpha[\mathcal{U}']$ need not be sound, but the iterability of $L[\mathcal{U}]$ and the countable completeness of U ensure that it is iterable. Hence lemma 3.25 implies that $L_\alpha[\mathcal{U}']$ is an iterated ultrapower of a mouse N, but this is impossible since any such mouse would be a member of K^{DJ}. $\dashv$

The second part of the proof will use the following analog of the condensation lemma for L:

3.37 Lemma. *Assume that there is no inner model with a measurable cardinal, and that $X \prec K^{DJ}$ is a class such that* $\mathrm{otp}(X \cap \lambda^{+K^{DJ}}) = \lambda^{+K^{DJ}}$ *for a proper class of cardinals λ of K^{DJ}. Then $X \cong K^{DJ}$.*

Sketch of proof. Let N be the transitive collapse of X, and suppose to the contrary that $N \neq K^{DJ}$. Then there is there is a mouse $M \in K^{DJ} - N$. Now use iterated ultrapowers to compare the models M and N, and let M_θ and N_θ be the final models of the iterated ultrapower of M and N, respectively. Then N_θ must be a proper initial segment of M_θ, since there is a set $x \subseteq \rho^M$ which is definable in M but not in N. Then x is definable in M_θ, but not in N_θ so N_θ is a proper initial segment of M_θ and it follows by corollary 3.31 that the iterated ultrapower on N does not drop. Since N does not contain any full ultrafilters it follows that the iterated ultrapower on N must be trivial, that is, $N = N_\theta$, and this implies that $\mathrm{otp}(X \cap \kappa^{+K^{DJ}}) = (\lambda^+)^N = (\lambda^+)^{M_\theta}$ for all cardinals λ. However this contradicts the hypothesis, since $\lambda^{+M_\theta} < \lambda^{+K^{DJ}}$ for all $\lambda > |M|$. This contradiction completes the proof of proposition 3.37. $\dashv$

The set Γ from Kunen's proof does satisfy the modified form of the hypothesis of lemma 3.37 obtained by replacing $\lambda^{+K^{DJ}}$ with the true λ^+. Hence the importance of the weak covering lemma:

3.38 Definition. A cardinal λ is countably closed if $\eta^\omega < \lambda$ for all $\eta < \lambda$, and a model M has the *countably closed weak covering property* if for all sufficiently large countably closed singular cardinals λ we have $(\lambda^+)^M = \lambda^+$.

3.39 Lemma. *Suppose that $0^\dagger$ does not exist, and that K satisfies the countably closed weak covering property. If there is a nontrivial elementary embedding $i\colon K \to M$ then $K = L[U]$, where U is a measure in $L[U]$ with* $\mathrm{crit}(U) \leq \mathrm{crit}(i)$.

Proof. First we show that $K \neq K^{DJ}$, so that there is a model with a measure. Suppose to the contrary that $K = K^{DJ}$. Then M also equals K^{DJ}, and we can apply the proof, sketched in chapter introduction/Th 1.13, of Kunen's corresponding result for L. This involves defining a continuously descending sequence of classes Γ_α, beginning with $\Gamma_0 = \operatorname{ran}(i)$ and setting $\Gamma_{\alpha+1} = \{\, x \in \Gamma_\alpha : i_\alpha(x) = x \,\}$ where i_α is the transitive collapse of $\Gamma_\alpha \prec K^{DJ}$. The sets Γ_α contain their limit points of sufficiently large cofinality, and if $\nu \in \Gamma_\alpha$ then $|\Gamma_\alpha \cap \nu^+| = \nu^+$. Since $\nu^+ = \nu^{+K^{DJ}}$ by the weak covering property, it follows by lemma 3.37 that $\Gamma_\alpha \cong K^{DJ}$ for each ordinal α. Now the same argument as for L shows that if we set $\kappa = \operatorname{crit}(i)$ and $\kappa_\alpha = \min(\Gamma_\alpha - \kappa)$ then the class of ordinals $(\kappa_\alpha : \alpha \in \mathrm{Ord})$ is a closed and unbounded class of indiscernibles for K^{DJ}. It follows that $\{\, \kappa_\alpha : \alpha < \omega_1 \,\}$ generates a normal K^{DJ}-measure U_{ω_1} on κ_{ω_1}, and since U_{ω_1} is countably complete it follows by lemma 3.36 that U_{ω_1} is a measure in $L[U_{\omega_1}]$. Thus $K = L[U]$, where U is a measure on some $\kappa' \leq \kappa_{\omega_1}$.

Thus $K = L[U]$ for some measure U. To see that $\operatorname{crit}(U) \leq \operatorname{crit}(i)$, assume to the contrary that $\kappa = \operatorname{crit}(U) > \lambda = \operatorname{crit}(i)$ and observe that it is true for $L[U]$, as for L, that $\mathcal{H}^{L[U]}(\Gamma) \cong L[U]$ for any proper class Γ of ordinals. Now $M = L[U']$, with $\kappa' = \operatorname{crit}(U') \geq \kappa = \operatorname{crit}(U)$, so there is an iterated ultrapower $j\colon K \to M$. Let $\Gamma = \{\, \nu : i(\nu) = j(\nu) \,\}$. Then Γ is a proper class, which contains its limit points of cofinality greater than κ such that $\lambda \notin \mathcal{H}^K(\Gamma) = \mathrm{Ord} \cap \Gamma$. I claim that, to the contrary, $\kappa \subseteq \mathcal{H}^K(\Gamma')$ for any such class Γ: Otherwise we can find such a Γ' so that $\mathcal{H}^K(\Gamma') \cap \kappa$ is as small as possible. Then let $k\colon L[U'] \cong \mathcal{H}^K(\Gamma')$ be the transitive collapse. Since $\operatorname{crit}(U)$ is as small as possible, $\operatorname{crit}(U') = \operatorname{crit}(U) = \kappa$. However there is some $\eta < \alpha$ so that $k(\eta) > \eta$. Then if $\Gamma'' = \{\, \nu : k(\nu) = \nu \,\}$ then $k(\eta) \in \kappa \cap \mathcal{H}^K(\Gamma') - \mathcal{H}^K(\Gamma'')$, contrary to the choice of Γ'. ⊣

Part 1 of the proof

The proof of the Dodd-Jensen covering lemma can be divided into two parts. The first part is a direct generalization of the proof of the covering lemma for L: it involves defining the basic construction for suitable sets $X \prec K_\kappa$, and showing that the class of suitable sets is unbounded. One of the two major novelties in this stage of the proof is the possible use of an iterated ultrapower in the construction. The second part of the proof, which has no analog for L, is used to analyze the indiscernibles generated by this iterated ultrapower. In the case when this iterated ultrapower is infinite, these indiscernibles will yield a sequence C which is Prikry generic over $K = L[U]$.

The other major novelty arises from the fact that lemma 3.39, which is needed in the proof, has the hypothesis that K satisfies the countably closed weak covering property. Thus we will, during part one of the proof, simultaneously prove two results, the first of which is the hypothesis to the second.

3.40 Lemma. *Assume that $0^\dagger$ does not exist.*

1. *The core model K satisfies the countably closed weak covering property.*

2. *If K satisfies the countably closed weak covering property then it also satisfies the full Dodd-Jensen covering lemma, theorems 1.12 and 1.13.*

Most of this proof will be reused in proving the covering lemma for sequences of measures; however certain segments of the proof are substantially simplified by our assumption that $0^\dagger$ does not exist, which is equivalent to the assumption that no premouse has more than one full ultrafilter.

The following definition will be valid up to a strong cardinal. We write K_κ for $L_\kappa[\mathcal{U}] = K \cap V_\kappa$.

3.41 Definition. Let $X \prec_1 K_\kappa$, with transitive collapse $\pi\colon \overline{K} \to X$. We say that X is *suitable* if the following two conditions hold for every integer n, ordinal $\beta \leq \kappa$, and premouse $M = L_\alpha[\mathcal{U}']$ such that $\widetilde{M} = \mathrm{ult}_n(M, \pi, \beta)$ is defined and $\mathcal{U}' \restriction \bar{\beta} = \mathcal{U} \restriction \bar{\beta}$ where β is the least ordinal such that $\pi(\bar{\beta}) \geq \beta$:

1. The structure $\widetilde{M}$ is iterable.

2. If U is a ultrafilter in $\widetilde{M}$ which is a K-ultrafilter on some $\eta < \kappa$, then $\mathrm{ult}(K, U)$ is well founded.

As in the proof of the covering lemma for L, we say that $X \prec_1 K_\kappa$ is *countably closed* if $X = Y \cap K_\kappa$, where ${}^\omega Y \subseteq Y$ and $Y \prec \mathrm{H}_\tau$ for some $\tau > \kappa$.

3.42 Lemma. *(i) Every countably closed set $X \prec_1 K_\kappa$ is suitable, and (ii) the class of suitable sets X is closed under increasing unions of uncountable cofinality, and is unbounded in $H_\delta(K_\kappa)$ for any uncountable cardinal δ.*

Sketch of proof. The only difference between the proof of this lemma and the corresponding lemma for L comes in the definition 3.16 of a witness to the unsuitability of X. Clause 3.16(2), stating that $\operatorname{dir\,lim}(\pi^{-1}(w)) = \mathfrak{C}_n(L_\alpha)$ for some ordinal α, is changed to require that $\operatorname{dir\,lim}(\pi^{-1}(w))$ be the Σ_n code for some mouse. Clause 3.16(3), stating that the structure $\widetilde{M}_0$ of which $\operatorname{dir\,lim}(w)$ is the Σ_n-code is not well founded, must be changed to cover the two clauses of definition 3.41: either $\widetilde{M}_0$ is not iterable, or else there is an ultrafilter $U \in \widetilde{M}_0$ on some $\eta < \kappa$ such that $\mathrm{ult}(K, U)$ is not well founded. $\dashv$

We are now ready to describe the basic construction. As in the proof of the covering lemma for L, we are given a suitable set $X \prec_1 K_\kappa$, and we let $\pi\colon \overline{K} = L_{\bar{\kappa}}[\overline{\mathcal{U}}] \cong X \prec_1 K_\kappa$ be the transitive collapse. We assume that X is not transitive, so that π is not the identity, and furthermore we assume that either X is countably closed or else K satisfies the countably closed weak covering property.

In the proof for L, the next step was to set $M = L_\alpha$, where $\alpha \geq \bar\kappa$ was the least ordinal such that there is a bounded subset of $\bar\kappa$ in $L_\alpha - L_{\bar\kappa}$. This doesn't work here, because $\overline{\mathcal{U}}$ need not be equal to $\mathcal{U}{\restriction}\bar\kappa$. Hence we modify the construction by using iterated ultrapowers to compare the models $\overline{K} = L_{\bar\kappa}[\overline{\mathcal{U}}]$ and $K = L[\mathcal{U}]$. A key step of the proof is showing (see lemma 3.43(2)) that $\overline{K}$ is never moved in this comparison, so that the final model of the iterated ultrapower on K is a model $M_\theta = L_{\alpha_\theta}[\mathcal{U}_\theta]$ such that $\overline{\mathcal{U}} = \mathcal{U}_\theta{\restriction}\bar\kappa$.

Thus we obtain the following variant of diagram I.5, where each of the subset symbols indicate containment as an initial segment:

$$\begin{array}{ccccc} K \overset{i}{\rightsquigarrow} M_\theta \supseteq & M & \xrightarrow{\tilde\pi} & \mathrm{ult}(M,\pi,\kappa) = \widetilde{M} \\ & \big\uparrow & & \big\uparrow \\ \overline{K} = & L_{\bar\kappa}[\overline{\mathcal{U}}] & \xrightarrow{\pi} & X \prec K_\kappa \end{array} \tag{I.9}$$

We will write $M = L_\alpha[\mathcal{U}^M]$ and $\widetilde{M} = L_{\tilde\alpha}[\widetilde{\mathcal{U}}]$. As in diagram (I.5), $\overline{K}$ is the transitive collapse of the set X and π is the inverse of the collapse map. The iterated ultrapower is indicated by the wavy line from K to M_θ. Since this iteration always drops, the wavy line does not represent an embedding.

Once the model $M_\theta = L_{\alpha_\theta}[\mathcal{U}^{M_\theta}]$ has been constructed, complete the diagram like diagram I.5: Let $M = L_\alpha[\mathcal{U}^{M_\theta}]$, where (α, n) is the largest pair (α, n) such that $\mathrm{ult}_n(L_\alpha[\mathcal{U}^{M_\theta}], \pi)$ is defined. Thus α is the least ordinal such that there is a bounded subset x of κ which is definable in $L_\alpha[\mathcal{U}_\theta]$ but is not a member of $\overline{K}$. Finally, set $\widetilde{M} = \mathrm{ult}_n(M, \pi, \kappa)$.

3.43 Lemma. *Assume $\neg 0^\dagger$. Let $X \prec_\kappa K_\kappa$ be a suitable set which is not transitive, so that the collapse $\pi\colon \overline{K} \cong X \prec_1 K_\kappa$ is not the identity. Finally, assume that either X is countably complete or else K has the countably closed weak covering property.*

1. *$\mathcal{P}^K(\eta) \not\subseteq \overline{K}$, where $\eta = \mathrm{crit}(\pi)$.*
2. *In the comparison of K and $\overline{K}$, the iterated ultrapower on the model $\overline{K}$ is trivial.*
3. *Either $\overline{K}$ is an initial segment of K, or else $1 \in D$, so that the iterated ultrapower on K drops immediately.*
4. *$\overline{K}$ is an initial segment of the final model M_θ of the iteration of K.*
5. *$\widetilde{M} \in K$.*

Sketch of proof. If clause (1) fails then $U' = \left\{ x \in \mathcal{P}^{\overline{K}}(\eta) : \eta \in \pi(x) \right\}$ is a K-ultrafilter. If X is countably complete then so is U', so lemma 3.36

implies that U' is a measure in $L[U']$. If K has the countably closed weak covering property then, since $\mathrm{ult}(K, U')$ is well founded by the definition of suitability, lemma 3.39 implies that $K = L[U]$ where $\mathrm{crit}(U) \leq \mathrm{crit}(U')$. Thus, under the hypothesis of either clause of lemma 3.40, $K = L[U]$ for some measure U with $\mathrm{crit}(U) \leq \mathrm{crit}(i)$; but this is impossible: if $\mathrm{crit}(U) < \mathrm{crit}(i)$ then it would follow that $0^\dagger$ exists, contrary to the the hypothesis, while if $\mathrm{crit}(U) = \mathrm{crit}(i)$ then $\mathrm{crit}(U)$ would be definable in K_κ as the only measurable cardinal, and hence would be in X.

Since $\overline{K}$ has no full measures, the iterated ultrapower on $\overline{K}$ is trivial unless it drops; however in that case lemma 3.29 and lemma 3.31 would imply that the iterated ultrapower on K does not drop and has a final model M_θ which is an initial segment of the final model N_θ of the iterated ultrapower on $\overline{K}$. This is absurd, since if τ is the least ordinal such that there is a set $x \subseteq \tau$ in $K - \overline{K}$, then x is in $M_\theta - N_\theta$. Thus the iterated ultrapower on $\overline{K}$ is trivial, and $\overline{K}$ is an initial segment of M_θ. This proves clause (2).

To verify clause (3), note that if $\overline{K}$ is not an initial segment of K then the iterated ultrapower on K is nontrivial; however any full ultrafilter in K with critical point smaller than η would also be in $\overline{K}$, and hence would not be used in the iteration. Thus the iterated ultrapower on K must drop immediately.

To complete the proof we need to check that $\widetilde{M} \in K$. If $\widetilde{M}$ has no full measure $U = \mathcal{U}_\gamma^{\widetilde{M}}$ with $\mathrm{crit}(U) < \kappa$, then $\widetilde{M}$ is iterable because of the suitability of X. Thus $\widetilde{M}$ is a mouse and hence $\widetilde{M} \in K$.

If, on the other hand, there is such a measure $U = \mathcal{U}_\gamma^{\widetilde{M}}$ in $\widetilde{M}$ then U is a K-ultrafilter, since $\widetilde{M}$ and K have the same bounded subsets of κ. Furthermore $\mathrm{ult}(K, U)$ is well founded by clause (2) of the definition of suitability. Also, the hypothesis implies that either U is countably closed because ${}^\omega X \subseteq X$, or else K has the countably closed weak covering property. Hence $U \in K$ either by lemma 3.36 or by lemma 3.39. Thus it again follows that $\widetilde{M}$ is a mouse which strongly agrees with K up to κ, and hence $\widetilde{M}$ is in K. $\dashv$

We can now finish the proof of clause 1 of lemma 3.40:

3.44 Corollary. *If* $0^\dagger$ *does not exist then* K *satisfies the countably closed weak covering property.*

Sketch of proof. Suppose to the contrary that λ is a countably closed singular cardinal, and that $\kappa = \lambda^{+K} < \lambda^+$. Then $\mathrm{cf}(\kappa) \leq \lambda$, and since λ is singular it follows that $\mathrm{cf}(\kappa) < \lambda$. Since λ is countably closed it follows that $\mathrm{cf}(\kappa)^\omega < \lambda$, so there is a set $Y \prec H_{\kappa^+}$ with $Y^\omega \subseteq Y$ and $|Y| = \mathrm{cf}(\lambda)^\omega < \kappa$ such that $Y \cap \kappa$ is cofinal in κ. Thus $X = Y \cap K_\kappa$ is countably closed. Then

$\pi^{-1}(\lambda)$ is the largest cardinal of the model $\overline{K}$ from diagram (I.9), so the iterated ultrapower i cannot have any critical points in the interval $(\pi^{-1}(\lambda), \bar{\kappa})$. If h is the Skolem function of M then it follows that $M = h``\pi^{-1}(\lambda)$. Now let $\tilde{h}$ be the function given by lemma 3.8. Then $X = \pi``(\overline{K} \cap h``\pi^{-1}(\lambda)) \subseteq \tilde{h}``\lambda$, but since $\tilde{h} \in K$ and X is cofinal in κ this contradicts the assumption that $\kappa = \lambda^{+K}$. Hence we must have $\lambda^{+K} = \lambda^{+}$. $\dashv$

With the completion of the proof of the countably closed weak covering lemma, we can concentrate on the full covering lemma.

Part 2: Analyzing the set of indiscernibles

Having constructed all of the elements of diagram I.9 we are ready to finish the proof of the covering lemma by looking at the indiscernibles generated by the iterated ultrapower i.

Fix, for the moment, an arbitrary suitable set X. Recall that we would like to find $f \in K$ and $\eta < \kappa$ such that either $X = f``(\eta \cap X)$ or else there is a Prikry sequence $C$ so that $X = f``(C \cup \eta \cap X)$. Furthermore, we want to show that the Prikry sequence C, if it is exists, is unique modulo finite differences.

If $M \neq M_\theta$ then M is a proper initial segment of the potential premouse M_θ. It follows that M is sound and is hence a mouse. In this case the proof proceeds exactly as in that of the covering lemma for L, and leads to the conclusion that $X = \tilde{h}``(X \cap \rho)$, where $\tilde{h}$ comes from lemma 3.8 and $\rho = \pi(\rho^M_{m+1})$ where m is least such that $\rho^M_{m+1} < \bar{\kappa}$.

Thus we can assume that $M = M_\theta$. Then lemma 3.43 states that $1 \in D$, so $D \neq \varnothing$. Let $\nu_0 + 1 < \theta$ be the largest member of D. Then M_θ is an iterated ultrapower (without drops) of the potential premouse $M^*_{\nu_0+1}$, which is an initial segment of M_{ν_0}. All of the remaining ultrapowers have the same degree n, and $M^*_{\nu_0+1}$ is n-sound. Let $\overline{C} = \{\, i_{\nu_0,\nu}(\kappa_{\nu_0}) : \nu_0 < \nu < \theta \,\}$, where $i_{\nu+0,\nu} \colon M^*_{\nu_0+1} \to M_\nu$. Then $\overline{C}$ is a sequence of indiscernibles for M_θ.

Let $\bar{\rho}$ be the Σ_n-projectum of M, which is equal to the Σ_n-projectum of $M^*_{\nu_0+1}$, and let $\bar{h}$ be the Σ_n-Skolem function of M_θ. Then

$$M_\theta = \bar{h}``(\bar{\rho} \cup \overline{C}) \tag{I.10}$$

by the soundness of $M^*_{\nu_0}$ and lemma 3.24. This statement can be strengthened:

$$\forall \xi \in \bar{\kappa} - \overline{C} \;\; \xi \in \bar{h}``(\bar{\rho} \cup (\overline{C} \cap \xi)). \tag{I.11}$$

Now let $\rho = \sup \pi``\bar{\rho}$ and $C = \pi``\overline{C}$. If $\tilde{h}$ is the function given by lemma 3.8 then it follows that $X = \tilde{h}``(\rho \cup C)$, and if $\xi \in X \cap \kappa$ then $\xi \in \tilde{h}``(\rho \cup (C \cap \xi + 1))$.

If $\overline{C}$ is finite then we can define $g(x) = \tilde{h}^X(x, C)$, so that $g \in K$ and $X = g``(X \cap \rho^X)$. Thus the first of the desired alternatives hold.

For the remainder of the proof, we will assume that C is infinite. We use a superscript X to designate the results of applying this construction to the arbitrary suitable set X.

3.45 Definition. Let $\mathbf{C}$ be the class of suitable sets X such that either C^X is finite or else $K = L[U]$, the set C^X is a Prikry sequence for U, and C^X is maximal in the sense that $C - C^X$ is finite whenever C is any other Prikry sequence for $L[U]$.

Notice that C^X and $C^{X'}$ differ only finitely for any two sets $X, X' \in \mathbf{C}$ such that C^X and $C^{X'}$ are both infinite.

The following lemma completes the proof of the Dodd-Jensen covering lemma:

3.46 Lemma. *If $0^\dagger$ does not exist then the class $\mathbf{C}$ is closed under increasing unions of uncountable cofinality, and is unbounded in $[K_\kappa]^\delta$ whenever κ is a cardinal of K and δ is an uncountable regular cardinal.*

The proof of this lemma will take up the rest of subsection 3.3. We already know that the class of unsuitable sets is nonstationary, and by the comments above, we can assume that C^X is infinite for all but a nonstationary set of sets X.

First we show, assuming lemma 3.46 is true for all cardinals $\mu < \kappa$, that $\operatorname{otp}(C^X) = \omega$ on all but a nonstationary set. Assume the contrary; then there is $\mu < \kappa$ such that the ω^{th} member C^X of C^X is equal to μ for a stationarily many sets X. Since the induction hypothesis states that the covering lemma holds for $C \cap K_\mu$, there is X with $\mu^X = \mu$ such that $X \cap K_\mu \in \mathbf{C}$, but this implies that $K = L[U]$ where U is a measure on μ. Now this measure U must be in X, and is generated by $C^X \cap \mu$. Thus the iterated ultrapower from diagram (I.9) which was used to generate $\overline{C}$ would not continue past $(\pi^X)^{-1}(\mu)$, and hence $\overline{C}^X \subseteq (\pi^X)^{-1}(\mu)$. This contradicts the assumption that $C^X \not\subseteq \mu$, and completes the proof that C^X has order type ω except on a nonstationary set.

The proof of lemma 3.46 is based on the following observation:

3.47 Proposition. *Suppose that $X = Y \cap K_\kappa$ is suitable, that $Y \prec_1 K_\lambda$ for some cardinal $\lambda > \kappa$, and that $z \in \mathcal{P}(\kappa) \cap Y$. Then $z \in \operatorname{ran}(\tilde{\pi}^X)$.*

Proof. Let $\pi^Y : N^Y \cong Y$ be the transitive collapse of Y, so that $N^Y \cap K_{\bar{\kappa}} = \overline{K}$ and $\pi^Y \restriction \overline{K} = \pi^X$. Let $\mathfrak{m} \in Y$ be the least mouse such that $z \in \mathfrak{m}$. Then $\bar{\mathfrak{m}} = (\pi^X)^{-1}(\mathfrak{m}) \in N^Y$, and since $X = Y \cap L_\kappa$ it follows that every bounded subset of $\bar{\kappa}$ in $\bar{\mathfrak{m}}$ is a member of N. Thus the mouse $\bar{\mathfrak{m}}$ is an initial segment of M^X and it follows by lemma 3.32 that $\bar{z} = (\pi^Y)^{-1}(z) \in M^X$ and hence $z = \tilde{\pi}^X(\bar{z}) \in \operatorname{ran} \tilde{\pi}^X$. $\dashv$

3.48 Corollary. *Suppose that X is suitable and $X = Y \cap K_\kappa$ where $Y \prec H_{\kappa^+}$ and $C^X \in Y$. Then $X \in \mathbf{C}$.*

Sketch of Proof. If C fails to satisfy definition 3.45, then by elementarity there is a witness to this failure which is a member of Y. Such a witness is either a set $x \subseteq \kappa$ in K showing that C^X is not a Prikry sequence, or else a second Prikry sequence C' such that $C' - C$ is infinite. The first is impossible, since by proposition 3.47 the set x would be in $\widetilde{M}^X$, and C^X is a set of indiscernibles for $\widetilde{M}^X$. The second is also impossible, since it would follow from $Y \prec H_{\kappa^+}$ that there is such a Prikry sequence C' which is a member of Y. Then $Y' \subseteq X$, so any member α of $C' - C$ is in $\tilde{h}^X$ "$(\tilde{\rho}^X \cup C \cap \alpha)$, and since $\tilde{h}^X \in K = L[U]$ it follows that $C' - C$ is finite since C' is a Prikry sequence. $\dashv$

3.49 Corollary. *If X is countably closed then $X \in \mathbf{C}$.* $\dashv$

Now we will deal with the general proof of lemma 3.46 for non-countably closed X. This proof uses the ideas of the proof of lemma 3.42, stating that the class of suitable sets is unbounded, but is significantly more difficult. We use the notation $a \subseteq^* b$ to mean that $a - b$ is finite, and $a =^* b$ to mean that $a \subseteq^* b$ and $b \subseteq^* a$.

3.50 Lemma. *If X_0, X_1 are suitable and $X_0 \subseteq X_1$, then $C^{X_1} \cap X_0 \subseteq^* C^{X_0}$.*

Proof. We will use subscripts 0 and 1 to distinguish objects defined from X_0 or X_1; for example we write $\pi_0 = \pi^{X_0}$ and $\pi_1 = \pi^{X_1}$. We will find a function h^*, definable in $M_1 = M^{X_1}$, such that $\xi \in h^*$ "ξ for all but boundedly many $\xi \in \pi_1^{-1}(C_1 - C_0)$. Since this can only hold for finitely many $\xi \in \pi_1^{-1}(C_1)$, this will imply that $C_1 \subseteq^* C_0$.

To this end, let ν be any member of $X_0 \cap (C_1 - C_0)$ and set $\nu_0 = \pi_0^{-1}(\nu)$. Then $\nu_0 \notin \overline{C}_0$, so $\nu_0 \in h_0$ "ν_0 where h_0 is the Skolem function of $M_0 = M^{X_0}$. Now let $\tau = \pi_1^{-1} \circ \pi_0 \colon \overline{K}_0 \to \overline{K}_1$, and let

$$\tilde{\tau} \colon M_0 \to M^* = \mathrm{ult}(M_0, \tau, \bar{\kappa}_1).$$

Then $\nu_1 = \tau(\nu_0) \in h^*$ "ν_1 where h^* is given by lemma 3.8. But M^* is sound above $\bar{\kappa}_1$, and agrees with $\bar{K}_1$ up to $\bar{\kappa}_1$, so by lemma 3.32 one of M^* and M_1 is an initial segment of the other. Since every bounded subset of $\bar{\kappa}$ in M^* is a member of $\bar{K}_1$, it must be that M^* is an initial segment of M_1 and it follows that h^* is definable in M_1 from some parameter q. Since $\nu \in X_0 \cap (C_1 - C_0)$ was arbitrary, it follows that $C_1 \cap X_0 \subseteq^* C_0$ is finite. $\dashv$

3.51 Corollary. *The class $\mathbf{C}$ is uncountably upward closed.*

Proof. Suppose that $X = \bigcup_{\xi<\eta} X_\xi$ is an increasing union of sets $X_\xi \in \mathbf{C}$ such that $\mathrm{cf}(\eta) > \omega$. Then X is suitable since the class of suitable sets is closed under uncountable increasing unions, and $C^X \subseteq X_\xi$ for some $\nu < \eta$

so $C^X \subseteq^* C^{X_\xi}$ by lemma 3.50. In particular, the fact that C^{X_ξ} is a Prikry sequence for the measure U implies that C^X is a Prikry sequence for the same measure.

To complete the proof that $X \in \mathbf{C}$ we need to show that C^X is maximal, and since C^{X_ξ} is maximal it is sufficient to show that $C^{X_\xi} \subseteq^* C^X$. Now if ν is any member of $C^{X_\xi} - C^X$ then $\nu \in \tilde{h}``\nu$, where $\tilde{h}$ is the function given by lemma 3.8. But since $C^{X_\xi}$ is a Prikry sequence over $K$ and $\tilde{h} \in K$, it follows that $C^{X_\xi} - C^X \subseteq \left\{ \nu \in C^{X_\xi} : \nu \in \tilde{h}``\nu \right\}$ is finite, so $C^{X_\xi} \subseteq^* C^X$. ⊣

The following lemma completes the proof of the Dodd-Jensen covering lemma. We give a proof which is somewhat different from that given by Dodd and Jensen in [10], as that proof does not easily adapt to larger core models.

3.52 Lemma. *If δ is a regular cardinal, and κ is a cardinal of K_κ then* $\mathbf{C}$ *is unbounded in $H_\delta(K_\kappa)$.*

Proof. As in the proof of the covering lemma for L, we work in the space $\mathrm{Col}(\delta, K_\kappa)$. If $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ then we will sometimes identify σ with $\mathrm{ran}(\sigma)$, especially when it appears as a superscript. Let S be the set of functions $\sigma \in \mathrm{Col}(\delta, K_\kappa)$ such that $\mathrm{cf}(\mathrm{dom}(\sigma)) > \omega$ and $\mathrm{ran}(\sigma)$ is suitable, but $\mathrm{ran}(\sigma) \notin \mathbf{C}$, and suppose for the sake of contradiction that S is stationary in $\mathrm{Col}(\kappa, K_\kappa)$. By lemma 3.21 there is $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ such that $\sigma \supseteq \sigma_0$ and $C^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_0$. Thus $C^\sigma \subseteq^* C^{\sigma_0}$ for all $\sigma \in S_0$ by lemma 3.50. Set $C_0 = C^{\sigma_0}$.

As in the proof of the covering lemma for L, we define, for each member of S_0, a witness $w(\sigma)$ to the fact that $\mathrm{ran}(\sigma) \notin \mathbf{C}$:

3.53 Claim. *There is a function w mapping each member σ of S_0 to a countable subset of $\mathrm{ran}(\sigma)$ such that for any $\sigma_1, \sigma_2 \in S_0$ such that $\sigma_1 \subseteq \sigma_2$ and $w(\sigma_2) \subseteq \mathrm{ran}(\sigma_1)$ we have $C^{\sigma_1} \subseteq^* C^{\sigma_2}$.*

First we show that the lemma follows from this claim. By applying lemma 3.21 a second time, we can find $\sigma_1 \in S_0$ and a stationary set $S_1 \subseteq S_0$ so that $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \mathrm{ran}(\sigma_1)$ for all $\sigma \in S_1$. If σ is any member of S_1 then $C^{\sigma_1} \subseteq^* C^\sigma$ by claim 3.53 and $C^\sigma \subseteq^* C^{\sigma_1}$ by lemma 3.50. Thus $C^\sigma =^* C^{\sigma_1}$ for all $\sigma \in S_1$.

Since S_1 is stationary, there is $\sigma \in S_1$ such that $\mathrm{ran}(\sigma) = Y \cap K_\kappa$ for some $Y \prec H_{\kappa^+}$ with $C^{\sigma_1} \in Y$. Since $C^\sigma =^* C^{\sigma_1}$ it follows that $C^\sigma \in Y$. This implies $\mathrm{ran}(\sigma) \in \mathbf{C}$ by lemma 3.48, contradicting the fact that $\sigma \in S_1 \subseteq S_0$. This contradiction shows that S_0 is not stationary, and hence $\mathbf{C}$ is unbounded.

Proof of claim 3.53. We will fix $\sigma \in S_1$ for the moment in order to define $w(\sigma)$. The critical point is that $C^\sigma \subseteq^* C^{\sigma_0}$, so that C^σ is determined, up to a finite set, by $D = C^{\sigma_0} - C^{\sigma}$. If D is finite then we can set $w(\sigma) = \varnothing$, so we will assume that D is infinite. Let $\langle\, d_k : k < \omega\,\rangle$ enumerate D in increasing order, and set $\bar{d}_k = \pi^{-1}(d_k)$. Then $\bar{d}_k \in h^M``d_k$, where h^M is the Skolem function for the premouse M of diagram (I.9).

To define the function w we modify the definition 3.16 of a witness to the unsuitability of X by replacing clause 3 with the statement that there is a function h which is Σ_n definable over $\operatorname{dir\,lim}(w)$ such that $d \in h``d$ for all $d \in D$.

To see that this witness function $w(\sigma)$ satisfies claim 3.53, let $\sigma_1 \subseteq \sigma$ be a member of S_0 with $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$. Write π_1 for π^{σ_1}, and set $\tau = \pi_1^{-1}\pi\colon \overline{K}^{\sigma_1} \to \overline{K}^{\sigma}$. If $\bar{\mathfrak{m}} = \operatorname{dir\,lim}\bigl(\pi_1^{-1}(w(\sigma))\bigr)$ then the map τ extends to an elementary embedding $\tilde{\tau}\colon \bar{\mathfrak{m}} \to \mathfrak{m}$, so $\bar{\mathfrak{m}}$ is also a premouse. The measure on $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is generated by the indiscernibles $\tau^{-1}(\overline{C}^{\sigma}) = \pi_1^{-1}(C^{\sigma})$, and since $C^{\sigma} \subseteq^* C^{\sigma_1}$ it follows that this measure is equal to the measure in M^{σ_1}. Thus $\bar{\mathfrak{m}}$ strongly agrees with M^{σ_1} up to $\bar{\kappa}_1$. Since both premice are sound above $\bar{\kappa}_1$ it follows that one is an initial segment of the other. Now if M^{σ_1} were a proper initial segment of $\bar{\mathfrak{m}}$, then there would be a bounded subset x of $\bar{\kappa}_1$ in $\bar{\mathfrak{m}} - \overline{K}^{\sigma_1}$. This is not the case, since every bounded subset of $\bar{\kappa}_1$ in $\bar{\mathfrak{m}}$ is in some $\bar{\mathfrak{m}}_k \in \overline{K}^{\sigma_1}$, so $\bar{\mathfrak{m}}$ must be an initial segment of M^{σ_1}. Hence, the Skolem function of $\bar{\mathfrak{m}}$ is definable in M^{σ_1}, and thus every sufficiently large member d of $\pi_1^{-1}(D)$ is in $h^{M^{\sigma_1}}``d$. It follows that $D \cap C^{\sigma_1}$ is finite, which is to say that $C^{\sigma_1} \subseteq^* C^{\sigma}$ as was to be proved. $\dashv$

This completes the proof of lemma 3.52, and hence of the Dodd-Jensen covering lemma, theorems 1.2 and 1.3. $\dashv$

3.4. Unsuitable Covering Sets

As we have seen, the proof of the covering lemma for L shows that, assuming $\neg 0^\sharp$, every suitable set is in L. This striking fact suggests that the proof may also have something to say about sets X which are not suitable. Some restrictions on X are certainly needed: for example, if X is a Cohen generic subset of some uncountable cardinal τ then $y \not\subseteq X$ for any infinite set $y \in L$, and $y \not\supseteq X$ for any coinfinite set $y \in L$. Thus we will retain the first order part of the definition of suitability: we assume that $X \prec_1 L_\kappa$ (or $X \prec_1 K_\kappa$ in the case of larger core models) but omit the second order condition that $\widetilde{M}^X$ is well founded and (in the case of K) that C^X is a maximal Prikry sequence. This one idea leads to two separate results: For L, or more generally if there is no ω_1-Erdős cardinal in K, it gives Magidor's covering lemma, 1.15, while in the presence of larger cardinals it gives theorem 1.16 stating that Jónsson and Ramsey cardinals relativize to K.

We recall the statement of Magidor's covering lemma 1.15 for L. This statement follows [25] in using the hypothesis that X is primitive recursively closed instead of $X \prec_1 L_\kappa$, but we do not verify that this weaker condition is sufficient.

3.54 Theorem. *Suppose that $0^\sharp$ does not exist and that X is a primitive recursively closed subset of L_κ, and let $\delta = \inf(\kappa - X)$. Then there are functions $h_i \in L$ for $i < \omega$ such that $X \cap \kappa = \bigcup_{i<\omega} h_i``(\delta)$*

Sketch of proof. Like the covering lemma, this theorem is proved by induction on κ. Suppose that $X \prec_1 L_\kappa$. If $\mathrm{cf}(\kappa) = \omega$, say $\kappa = \bigcup_{n<\omega} \kappa_n$, then $X = \bigcup_{n<\omega}(L_{\kappa_n} \cap X)$, so that the truth of the theorem for X follows from its truth for each of the sets $X \cap L_{\kappa_n}$ for $n < \omega$. Thus we can assume that $\mathrm{cf}(\kappa) > \omega$. In addition we can assume that κ is a cardinal in L, that X is cofinal in κ, and that $\kappa \not\subseteq X$ so that $\delta < \kappa$. Note that we do not assume that $|X| < \kappa$.

The proof begins exactly like that of the covering lemma, with the transitive collapse $\pi\colon N = L_{\bar\kappa} \cong X \subseteq L_\kappa$. Thus $\delta = \mathrm{crit}(\pi) < \kappa$.

If X is suitable then $X \in L$ by the proof of the covering lemma, so we can assume that X is not suitable. We recall the construction, given in lemma 3.17, of a witness to the to the unsuitability of X. There is a triple (α, n, β), with $\beta \le \bar\kappa < \alpha$, such that $\mathrm{ult}_n(L_\alpha, \pi, \beta)$ is defined but not well founded. Let (α, n, β) be the least such triple, in the lexicographic ordering, and pick $f_i \in L_\alpha$ and $a_i \in [\beta]^{<\omega}$ for $i < \omega$ so that $[a_{i+1}, f_{i+1}]_\pi E [a_i, f_i]_\pi$, where E is the membership relation in the ultrapower. Then $\beta \le \sup \bigcup_i a_i$, and since $\mathrm{cf}(\kappa) > \omega$ it follows that $\beta < \kappa$. We will show that there are functions $h_i \in L$ such that $X = \bigcup_{i<\omega} h_i``(X \cap \beta)$. The truth of the theorem for X then follows by applying the induction hypothesis to the set $X \cap \pi(\beta)$.

In order to simplify notation we will assume that $n = 0$ and that α is a limit ordinal. We make two observations:

1. We can choose $(\, f_i : i < \omega\,)$ so that $L_\alpha = \mathcal{H}_{\Sigma_1}^{L_\alpha}(\beta \cup \{\, f_i : i < \omega\,\})$. If this is not true for the original choice of functions f_i then let $\mathcal{M}' = L_{\alpha'}$ be the transitive collapse of $\mathcal{H}_{\Sigma_1}^{L_{\alpha_i}}(\beta \cup \{\, f_i : i < \omega\,\})$. Then $\alpha' \le \alpha$ and $\mathrm{ult}(L_{\alpha'}, \pi, \beta)$ is ill founded, so $\alpha' = \alpha$ by the minimality of α. The original functions f_i may be moved in the collapse, but we can replace them by their images under the collapse.

2. If $\alpha_i < \alpha$ is the least ordinal such that $f_i \in L_{\alpha_i}$ and $\mathcal{M}_i$ is the transitive collapse of $\mathcal{H}_{\Sigma_1}^{L_{\alpha_i}}(\bar\kappa \cup \{\, f_0, \dots, f_i\,\})$, then $\widetilde{\mathcal{M}}_i = \mathrm{ult}(\mathcal{M}_i, \pi, \kappa)$ is well founded. If it were not, then since $\mathrm{cf}(\kappa) > \omega$ it would follow that there is some $\eta < \bar\kappa$ such that $\mathrm{ult}(\mathcal{M}_{\eta,i}, \pi, \pi(\eta))$ is ill founded, where $\mathcal{M}_{\eta,i}$ is the transitive collapse of $\mathcal{H}_{\Sigma_1}^{L_{\alpha_i}}(\eta \cup \{\, f_0, \dots, f_i\,\})$. This is impossible, since $\mathcal{M}_{\eta,i} \in N$ and hence $\mathrm{ult}(\mathcal{M}_{\eta,i}, \pi, \pi(\eta))$ can be embedded into $\pi(\mathcal{M}_{\eta,i})$, which is well founded.

It follows that $\widetilde{\mathcal{M}}_i = \mathrm{ult}(\mathcal{M}_i, \pi, \kappa) \in L$ for each $i < \omega$, and $X = \pi``(\bar\kappa) =$

$\bigcup_{i<\omega} h_i``(X \cap \beta)$ where h_i is the function given by lemma 3.8. This completes the proof of theorem 3.54. $\dashv$

This argument, applied to K^{DJ} in the absence of a model with a ω_1-Erdős cardinal, yields Magidor's generalization of theorem 1.15 to K^{DJ}, while applied to larger core models K it yields the absoluteness to K of Jónsson and Ramsey cardinals, theorem 1.16. This extension of the argument to K requires that the iterated ultrapower constructed in definition 3.26 be modified by adding a second type of drop: Suppose that $M_\nu = L_{\alpha_\nu}[\mathcal{U}_\nu]$ has been defined, and let $\bar\beta_\nu$ be the largest ordinal such that $\mathcal{U}_\nu \restriction \bar\beta_\nu = \overline{\overline{\mathcal{U}}} \restriction \bar\beta_\nu$. The next model, $M_{\nu+1}$, is defined normally, following definition 3.26, except in the special case when $D \cap \nu = \varnothing$ and there is a triple (α, n, β) with $\beta \leq \pi(\beta_\nu)$ such that $\mathrm{ult}_n(L_\alpha[\mathcal{U}_\nu], \pi, \beta)$ is defined but not iterable. In this case put ν into D and set $M_{\nu+1} = L_\alpha[\mathcal{U}_\nu]$, where (α, n, β) is the least such triple. It is still true that N is an initial segment of the final model M_θ of this iteration: either as in the original proof because M_θ defines a bounded subset of $\bar\kappa$ which is not in N, or else because $\mathrm{ult}_n(M_\theta, \pi, \kappa)$ is not iterable, while $\mathrm{ult}(\overline{K}, \pi, \kappa) = K_\kappa$ is iterable.

Let C^X be the set of putative indiscernibles generated by this proof, that is, the image under π of the critical points (after the last drop) of the iterated ultrapower. Then we get, as in the proof of theorem 3.54, a set of functions $h_k \in K$ for $k < \kappa$ so that

$$\begin{aligned}\widetilde{M} &= \bigcup \{\, \mathcal{H}^{h_k}(\rho \cup C^X) : k \in \omega \,\} \\ &= \bigcup \{\, h_k``(\rho \cup \vec{c}) : k < \omega \wedge \vec{c} \in [C^X]^{<\omega} \,\}\end{aligned}$$

where $\rho = \inf(\kappa - X)$.

If C^X is finite or countable then this gives X as a countable union of sets in K, so we can assume that C^X is uncountable. There is no reason to expect C^X to be a set of indiscernibles for K, but C^X is a set of indiscernibles for any structure in the range of $\tilde\pi$. This observation, along with proposition 3.47, yields the following fact:

3.55 Proposition. *Suppose that $X = Y \cap K_\kappa$ where $Y \prec H_\lambda$ for some $\lambda > \kappa$, and that $\mathcal{A} \in Y \cap K$ is a structure with universe κ. Then C^X is a set of indiscernibles for $\mathcal{A}$.*

Furthermore, if $D \in K \cap Y$ is a closed and unbounded subset of κ then $C^X - D$ is bounded in $\sup(C^X)$. $\dashv$

We will use this proposition to show that any Jónsson cardinal is Ramsey in K. The argument that every δ-Jónsson cardinal is δ-Erdős in K is similar, as is Magidor's argument that theorem 1.15 holds for K^{DJ} unless there is a ω_1-Erdős cardinal in K^{DJ}.

Let $\mathcal{A}$ be any structure in K with universe κ. Since X is Jónsson there are sets Y and X as in the hypothesis of proposition 3.55 such that $|X| = \kappa$ but $\kappa \not\subseteq X$.

It follows from the construction of the set C^X that $\left|C^X\right| = \kappa$. To show that κ is Ramsey in K we will show that there is a $\rho < \kappa$ and a set $C \in K$ of indiscernibles for $\mathcal{A}$ such that $(C^X - \rho) \subseteq C$. To this end let U be the filter on κ generated by C^X, that is, $z \in U$ if and only if $C^X - z$ is bounded in κ. Let $\mathfrak{m}$ be the least mouse with projectum κ such that $\mathcal{A} \in \mathfrak{m}$. Then $\mathfrak{m} \in Y$, so that U is a normal ultrafilter on $\mathfrak{m}$. Furthermore $\mathrm{ult}(\mathfrak{m}, U)$ is iterable since U is countably complete, so $\mathrm{ult}(\mathfrak{m}, U) \in K$ and hence $U \cap \mathfrak{m} \in K$. Let h be the Skolem function of $\mathfrak{m}$ and define C to be the set of $\nu < \kappa$ such that, for each $k < \omega$ and each set $z \in \kappa^{1+k} \cap \tilde{h}``(\nu)$,

$$\begin{aligned} z \in U &\iff \nu \in z && \text{if } k = 0 \\ z \in U^{1+k} &\iff \left\{ \vec{\gamma} \in (\kappa - \nu)^k : \nu^\frown\vec{\gamma} \in z \right\} \in U^k && \text{if } k > 0. \end{aligned}$$

Then C is a set of indiscernibles for $\widetilde{M}^X$, and hence for $\mathcal{A}$. Furthermore $C \in K$, and $C^X - C$ is bounded in κ since $h \in Y$. Then C is the required set of indiscernibles in K for $\mathcal{A}$, and since $\mathcal{A}$ was arbitrary this completes the proof that every Jónsson cardinal is Ramsey in K.

4. Sequences of Measures

This section concerns the covering lemma in the presence of models containing large cardinals. Most of the section will concentrate on the core model for sequences of measures; the remainder will describe, with less detail, what is known about the covering lemma up to a strong cardinal and then for overlapping extenders in the Steel core model up to and beyond a Woodin cardinal. We begin with with a general survey, which is followed by a precise statement of the covering lemma for sequences of measures and some indications as to its proof.

The two large cardinal properties which critically affect the statement of the covering lemma are measurable cardinals and Woodin cardinals. Measurable cardinals are critical because they provide, via Prikry forcing, the first counterexample to the full covering property. Woodin cardinals are critical because they provide, via stationary tower forcing, a counterexample to the weak covering property.

The Covering Lemma and Sequences of Measures

The Dodd-Jensen covering lemma elegantly accommodates the covering lemma to models $L[U]$ with a single measure; indeed the hypotheses $\neg\exists 0^\dagger$ and $K = L[U]$ are as well understood as are the hypothesis $\neg\exists 0^\sharp$ of the

Jensen covering lemma. The situation for larger core models is both more complicated and less elegant. We begin this section by describing some of these complications, in order of the size of the core model at which they first appear.

The first three observations are relevant even in models in which $o(\kappa) \leq 1$ for all κ, that is, when no cardinal has more than one measure. To simplify the notation for this case, we write $\vec{\kappa} = (\,\kappa_\nu : \nu < \theta\,)$ for the sequence of measurable cardinals in K, and let $K = L[\mathcal{U}]$ where $\mathcal{U} = (\,U_\nu : \nu < \theta\,)$ and U_ν is a measure on κ_ν. A system of indiscernibles for this model K is a sequence $\mathcal{C} = (\,C_\nu : \nu < \theta\,)$, with $C_\nu \subseteq \kappa_\nu$. Each set C_ν is either finite or a Prikry sequence, but in addition the sequence $\mathcal{C}$ as a whole is uniformly a system of indiscernibles:

$$\forall \vec{x} \in K \ \Big((\forall \nu < \theta \, x_\nu \in U_\nu) \implies \Big| \bigcup \{\, C_\nu - x_\nu : \nu < \theta \,\} \Big| < \omega \Big). \qquad \text{(I.12)}$$

This leads to our first observation:

1. The sets C_ν need not be infinite: formula (I.12) is meaningful even if some or all of the sets C_ν are finite.

The only constraint on the function $f(\nu) = |C_\nu|$ when $o(\nu) = 1$ for all ν is that $\operatorname{otp} C_\nu \leq \omega$. For any predetermined function f, there is a straightforward modification of Prikry forcing which can be used to obtain a sequence such that $|C_\nu| = f(\nu)$ for all $\nu \in \operatorname{dom}(\mathcal{U})$.

As a consequence the relation between $L[\mathcal{U}]$ and $L[\mathcal{U}, \mathcal{C}]$ is more complicated than that between $L[U]$ and $L[U, C]$:

2. The function $f(\nu) = |C_\nu|$ need not be a member of K.

As an example, suppose that $\theta \geq \omega$ and let $a \subseteq \omega$ be a real which is Cohen generic over K. Then each of the measures U_n can be extended to a measure in $K[a]$, so we can modify Prikry forcing to obtain a system $\mathcal{C}$ of indiscernibles for $K[a]$ such that $|C_n| = 1$ (or, alternatively, a $|C_n| = \omega$) for each $n \in a$ and $C_n = \varnothing$ for each $n \in \omega - a$. Thus $a \in K[\mathcal{C}]$. If $|C_n| = \omega$ for each $n \in \omega$ then a is definable in $K[\mathcal{C}]$ as $\{\, n \in \omega : \operatorname{cf}(\kappa_n) = \omega \,\}$. If $|C_n| = 1$ for $n \in a$ then the covering lemma can be used to show that the system $\mathcal{C}$, and hence the set a, is definable in $K[\mathcal{C}]$ up to a finite set.

Note that a can be any set so long as the measures U_n can be extended to measures in $K[a]$. It is not known whether $\mathcal{C}$ is necessarily set generic over $L[\mathcal{U}]$.

In the Dodd-Jensen covering lemma for $L[U]$, the Prikry sequence C, if it exists, does not depend on the set x to be covered. This is not true for longer sequences:

3. If there is an inaccessible limit of measurable cardinals in K, then there is a cardinal preserving generic extension $K[G]$ of K in which

each measure in K has a Prikry sequence, but there is no sequence $\mathcal{C} = (\, C_\nu : \nu < \kappa \,)$ of Prikry sequences which satisfies (I.12) [29, theorem 1.3].

An inaccessible limit of measurable cardinals is needed to obtain such a sequence: it is shown in [33, thm 4.1] that if there is no model with an inaccessible limit of measurable cardinals then, as in the Dodd-Jensen covering lemma, there is a single sequence $\mathcal{C}$ which can be used to cover any set x.

Since the remaining observations only apply in the presence of cardinals κ with $o(\kappa) > 1$, we now revert to the notation for sequences of measures described in chapter $\boxed{\text{introduction}}$ and in the last section: the core model K is a structure of the form $L[\mathcal{U}]$, where $\mathcal{U}$ is a sequence of filters such that each member $\mathcal{U}_\gamma$ of the sequence is a normal measure on $L[\mathcal{U}{\restriction}\gamma] \cap \mathcal{P}(\kappa)$, where $\gamma = \kappa^{++}$ in $L[\mathcal{U}{\restriction}\gamma]$. Not all of the filters $\mathcal{U}_\gamma$ are full measures in K but we only need to consider those measures which are are full.

We frequently write $\mathcal{U}(\alpha, \beta)$ for the βth full measure on α in $L[\mathcal{U}]$; that is, $\mathcal{U}(\alpha, \beta) = \mathcal{U}_{\gamma_\beta}$ where $(\, \gamma_\nu : \nu < o(\alpha) \,)$ is the increasing enumeration of the ordinals γ such that $\mathcal{U}_\gamma$ is a full measure on α in $L[\mathcal{U}]$. We write $o(\alpha)$, as above, for least ordinal β such that $\mathcal{U}(\alpha, \beta)$ is undefined. The sequence $\mathcal{U}$ has the following *coherence* property: if $i\colon K \to \operatorname{ult}(K, \mathcal{U}(\alpha, \beta))$ then $o^{i(\mathcal{U})}(\alpha) = \beta$ and $i(\mathcal{U})(\alpha, \beta') = \mathcal{U}(\alpha, \beta')$ for all $\beta' < \beta$.

For the next three observations we assume that the core model does not contain any extenders, so that K always satisfies $o(\alpha) \leq \alpha^{++}$.

Corresponding to a sequence $\mathcal{U}$ of measures we will use $\mathcal{C}$ to denote a system of indiscernibles: if $\gamma \in \operatorname{dom}(\mathcal{C})$ then $\mathcal{C}_\gamma \subseteq \operatorname{crit}(\mathcal{U}_\gamma)$ is a set of indiscernibles for the measure $\mathcal{U}_\gamma$ (or, in the other notation, $\mathcal{C}(\alpha, \beta) \subseteq \alpha$ is a set of indiscernibles for $\mathcal{U}(\alpha, \beta)$). The precise definition of a system of indiscernibles will be given later, in definitions 4.11 thru 4.14 and in the covering lemma, theorem 4.15.

4. The sets $\mathcal{C}(\kappa, \beta)$ may have order type greater than ω. In general, the set $\bigcup_{\beta < o(\kappa)} \mathcal{C}(\kappa, \beta)$ of indiscernibles for measures on a cardinal κ is a closed subset of κ which may have any order type up to $\min(\kappa, \omega^{o(\kappa)})$.

In [24] Magidor generalizes Prikry forcing in order to add such a sequence of indiscernibles and hence change the cofinality of a cardinal κ to any smaller regular cardinal λ, provided that $o(\kappa) \geq \lambda$ in the ground model. This forcing is discussed briefly in chapter $\boxed{\text{introduction}}$ and extensively in chapter $\boxed{\text{gitik}}$.

For longer sequences of measures, and in particular when $o(\kappa) > \kappa$, it is important in applications that the domain of the sequence $\mathcal{C}^X$ is contained in the precovering set X. For this reason, we assume a slightly different context for the covering lemma for sequences than was used for the Dodd-Jensen covering lemma. Let $\kappa = \sup(x)$ be a cardinal of K, where x is the set which we are trying to cover. We will look for a precovering set $X \supseteq x$ such that $X \prec_1 K_{\breve{\kappa}}$, where $\breve{\kappa}$ is the least cardinal of K such that $\breve{\kappa} \geq \max(\kappa, o(\kappa))$.

This requirement that $\operatorname{dom}(\mathcal{C}^X) \subseteq X$ leads to two somewhat technical problems in the study of longer sequences of measures:

5. It need not be that every suitable set X can be written as $X = h``(\rho; \mathcal{C}^X)$ where $\rho = \inf(\kappa - X)$ and $h \in K$.

Recall that the strong version of the Jensen covering lemma states that if $0^\sharp$ does not exist then every suitable set X is in L, and the Dodd-Jensen covering lemma for $L[U]$ states (assuming there is a Prikry sequence C) that any such set can be written as $h``(\rho \cup C)$ where $\rho = \inf(\kappa - X)$.

The covering lemma for longer sequences states that $X = h``(\rho \cap X; \mathcal{C}^X)$ for some $\rho < \kappa$, however the induction used to show that ρ can be taken to be $\inf(\kappa - X)$ breaks down for sequences of measures: it depends on the fact that $X \cap K_\rho$ is suitable as a subset of K_ρ, but in the case of sequences of measures it may be that ρ is measurable, but is not a member of X. In this case $X' = X \cap K_{\breve{\rho}}$ is not suitable since $\operatorname{dom}(\mathcal{C}^{X'})$ is not contained in X'.

So far this limitation has not caused problems in applications, nor has the next difficulty:

6. If $o(\alpha) \geq \alpha^+$ then it is not known whether countable completeness of a set X is enough to ensure that X is a suitable precovering set. What is known is stated in theorem 4.15, which requires that the set X be $\operatorname{cf}(\kappa)$-closed. In particular it is not known whether there always exist suitable precovering sets of size less than $(\operatorname{cf}(\kappa))^+$.

The problem, again, comes from the requirement that $\operatorname{dom}(\mathcal{C}) \subseteq X$, but it this case it is the measures on κ which are in question. These measures are generated by cofinal subsets of $\bigcup_\beta \mathcal{C}(\kappa, \beta)$, so the assumption that X contains its subsets of size at most $\operatorname{cf}(\kappa)$ implies that these subsets, and hence the corresponding measures, are in X.

Extenders

If $o(\kappa) > \kappa^{++}$ then the core model is built using extenders, and we will write $K = L[\mathcal{E}]$ to denote the core model. Below $0^\P$, the sharp for a strong cardinal, the extenders do not overlap and the covering lemma as stated for sequences of measures remains true with two modifications. One of these is primarily notational, but the following situation is unexpected:

7. If $\mathrm{cf}(\kappa) = \omega$ and $\{\, \alpha < \kappa : o(\alpha) \geq \alpha^{+n} \,\}$ is unbounded in κ for all $n < \omega$, then the fact that a set X is countably closed does not ensure that X contains all of the extenders on κ which are generated by the system $\mathcal{C}^X$ of indiscernibles for X.

   If $\mathrm{cf}(\kappa) \geq \omega_1$ and ${}^{\mathrm{cf}(\kappa)}X \subseteq X$, however, then X does contain all such extenders. As a result the covering lemma up to $0^{\P}$ is similar to the result of substituting "ω_1" and "countable" for "ω" and "finite" in the covering lemma for sequences of measures.

Both parts of this observation are due to Gitik. In [16] he defines a game which can be used to reconstruct a extender E on a cardinal of uncountable cofinality from the sequences of ordinals which generate the constituent ultrafilters, and in [19] he constructs a model in which this is not possible for extenders on a cardinal of cofinality ω.

The set $0^{\P}$ marks the introduction of overlapping extenders, and thus of a dramatic shift in our understanding of the covering lemma:

8. If $0^{\P}$ exists then we cannot prove much more than the weak covering lemma and the absoluteness theorem for Jónsson and Ramsey cardinals (theorem 1.16).

The basic construction of the proof of the covering lemma does still go through for overlapping extenders, with considerably increased technical difficulties,[37, 36] but it uses iteration trees rather than the linear iteration of definition 3.26. The indiscernibles generated by such iterations are very poorly understood, and the proofs for the known results above $0^{\P}$ rely on avoiding indiscernibles rather than on analyzing them.

9. No core model with cardinals very much larger than $0^{\P}$ is known to exist without an additional assumption that there is an inaccessible cardinal in the universe. It is known that there is no core model for a Woodin cardinal.

As was pointed out at the beginning of this section, even the weak covering lemma is false for any model with a Woodin cardinal δ, since nonstationary tower forcing can be used to collapse successors of many singular cardinals below δ. The situation between $0^{\P}$ and a Woodin cardinal is still under investigation.

The use of $0^{\P}$ as the dividing line is an oversimplification: it is possible to use tricks to push some of the results somewhat further. More importantly, there are some suggestions that it is the presence of actual overlapping extenders in K which cause the difficulty, not partial extenders such as those which appear in the countable mouse $0^{\P}$. Schimmerling and Woodin have shown that in certain special cases the core model can be proved to have the full covering property, even though it contains inner models with several Woodin cardinals. See [39], where Schimmerling and Woodin show that this result is not limited to the Steel core models, but has consequences for the existence of core-like models.

This concludes our summary. The next subsection contains a more detailed discussion of the covering lemma for sequences of measures.

4.1. The core model for sequences of measures

See chapter $\boxed{\text{introduction}}$ for a discussion of the inner models for sequences of measures, and section 3.3 of this chapter on the Dodd-Jensen core model for its discussion of the core model K in particular. Recall that $K = L[\mathcal{U}]$, where $\mathcal{U}$ is a coherent sequence with members $\mathcal{U}_\gamma$ which are $L_\gamma[\mathcal{U}{\restriction}\gamma]$-measures. We will define the sequence $\mathcal{U}$, and hence the core model, by recursion on γ. The main problem in designing this recursion is to ensure that the final model $L[\mathcal{U}]$ is iterable: when $\mathcal{U}{\restriction}\gamma$ has been defined, then the decision whether to set $\mathcal{U}_\gamma = U$ for some measure U must take into account the requirement that any iterated ultrapower of the as yet undefined model $L[\mathcal{U}]$ must be well founded. This is accomplished by defining two core models: the first, the countably complete core model K^{c}, has the weak covering property and is iterable because its measures are countably complete; the second, the true core model K, has the full covering property and is iterable because it is an elementary substructure of K^{c}.

4.1 Definition. Either the core model K, or the countably complete core model K^{c}, are defined as $L[\mathcal{U}]$ where the sequence $\mathcal{U}$ is defined by recursion on γ as follows. Assume that $\mathcal{U}{\restriction}\gamma$ has already been defined:

1. If there is a mouse $M = L_{\gamma'}[\mathcal{U}']$ with $\mathcal{U}'{\restriction}\gamma = \mathcal{U}{\restriction}\gamma$ such that M has projectum less than γ and no full measures other than those already in $\mathcal{U}$, then set $\mathcal{U}{\restriction}\gamma' = \mathcal{U}'$.
2. If there is no mouse as in clause 1, and if $L_\gamma[\mathcal{U}] \models \gamma = \kappa^{++}$ for some $\kappa < \gamma$ such that there is a $L_\gamma[\mathcal{U}]$-ultrafilter U on κ with $i^U(\mathcal{U}){\restriction}\gamma+1 = \mathcal{U}{\restriction}\gamma$, then set $\mathcal{U}_\gamma = U$, provided it satisfies an iterability condition depending on which model is being constructed:
    (a) For the model K^{c}, the ultrafilter U is added to the sequence only if U is countably complete and $\mathrm{cf}(\mathrm{crit}(U)) = \omega_1$.
    (b) For the true core model K, the ultrafilter U is added to the sequence only if $\mathrm{ult}(L[\mathcal{W}], U)$ is well founded for every iterable inner model $L[\mathcal{W}]$ such that $\mathcal{W}{\restriction}\gamma = \mathcal{U}$.
3. If neither of the previous clauses apply then $\mathcal{U}_\gamma = \varnothing$.

The construction in clause 1 apparently depends on the choice of the mouse M to be added; however if $L_{\alpha_0}[\mathcal{W}_0]$ and $L_{\alpha_1}[\mathcal{W}_1]$ are any two such possible choices then one is an initial segment of the other. Thus clause 1 can be equivalently restated by specifying that $\mathcal{U}{\restriction}\gamma$ is to be extended to the longest good sequence $\mathcal{U}' \supseteq \mathcal{U}$ such that $L[\mathcal{U}']$ is iterable, the largest cardinal in $L_{\text{len}(\mathcal{U}')}[\mathcal{U}']$ is smaller than γ, and there are no full measures in $\mathcal{U}' - \mathcal{U}$.

Notice that it is only the measures added by clause 2 which are problematic with respect to iterability. The filters added to the sequence by clause 1 are not full measures except inside the mouse M, and hence never enter into an iterated ultrapower unless it drops to M. In this case the remaining portion of the iterated ultrapower is an iterated ultrapower of M, which is iterable by definition.

In the case of K^{c}, the countable completeness of the full measures added in clause 2 insures that the final model is iterable. It can be shown, either by the use of *bicepheli* or with the help of the covering lemma, that there is never more than one measure which satisfies the requirements to be chosen as $\mathcal{U}_\gamma$.

In order to show that K is iterable, we give an alternate definition of K as an elementary substructure of K^{c}. This definition, which depends on the weak covering lemma for K^{c}, begins with the following preliminary definitions, which generalize the fact that the weak covering lemma for K^{DJ} implies any that elementary substructure $X \prec K^{DJ}$ with $|X \cap \lambda^+| = \lambda^+$ is isomorphic to K^{DJ}.

4.2 Definition. An iterable potential premouse $M = L[\mathcal{U}]$ is said to be *universal* if, whenever M is compared with any other iterable potential premouse M', the iterated ultrapower on M' does not drop and the final model in that iteration is a (possibly proper) initial segment of the final model of the iteration on M.

Note that it follows that M must be a proper class, since if $M = L_\alpha[\mathcal{U}]$ is an iterable potential premouse which is a set then lemma 3.25 implies that M is the iterated ultrapower of a mouse $M' = L_{\alpha'}[\mathcal{U}']$. Thus $L[\mathcal{U}']$ is an iterable potential premouse, and M comes out shorter than $L[\mathcal{U}']$ when they are compared because the iteration on $L[\mathcal{U}']$ consists of an initial drop to the mouse M', followed by the iterated ultrapower of M' to M.

4.3 Proposition. *If M is a iterable class premouse and ${\lambda^+}^M = \lambda^+$ for a stationary class of cardinals then M is universal.*

Proof. If M is not universal then there is an iterable potential premouse M' and iterated ultrapowers $i\colon M \to P$ and $i'\colon M' \to P'$ such that P is a proper initial segment of P'. Thus the class of ordinals of M is the class Ω of actual ordinals, and since M' is longer, we have $\Omega \in M'$. The iteration i does not drop by lemma 3.29. We can assume that j also does not drop, since if it did drop we could use only the tail of the iteration starting after the last drop. If ν_0 and x are chosen so that $\Omega = i'_{\nu_0,\Omega}(\kappa)$ then the class Γ of ordinals λ such that $\lambda = i_{0,\lambda}(\lambda) = i'_{\nu_0,\lambda}(\kappa)$ is closed and unbounded. For any $\lambda \in \Gamma$, we have $\lambda^{+M} \leq \lambda^{+M_\lambda} = \lambda^{+M'_\lambda} = i'_{\nu_0,\lambda}(\kappa^{+M'_{\nu_0}}) < \lambda^+$. This contradicts the hypothesis of the theorem, which implies that there is $\lambda \in \Gamma$ with $\lambda^{+M} = \lambda^+$. ⊣

We will prove the following lemma later, simultaneously with the full covering lemma:

4.4 Lemma (Weak Covering Lemma for K^c). *Suppose that there is no inner model of $\exists\kappa\, o(\kappa) = \kappa^{++}$ and that λ is a singular strong limit cardinal. Then $\lambda^+ = (\lambda^+)^{K^c}$.*

4.5 Definition. A class Γ is *thick* if Γ is a proper class, there is some τ such that Γ contains its limit points of cofinality greater than τ, and $|\Gamma \cap \lambda^+| = \lambda^+$ for all sufficiently large singular strong limit cardinals $\lambda \in \Gamma$.

4.6 Lemma. *Assume that K^c satisfies the weak covering lemma, that λ is a limit cardinal and Γ is thick. Then for each $x \in \mathcal{P}^{K^c}(\lambda)$ there is $y \in \mathcal{H}^{K^c}(\Gamma \cup \lambda)$ with $x = y \cap \lambda$.*

Proof. Let $\pi\colon N \cong \mathcal{H}^{K^c}(\Gamma \cup \lambda + 1)$ be the transitive collapse. Then $\xi^{+N} \geq |\lambda^+| = \lambda^+$ for all singular limit cardinals of sufficiently large cofinality, so N is universal by lemma 4.3. It follows that the comparison of N with K^c will result in iterated ultrapowers with no drops and a common final model P. Furthermore, the critical point of each of the associated embeddings $i\colon N \to P$ and $j\colon K^c \to P$ is at least λ, since N and K^c agree on all measures with critical point below λ. Thus $x \in N$, and the set $y = \pi(x)$ satisfies the requirements. ⊣

4.7 Lemma. *If K^c satisfies lemma 4.4, Γ is thick and $\lambda > \omega_1$ is a cardinal of cofinality ω_1 then $\lambda \in \mathcal{H}^{K^c}(\lambda \cup \Gamma)$.*

Proof. Let $X = \mathcal{H}^{K^c}(\lambda \cup \Gamma)$, and suppose to the contrary that $\lambda \notin X$. Let $\pi\colon N \cong X$ be the transitive collapse, and let $U = \{\, x \subseteq \lambda : \lambda \in i(x) \,\}$. Then U is a K^c-ultrafilter, since $\mathcal{P}^{K^c}(\lambda) \subseteq N$. Furthermore U is countably complete since $\mathrm{cf}(\lambda) > \omega$ and $\mathrm{cf}(\lambda^{+K^c}) = \mathrm{cf}(\lambda^+) = \lambda^+ > \omega$.

Thus $U \in K^c$, but this is impossible since $o(U) = o^{K^c}(\lambda)$. The contradiction shows that $\lambda \in X$. ⊣

4.8 Lemma. *If K^c satisfies lemma 4.4 then K is isomorphic to the class $X = \bigcap \{\, \mathcal{H}^{K^c}(\Gamma) : \Gamma$ is thick$\,\}$.*

Sketch of proof. Let $\pi\colon \widetilde{K} \cong X$ be the transitive collapse of X. First we show that X is a proper class. Suppose to the contrary that X is a set and let $\kappa = \sup X$. Now let $(\, \Gamma_\nu : \nu < \theta \,)$ be a descending sequence of thick classes such that $X = \mathcal{H}^{K^c}(\Gamma_0) \cap V_{\kappa+1}$, the sequence of ordinals $\kappa_\nu = \inf(\mathcal{H}^{K^c}(\Gamma_\nu) - \kappa)$ is strictly increasing, and $\lambda = \sup\{\, \kappa_\nu : \nu < \theta \,\}$ is a strong limit cardinal of cofinality ω_1. We will show that $\lambda \in X$, contradicting the choice of κ.

Suppose to the contrary and pick $\Gamma_\theta \subseteq \bigcap_{\nu<\theta} \Gamma_\nu$ so that $\lambda \notin \mathcal{H}^{K^c}(\Gamma_\theta)$. By lemma 4.7 there is a parameter $a \in \Gamma_\theta$, an ordinal $\xi < \lambda$, and a formula φ such that λ is the unique η such that $K^c \models \varphi(a, \xi, \eta)$. Let $\tau \in K$ be the Skolem function for φ (with parameter a) so that $K^c \models \forall\iota\, (\exists\eta\, \varphi(a, \iota, \eta) \implies \varphi(a, \iota, \tau(\iota))$.

Now, notice that if $\nu < \theta$ then $\tau``\kappa_\nu \cap \lambda \subseteq \kappa_\nu$: otherwise there is some $\nu' > \nu$ and $\xi' < \kappa_\nu$ so that $\kappa_\nu \leq \tau(\xi') < \kappa_{\nu'}$. Then the least such ξ' is definable from κ_ν, $\kappa_{\nu'}$ and a, so $\xi' \in \mathcal{H}^{K^c}(\Gamma_\nu) \cap \kappa_\nu = X \cap \kappa_\nu$, but this is impossible since in that case $\tau(\xi') \in \mathcal{H}^{K^c}(\Gamma_{\nu'})$.

It follows that $\lambda = \min(\tau``\kappa_\nu - \kappa_\nu) \in \mathcal{H}^{K^c}(\Gamma_\nu)$. Now let ξ_0 be the least ordinal ξ such that $\lambda = \tau(\xi)$. Then $\xi_0 \in \mathcal{H}^{K^c}(\Gamma_\nu)$ for all $\nu < \theta$, and hence $\xi_0 \in X$. Then $\lambda = \tau(\xi_0) \in \mathcal{H}^{K^c}(\Gamma_\theta)$, contrary to the choice of Γ_θ.

Now we show that $\widetilde{K} = K$. Else, let κ be the least ordinal such that there is a measure U on κ in K which is not in $\widetilde{K}$ and let Γ be a thick class such that $\mathcal{H}^{K^c}(\Gamma) \cap V_{\kappa+2} = X \cap V_{\kappa+2}$. Now consider the following diagram:

$$\begin{tikzcd} \widetilde{K} \arrow[r, "i^U"] \arrow[dr, "j"'] & \mathrm{ult}(\widetilde{K}, U) \arrow[d, "k"] \\ & \overline{N} \end{tikzcd} \tag{I.13}$$

where $\overline{N}$ is the common final model of the iterated ultrapowers coming from the comparison of the universal models $\widetilde{K}$ and $\mathrm{ult}(\widetilde{K}, U)$, and j and k are the embeddings from these iterated ultrapowers. Let Γ' be the set of $x \in \Gamma$ such that $ki^U\pi^{-1}(x) = j\pi^{-1}(x)$. Then Γ' is thick, but $\pi(\kappa) \notin \mathcal{H}^{K^c}(\Gamma')$, contradicting the assumption that $\pi(\kappa) \in X$.

$\dashv$

4.9 Theorem. *If K^c satisfies the weak covering lemma then any universal iterable potential premouse M is an iterated ultrapower of K.*

The hypothesis that M is universal cannot be eliminated from this theorem: The model N constructed in the proof of Jensen's theorem 3.35 provides a counterexample in which N is a set, and a similar argument, starting with the assumption that there is a measurable cardinal κ in K such that $(K_{\kappa^+})^\sharp$ exists, gives a counterexample in which N is a proper class

Sketch of proof. Since both K and M are universal, the comparison of these two models gives iterated ultrapowers with no drops and with embeddings $i\colon K \to P$ and $j\colon M \to P$ into a common model P. We need to show that j is the identity, so that $P = M$. Suppose the contrary, and let ν be least such that $j_{\nu,\nu+1}$ is not the identity. Then $M_{\nu+1} = \mathrm{ult}(M_\nu, U) = \mathrm{ult}(M, U)$ for some measure U in M. The models M and N_ν agree up to $\kappa = \mathrm{crit}(U)$, so that U is a N_ν-ultrafilter (or else i would drop). Furthermore, $\mathrm{ult}(N_\nu, U)$ is well founded since otherwise $\mathrm{ult}(P, U)$ would be illfounded, which it is not since P is an iterated ultrapower of M. Now compare N_ν and $\mathrm{ult}(N_\nu, U)$, letting $s\colon N_\nu \to Q$ and $t\colon \mathrm{ult}(N_\nu, U) \to Q$ be the resulting embeddings, and let $\Gamma = \left\{ \eta : ti^U i_\nu(\eta) = si_\nu(\eta) \right\}$.

$$
\begin{tikzcd}
K \arrow[rr, "i_\nu"] & & N_\nu \arrow[r, "i^U"] \arrow[d, "i_{\nu,\theta}"'] \arrow[dr, "s"] & \mathrm{ult}(N_\nu, U) \arrow[d, "t"] \\
M \arrow[r, equal, "j_\nu"] & M_\nu \arrow[r, "j_{\nu,\theta}"'] & P & Q
\end{tikzcd}
$$

Then Γ is thick, and hence $K = \mathcal{H}^K(\Gamma)$. It follows that $N_\nu = \mathcal{H}^{N_\nu}(\Gamma \cup \kappa)$. Thus $ti^U(\kappa) = s(\kappa)$ and $ti^U{\restriction}\mathcal{P}(\kappa) = s{\restriction}\mathcal{P}(\kappa)$. Since $i^U(\kappa) > \kappa$ it follows that $s(\kappa) > \kappa$, so that the iteration s begins with an ultrapower by some measure $U' = \mathcal{U}^{N_\nu}_\gamma \in N_\nu$ with critical point κ. But then $U = U'$, since $x \in U \iff \kappa \in ti^U(x) = s(x) \iff x \in U'$. This contradicts the assumption that $U \notin N_\nu$. $\dashv$

4.2. The covering lemma up to $o(\kappa) = \kappa^{++}$

We use the following setting for the covering lemma for sequences of measures: We take κ to be a cardinal of K which is singular in V, and we consider covering sets $X \prec_1 K_{\breve\kappa}$ where $\breve\kappa$ is a cardinal of K with $\breve\kappa \geq \max(\kappa, o(\kappa))$. The covering lemma asserts that for suitable sets X there is a system $\mathcal{C}^X$ of indiscernibles, a function h in K, and a $\rho < \kappa$ such that $X = h``(X \cap \rho; \mathcal{C})$, the smallest set containing $X \cap \rho$ and closed under h and $\mathcal{C}$. We will call such a set X a *precovering set*.

Some definitions are required before we can give a precise statement of the covering lemma. Here is a general definition for a system of indiscernibles:

4.10 Definition. 1. If U is a measure then $\mathrm{crit}(U)$ is the cardinal κ such that U is a measure on κ.

2. If $\gamma < \gamma'$, with $\gamma' \in \mathrm{dom}(\mathcal{U})$, then $\mathrm{Coh}_{\gamma,\gamma'}$ is the least function f in the ordering of $L[\mathcal{U}]$ such that $\gamma = [f]_{\mathcal{U}_{\gamma'}}$ in $\mathrm{ult}(L[\mathcal{U}], \mathcal{U}_{\gamma'})$.

4.11 Definition. If $\mathcal{U}$ is a sequence of measures then a *system of indiscernibles* for $M = L[\mathcal{U}]$ is a function $\mathcal{C}$ such that

1. $\mathrm{dom}(\mathcal{C}) \subseteq \mathrm{dom}(\mathcal{U})$, and $\mathcal{C}_\gamma \subseteq \mathrm{crit}(\mathcal{U}_\gamma)$ for all $\gamma \in \mathrm{dom}(\mathcal{C})$.
2. For any function $f \in M$ there is a finite set a of ordinals such that the following conditions hold for any $\nu \in \mathcal{C}_\gamma$ and any set $x \in f``[\nu \cup \{\mathrm{crit}(\mathcal{U}_\gamma)\}]$
   (a) If $\nu > \sup(a \cap \mathrm{crit}(\mathcal{U}_\gamma))$ then $\nu \in x \iff \big(x \cap \mathrm{crit}(\mathcal{U}_\gamma)\big) \in \mathcal{U}_\gamma$.
   (b) If $\nu < \nu' \in \mathcal{C}(\gamma)$ and $a \cap [\nu, \nu'] = \varnothing$ then $\nu \in x \iff \nu' \in x$.

The indiscernible sequences rising from the covering lemma have some additional structure:

4.12 Definition. If $\mathcal{C}$ is a system of indiscernibles for M then $\mathcal{C}$ is said to be *h-coherent* if the following conditions hold:

1. For all $\nu \in \bigcup_\gamma \mathcal{C}_\gamma$ there is unique $\gamma \in h``(\nu)$ such that $\nu \in \mathcal{C}_\gamma$.
2. Suppose that $\nu \in \mathcal{C}_\gamma \cap \mathcal{C}_{\gamma'}$ and $\gamma \in h``\nu$. If $\gamma' \neq \gamma$ then $\mathrm{crit}(\mathcal{U}_\gamma) < \mathrm{crit}(\mathcal{U}_{\gamma'})$, and $\mathrm{crit}(\mathcal{U}_{\gamma'}) \in \mathcal{C}_{\gamma''}$ for some $\gamma'' < \gamma$ with $\mathrm{crit}(\mathcal{U}_{\gamma''}) = \mathrm{crit}(\mathcal{U}_\gamma)$.
3. Suppose $\gamma_\nu = \mathrm{Coh}_{\gamma',\gamma}(\nu)$, where $\gamma' < \gamma$ with $\mathrm{crit}(\mathcal{U}_{\gamma'}) = \mathrm{crit}(\mathcal{U}_\gamma)$; and suppose that $\nu \in \mathcal{C}_\gamma$ with $\gamma' \in h``\nu$. Then $\mathcal{C}_{\gamma_\nu} = \mathcal{C}_{\gamma'} \cap (\nu - \nu')$, where $\nu'$ is least such that $\gamma \in h``\nu'$.

The function $h(x;\mathcal{C})$ provides a weak sense in which a precovering set X is generated by a function $h \in K$ and a sequence $\mathcal{C}$ of indiscernibles:

4.13 Definition. Suppose that $\mathcal{C}$ is a system of indiscernibles and x is a set. Then $h``(x;\mathcal{C})$ is the smallest set $X$ such that $x \subseteq X$ and $X = h``(X \cup \bigcup_{\gamma \in X} \mathcal{C}_\gamma)$.

Definition 4.13 is too weak, since it does not provide any bounds on the size of the sets $\mathcal{C}_\gamma$. The functions defined below are used in clause (4) of theorem 4.15 to describe a stronger sense in which X is generated by $\mathcal{C}$:

4.14 Definition. If $\mathcal{C}$ is a g-coherent system of indiscernibles, and X is a set then we define

1. $s^{\mathcal{C}}(\gamma, \xi)$ is the least member of $\mathcal{C}_\gamma - (\xi + 1)$.
2. $s^{\mathcal{C}}_*(\gamma, \xi)$ is the least member of $\bigcup_{\gamma' \geq \gamma} \mathcal{C}_{\gamma'} - (\xi + 1)$.
3. An ordinal ξ is an *accumulation point* of $\mathcal{C}$ in X for γ if $\gamma, \xi \in X$, and $\bigcup_{\gamma'' \geq \gamma'} \mathcal{C}_{\gamma''}$ is unbounded in $X \cap \xi$ for all $\gamma' < \gamma$ in $X \cap g``\nu$.
4. $a^{\mathcal{C},X}(\gamma, \xi)$ is the least accumulation point of $\mathcal{C}$ in X for γ above ξ.

This definition of an accumulation point does not seem to be entirely satisfactory, since it depends on the set X and the function g as well as on the system $\mathcal{C}$; however Clause 5 of theorem 4.15 gives a sense in which the functions $s^{\mathcal{C}}$ and $a^{\mathcal{C},X}$ are, up to finite differences, independent of g, X and $\mathcal{C}$.

4.15 Theorem (Covering for sequences of measures). *Assume there is no model of $\exists\kappa\, o(\kappa) = \kappa^{++}$. Let κ be a cardinal of the core model K, and let $\breve{\kappa}$ be a cardinal of K such that $\breve{\kappa} \geq \max(\kappa, o(\kappa))$. Finally, let X be a set such that $\kappa \not\subseteq X = Y \cap K_{\breve{\kappa}}$ for some set Y such that $Y \prec_1 H_{\breve{\kappa}}$ and ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$.*

Then there is an ordinal $\rho < \kappa$, a function $h \in K$, and a function $\mathcal{C}$ such that

1. *$\mathcal{C}$ is an h-coherent system of indiscernibles for K.*
2. *$\mathrm{dom}(\mathcal{C}) \subseteq X$ and $\bigcup_\gamma \mathcal{C}_\gamma \subseteq X$.*
3. *$X = h``(X \cap \rho; \mathcal{C})$, and hence $X \subseteq h``(\rho; \mathcal{C})$.*
4. *For all $\nu \in X \cap \kappa$, either $\nu \in h``(X \cap \nu)$ or else there is γ such that $\nu \in \mathcal{C}_\gamma$. In the latter case there is $\xi \in X \cap \nu$ such that either*
   (a) $\nu = s^{\mathcal{C}}(\gamma, \xi) = s^{\mathcal{C}}_*(\gamma, \xi)$, or else
   (b) $\nu = a^{\mathcal{C},X}(\gamma', \xi)$ for some $\gamma' > \gamma$.

   Furthermore, clause (a) holds if ν is a limit point of X.
5. *If X' is another set satisfying the hypothesis of the theorem, then there is a finite set a of ordinals such that for any $\xi, \gamma \in X \cap X'$ with $[\xi, \gamma) \cap a = \varnothing$ we have*

$$s^{\mathcal{C}}(\gamma,\xi) = s^{\mathcal{C}'}(\gamma,\xi)$$
$$s^{\mathcal{C}}_*(\gamma,\xi) = s^{\mathcal{C}'}_*(\gamma,\xi)$$
$$a^{\mathcal{C},X}(\gamma,\xi) = a^{\mathcal{C}',X'}(\gamma,\xi)$$

   whenever either is defined.

In order to compare theorem 4.15 with the Dodd-Jensen covering lemma, notice that if $K = L[U]$ then $\mathcal{C}$ contains only a single set C of indiscernibles for the unique measure U. Then clause 4 asserts that $\mathrm{otp}(C) \leq \omega$, and clause 5 asserts that C is maximal.

4.16 Remark. As with the Dodd-Jensen covering lemma, the hypothesis ${}^{\mathrm{cf}(\kappa)}X \subseteq X$ can be weakened: If $\mathrm{cf}(\kappa) < \delta < \kappa$ and δ is the successor of a regular cardinal then there is an unbounded class $\mathbf{C} \subseteq \mathcal{P}_\delta(K_{\breve{\kappa}})$ of sets X satisfying the conclusion of theorem 4.15 such that if $\vec{X}$ is an increasing chain of members of $\mathbf{C}$ such that $\mathrm{cf}(\kappa) < \mathrm{cf}(\mathrm{len}(\vec{X})) < \kappa$ then $\bigcup \vec{X} \in \mathbf{C}$.

4.17 Remark. The assumption that ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ is used to ensure that the measures on κ generated by $\mathcal{C}$ are members of X. As was pointed out in observation 6, this assumption can be weakened to ${}^{\omega}Y \subseteq Y$ if $o(\kappa) < \kappa^{+}$.

Similarly, if $o(\kappa) < \kappa^{+}$ then remark 4.16 can be improved to state that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality.

4.18 Remark. If every measurable limit point of X is a member of X then the condition $\rho < \kappa$ can be strengthened to $\rho = \inf(\kappa - X)$, so that $X = h``(\rho; \mathcal{C})$. In particular, $\rho = \inf(\kappa - X)$ whenever $o(\alpha) < \inf(\kappa - X)$ for all $\alpha < \kappa$.

Introduction to the proof

Before beginning to sketch the proof of the covering lemma we pause to look at three complications and digressions:

1. It was pointed out earlier that in order to ensure that $\mathrm{dom}(\mathcal{C}) \subseteq X$ we are assuming that $X \prec_1 K_{\check{\kappa}}$, rather than $X \prec_1 K_{\kappa}$ as in the last section. This change, however, appears only in the very last step of the proof: until then we work only with $X \cap K_{\kappa}$ and use arguments which closely parallel those of the Dodd-Jensen covering lemma.

Similarly, this final step is the only place where the closure condition ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$ is used: up until then countable closure, ${}^{\omega}Y \subseteq Y$, is all that is needed.

2. The proof we give is for the $\mathrm{cf}(\kappa)$-closed case, with ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$, as in the statement of lemma 4.15. With one exception, the extension of the proof to the stronger result of remark 4.16 is relatively straightforward, using the ideas outlined in the proof of the Dodd-Jensen covering lemma. The exception is Lemma 4.21, and we will digress from the main line of the proof to state lemma 4.24, the analogue of lemma 4.21 for the unclosed case, and to sketch its proof. The reader may, if desired, skip this digression.

3. As was explained in section 4.1 an essential complication arises from the special role which the weak covering lemma plays in the definition of the core model. Beyond $0^{\dagger}$, the core model K is constructed in two stages: The first stage constructs the *countably complete* core model K^{c}, for which iterability is guaranteed (below $0^{\P}$) by the fact that every full measure of K^{c} is countably complete in V. After lemma 4.4 is proved for K^{c}, the true model K is shown to be an elementary substructure of K^{c}, so that the iterability of K^{c} implies the iterability of K.

Part one of the proof

Here we give part one of the proof of the covering lemma for the true core model K. At the end of this subsection we will show how to adapt this proof to prove the weak covering lemma 4.4 for K^{c}. As in the proof of the Dodd-Jensen covering lemma we begin with the following diagram:

$$
\begin{tikzcd}
M_0 \arrow[r, squiggly, "i_{0,\theta}"] & M_\theta \supseteq M \arrow[r, "\tilde{\pi}"] & \widetilde{M} = \mathrm{ult}(M, \pi, \kappa) \\
& \overline{K} \arrow[r, "\pi"] \arrow[u, hook] & X \cap K_\kappa \prec_1 K_\kappa \arrow[u, hook]
\end{tikzcd} \tag{I.14}
$$

The construction of this diagram is identical to the construction for the Dodd-Jensen covering lemma: M_θ is obtained as the last model of the iterated ultrapower $i_{0,\theta}$ of K arising from the comparison of $M_0 = K$ with the transitive collapse $\overline{K}_{\bar{\kappa}}$ of $X \cap K_\kappa$; and $\widetilde{M} = \mathrm{ult}_n(M, \pi, \kappa)$ where M is the largest initial segment of M_θ, and n is the largest integer, such that the ultrapower is defined.

The proof of the analogue of lemma 3.15, which states that the construction of diagram (I.14) succeeds, is the same as for the Dodd-Jensen covering lemma except for the final clause:

4.19 Claim. *Either $\theta = 0$ and M is a proper initial segment of $M_0 = K$, or else $1 \in D$, that is, the iteration drops immediately.*

Proof. Set $\eta = \mathrm{crit}(\pi)$. Then, as in the proof of lemma 3.15, $\mathcal{P}(\eta) \cap K \not\subseteq \overline{K}$ since otherwise the filter $U = \{\, x : \eta \in \pi(x) \,\}$ would be a K-ultrafilter, and hence a member of K. This is impossible because the embedding π shows that $U_\gamma \lhd U$ for every measure U_γ on η in K.

Now let ρ be least such that $\mathcal{P}(\rho) \cap K \not\subseteq \overline{K}$. Then $\rho \leq \eta$ and, again as in the proof of lemma 3.15, claim 4.19 follows if we can show that there is no measure $U = \mathcal{U}_\gamma$ in $K - \overline{K}$ with $\rho > \tau = \mathrm{crit}(U)$. If there is such a measure then evidentially $\eta > \tau^{+\overline{K}}$, since otherwise $\tau^{+K} > \tau^{+\overline{K}}$, which contradicts the assumption that $\mathcal{P}^K(\tau) \subseteq \overline{K}$. Then $\eta \geq \tau^{++\overline{K}}$, and since $K \models o(\tau) < \tau^{++}$ it follows that $o^{\overline{K}}(\tau) = o^K(\tau) \in \overline{K}$, which contradicts the assumption that $U = \mathcal{U}_\gamma \notin \overline{K}$. $\dashv$

This completes part one of the proof of the covering lemma for K, and we are now ready to prove the weak covering lemma 4.4 for K^{c}:

Proof of lemma 4.4. The proof is similar to the proof of the weak covering lemma for K^{DJ}. Suppose to the contrary that λ is a singular cardinal with $\mu^\omega < \lambda$ for all $\mu < \lambda$, and that $\kappa = \lambda^{+K^{\mathrm{c}}} < \lambda^+$. Then there is a set $X \prec_1 K_\kappa$, cofinal in κ, such that $\lambda \not\subseteq X$, ${}^\omega X \subseteq X$ and if $\eta = \min(\lambda - X)$ then $\mathrm{cf}(\eta) = \omega_1$. The final condition is obtained by constructing X as the union of an increasing chain of sets of length ω_1.

Now apply the construction above of part one of the proof to the set X, using K^{c} for K. The constraint $\mathrm{cf}(\eta) = \omega_1$ is needed to ensure that the measure U of the first paragraph of the proof of claim 4.19 would be in K^{c} if it existed.

Now, as in the proof of the weak covering lemma for K^{DJ}, the fact that there are no measurable cardinals of K^{c} in the interval $(\lambda, \kappa]$ implies that κ is singular in K^{c}, contradicting the fact that $\kappa = \lambda^{+K^{\mathrm{c}}}$. ⊣

We now turn to the main subject of this section, the analysis of indiscernibles which will complete the proof of the full covering lemma.

Part two of the proof: Analyzing the indiscernibles.

As in the proof of the Dodd-Jensen lemma, the model $\widetilde{M} = \mathrm{ult}_n(M, \pi, \kappa)$ of diagram (I.14) is a mouse in K. It follows that $\widetilde{M}$ is an initial segment of K; that is, $\widetilde{M} = L_{\tilde{\alpha}}[\mathcal{U}\restriction\tilde{\alpha}]$ for some ordinal $\tilde{\alpha} < \kappa^+$.

Still following the proof of the Dodd-Jensen covering lemma, let ν_0 be the largest member of the set D of drops, and let $\bar{\rho} < \bar{\kappa}$ be the Σ_n-projectum of $M^*_{\nu_0}$ and hence of M_ν for every ordinal ν in the interval $\nu_0 < \nu \leq \theta$. The ordinal ρ required by lemma 4.15 must satisfy

$$\rho \geq \pi(\bar{\rho}). \tag{A}$$

In the proof of the Dodd-Jensen covering lemma we could set $\rho = \pi(\bar{\rho})$, but in the present proof there are several other things which can go wrong, and each of these will determine a separate lower bound for ρ. Rather than specifying ρ at this point we will, at various points during the course of the proof, specify a series (A)–(E) of lower bounds on ρ. At any point in the proof we will assume that ρ is an ordinal less than κ which satisfies all the lower bounds specified up to that point.

Define $\widetilde{\mathcal{C}}$ with $\mathrm{dom}(\widetilde{\mathcal{C}}) = \pi``\mathrm{dom}(\mathcal{C})$ by setting $\widetilde{\mathcal{C}}(\tilde{\pi}(\gamma)) = \pi``[\overline{\mathcal{C}}(\gamma)] - \rho$ for each $\gamma \in \mathrm{dom}(\mathcal{C})$. This is nearly the desired set of indiscernibles: it is an $\tilde{h}$-coherent system of indiscernibles for $\widetilde{M}$, and clearly $\widetilde{M} = \tilde{h}(X \cap \rho; \widetilde{\mathcal{C}})$ so that $X \cap K_\kappa = K_\kappa \cap \pi``h(\bar{\rho} \cap X; \mathcal{C}) = K_\kappa \cap \tilde{h}(X \cap \rho; \widetilde{\mathcal{C}})$.

However the system $\widetilde{\mathcal{C}}$ of indiscernibles for $\widetilde{M}$ is not a system of indiscernibles for K: $\mathcal{U}_\gamma$ is a full measure in $\widetilde{M}$ for every $\gamma \in \mathrm{dom}(\widetilde{\mathcal{C}})$, but if $\gamma > \kappa$ then $\mathcal{U}_\gamma$ is only a partial measure in K since $\mathrm{crit}(\mathcal{U}_\gamma) = \kappa < \mathrm{Ord}(M) < \kappa^{+K}$. We will show, however, that the system $\widetilde{\mathcal{C}}$ is a system of indiscernibles for K in the weaker sense that it generates full K-measures $\mathcal{U}^*_\gamma \supseteq \mathcal{U}_\gamma$. The measures $\mathcal{U}^*_\gamma$ are members of K, so we can define a function $\sigma\colon \mathrm{dom}(\widetilde{\mathcal{C}}) \to o(\kappa)$ by setting $\sigma(\gamma) = \gamma'$ where γ' is the least ordinal such that $\mathcal{U}^*_\gamma = \mathcal{U}_{\gamma'}$. Then the desired system $\mathcal{C}$ of indiscernibles for K is defined by setting $\mathcal{C}_{\sigma(\gamma)} = \widetilde{\mathcal{C}}_\gamma$. Finally, we will use the assumption ${}^{\mathrm{cf}(\kappa)}X \subseteq X$ to show that the range of σ is contained in X, and obtain the required function h by combining $\tilde{h}$ with a function g obtained by applying the covering lemma to $X \cap (\breve{\kappa} - \kappa)$.

Now we go to the first part of this proof, showing that $\widetilde{\mathcal{C}}$ is a system of indiscernibles for $\widetilde{M}$. Here and later we use the idea of an *indiscernible sequence*:

4.20 Definition. A sequence $\vec{\alpha} = (\,\alpha_n : n < \omega\,)$ is an *indiscernible sequence* for $\vec{\gamma} = (\,\gamma_n : n < \omega\,)$ in $\widetilde{\mathcal{C}}$ if $\vec{\alpha}$ is strictly increasing, $\alpha_n \in \widetilde{\mathcal{C}}(\gamma_n)$ for all but finitely many $n < \omega$, and either (i) $\sup_n \gamma_n = \sup_n \alpha_n$, or (ii) $\forall n \in \omega\ \mathrm{crit}(\mathcal{U}_{\gamma_n}) = \sup_n \alpha_n$.

The following lemma corresponds to the argument that C^X is a Prikry sequence in the proof of the Dodd-Jensen covering lemma.

4.21 Lemma. *If $\vec{\alpha}$ is an indiscernible sequence for $\vec{\gamma}$ in $\widetilde{\mathcal{C}}$ then for any function $f \in K$ there is $n_0 < \omega$ such that the following hold for all $n > n_0$ and all $\xi < \alpha_n$:*

1. *If $n < n' < \omega$ and $\gamma_n = \gamma_{n'}$ then $\alpha_n \in f(\xi) \iff \alpha_{n'} \in f(\xi)$.*
2. *If $\gamma_n < \kappa$ then $\alpha_n \in f(\xi)$ if and only if $f(\xi) \cap \mathrm{crit}(\mathcal{U}_{\gamma_n}) \in \mathcal{U}_{\gamma_n}$.*

Proof. Suppose to the contrary there are indiscernible sequences $\vec{\alpha}$ and $\vec{\gamma}$ for which the conclusion does not hold. The sequences $\vec{\alpha}$ and $\vec{\gamma}$ are subsets of $X = Y \cap K_{\check{\kappa}}$, and since ${}^{\omega}Y \subseteq Y$ it follows that $\vec{\alpha}, \vec{\gamma} \in Y$. By elementarity there is a counterexample f to the lemma which is a member of Y, but then $f \in \operatorname{ran} \tilde{\pi}$ by proposition 3.47. Thus $\tilde{\pi}^{-1}(f)$ is in M and contradicts the fact that $\overline{\mathcal{C}}$ is a system of indiscernibles for M. ⊣

4.22 Remark. Clause 1 of lemma 4.21 is only needed in the case that $\vec{\gamma}$ is essentially constant, or more precisely when there is $n_0 < \omega$ such that $\gamma_n \leq \gamma_{n_0}$ for all $n > n_0$, and $\gamma_n = \gamma_{n_0}$ for cofinally many $n < \omega$. Otherwise, if $\kappa < \gamma_n < \limsup_m \gamma_m$ for all sufficiently large $n < \omega$, then we use clause 2 as follows: By taking a cofinal subsequence if necessary, we can assume that $\kappa < \gamma_0$ and $\vec{\gamma}$ is strictly increasing. Then $\vec{\alpha}$ is an indiscernible sequence satisfying clause 2 for the sequence $\vec{\gamma}'$ defined by $\gamma'_n = \mathrm{Coh}_{\gamma_n, \gamma_{n+1}}(\alpha_{n+1})$, and $\vec{\gamma}'$ satisfies $\gamma'_n < \alpha_{n+1} < \kappa$ for each $n \in \omega$.

Clause 2 of Lemma 4.21 suggests a definition of *indiscernible sequence* which is independent of any particular system of indiscernibles:

4.23 Definition. We say that $\vec{\alpha}$ is an indiscernible sequence for $\vec{\gamma}$ if (i) $\vec{\alpha}$ is strictly increasing, (ii) either $\sup_n \alpha_n = \sup_n \gamma_n$ or else $\mathrm{crit}(\mathcal{U}_{\gamma_n}) = \sup_n \alpha_n$ for all n, and (iii) for any function $f \in K$ there is $n_0 < \omega$ such that for all $n > n_0$ and all $\xi < \alpha_n$

$$\alpha_n \in f(\xi) \text{ if and only if } f(\xi) \cap \mathrm{crit}(\mathcal{U}_{\gamma_n}) \in \mathcal{U}_{\gamma_n}.$$

This definition will be used to modify the domain of $\widetilde{\mathcal{C}}$ to obtain a system of indiscernibles $\mathcal{C}$ such that every sequence of indiscernibles in $\mathcal{C}$ is a sequence of indiscernibles in the sense of definition 4.23. First, however, we digress to look at the analog of lemma 4.21 for the case when X is not countably closed.

Digression for non-countably closed sets X. As mentioned on page 61, lemma 4.21 is the one point in the proof of theorem 4.15 where a new idea is needed, beyond those presented in the proof of the Dodd-Jensen covering lemma, in order to prove remark 4.16 which states the covering lemma without assuming that $X = Y \cap K_{\breve{\kappa}}$ for some $\mathrm{cf}(\kappa)$-closed set Y. Lemma 4.24 below substitutes for lemma 4.21 in this case. Lemma 4.24 and its proof may be skipped without affecting the proof of the covering lemma as stated in theorem 4.15.

4.24 Lemma. *Suppose that $\delta = \tau^+$ where τ is a uncountable regular cardinal, and let $\mathbf{C}$ be the set of $X \in \mathcal{P}_\delta(K_{\breve{\kappa}})$ which satisfy lemma 4.21. Then $\mathbf{C}$ is unbounded in $X \in \mathcal{P}_\delta(K_{\breve{\kappa}})$ and is closed under unions of increasing sequences of uncountable cofinality.*

Note that the requirement on δ is stronger than is needed for the corresponding results in the proof of the Dodd-Jensen lemma, for which δ could be any uncountable cardinal.

Proof sketch. The proof of the following analogue of lemma 3.50 is straightforward:

4.25 Lemma. *If $X_0 \subseteq X_1$, and $\vec{\alpha}$ and $\vec{\gamma}$ are sequences with range contained in X_0 such that $\vec{\alpha}$ is an indiscernible sequence for $\vec{\gamma}$ in $\mathcal{C}^{X_1}$, then $\vec{\alpha}$ is also an indiscernible sequence for $\vec{\gamma}$ in $\mathcal{C}^{X_0}$.* $\dashv$

As in the Dodd-Jensen covering lemma, it easily follows that $\mathbf{C}$ is closed under increasing unions of uncountable cofinality. Thus we only need to prove that $\mathbf{C}$ is unbounded.

Let S be the set of $\sigma \in \mathrm{Col}(\delta, K_\lambda)$ such that $\mathrm{cf}(\mathrm{dom}(\sigma)) = \tau$ and $\mathrm{ran}(\sigma)$ fails to satisfy lemma 4.21. As in the Dodd-Jensen covering lemma, we will finish the proof of lemma 4.24 by showing that S is nonstationary. Suppose towards a contradiction that S is stationary, and for each function $\sigma \in S$ let $\vec{\alpha}^\sigma$ and $\vec{\gamma}^\sigma$ be sequences which witness that lemma 4.24 fails for $X^\sigma = \mathrm{ran}(\sigma)$; that is, $\vec{\alpha}^\sigma$ is an indiscernible sequence for $\vec{\gamma}^\sigma$ in $\mathcal{C}^\sigma$, but $\vec{\alpha}$ and $\vec{\gamma}$ fail to satisfy one of clauses 1 or 2 of lemma 4.21. Now continue following the proof of lemma 3.52, which was the analog in the proof of the Dodd-Jensen covering lemma of lemma 4.24: Set $A^\sigma = \vec{\alpha}^\sigma \cup \{\, \gamma^\sigma_n : \gamma^\sigma_n < \kappa \,\}$ and find $\sigma_0 \in S$ and a stationary set $S_0 \subseteq S$ so that $\sigma \supseteq \sigma_0$ and $A^\sigma \subseteq \mathrm{ran}(\sigma_0)$ for all $\sigma \in S_0$.

Recall that the key point in the proof of lemma 3.52 was that, because $C^\sigma \subseteq^* C^{\sigma_0}$ for every $\sigma \in S_0$, each of the sets C^σ were determined (up to a finite set) by the subset $D^\sigma = C^{\sigma_0} - C^\sigma$ of C^{σ_0}. The key step in the current proof is to use Fodor's lemma and the hypothesis that $\delta = \tau^+$ to find a set Z which fills this role of C^{σ_0}. Towards this end, choose an function $k\colon \tau \cong \operatorname{dom}(\sigma_0) < \tau^+$. Since $\operatorname{cf}(\tau) > \omega$ there is, for each $\sigma \in S$, an ordinal $\xi^\sigma < \tau$ such that $A^\sigma \subseteq \sigma_0 k``(\xi^\sigma)$. By Fodor's lemma there is a stationary subset $S_0' \subseteq S_0$ such that $\xi^\sigma$ is constant, say $\xi^\sigma = \xi$ for each $\sigma \in S_0'$. Set $Z = (\sigma_0 \circ k)``\xi$, so that $|Z| < \tau$ and $A^\sigma \subseteq Z$ for every $\sigma \in S_0'$.

The rest of the proof parallels the proof of lemma 3.52 for the Dodd-Jensen lemma. First define, for each $\sigma \in S_0'$, a set $w(\sigma)$ which witnesses that the restriction of $\widetilde{\mathcal{C}}$ to Z is as large as possible. This set is obtained by modifying definition 3.16 as follows: Set the support β^σ of $w(\sigma)$ to be $\beta^\sigma = \max(\sup(Z), \rho^\sigma + 1) < \kappa$, and replace the requirement that $w(\sigma)$ be countable with the condition $|w(\sigma)| = |Z|$. Finally, modify Clause (3) of definition 3.16 to state that $\mathfrak{m}^\sigma = \operatorname{dir\,lim}(w(\sigma))$ is the Σ_n-code of a mouse of K, and there is a function $f = f^\sigma$ which is Σ_1-definable in $\mathfrak{m}^\sigma$ such that (i) for any $\alpha, \gamma \in Z$ such that $\gamma < \kappa$ and $\alpha \notin \widetilde{\mathcal{C}}^\sigma(\gamma)$, there is a set $x \in f``\alpha$ such that $x \in \mathcal{U}(\gamma)$ but $\alpha \notin x$, and (ii) for any $\alpha < \alpha'$ in $Z$ which are not members of the same set $\mathcal{C}^\sigma(\gamma)$, there is $x \in f``(\alpha \cap w(\sigma))$ such that $\alpha \in x$ and $\alpha' \notin x$.

Thus $w(\sigma)$ gives a complete description of the restriction of $\mathcal{C}^\sigma$ to Z.

Now, since $|w(\sigma)| = |Z| < \tau$ and $\operatorname{cf}(\operatorname{dom}(\sigma)) = \tau$ for every member σ of S_0', lemma 3.21 implies that there is $\sigma_1 \in S_0'$ and a stationary set $S_1 \subseteq S_0'$ such that if $\sigma \in S_1$ then $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$. By shrinking S_1 further, if necessary, we can ensure that $\beta^\sigma = \beta$ is constant for $\sigma \in S_1$.

Now since S_1 is unbounded and $\vec{\alpha}^{\sigma_1}$ is not an indiscernible sequence for $\vec{\gamma}^{\sigma_1}$, there is some $\sigma \in S_1$ with a function $f \in \operatorname{ran}(\sigma)$ which witnesses that $\vec{\alpha}^{\sigma_1}$ is not an indiscernible sequence for $\vec{\gamma}^{\sigma_1}$. It follows that $\vec{\alpha}^{\sigma_1}$ is not an indiscernible sequence in $\widetilde{\mathcal{C}}^\sigma$, and by the definition of $w(\sigma)$ it follows that there is a function f' which is Σ_1-definable in $\operatorname{dir\,lim}(w(\sigma))$ which witnesses this failure. Now $w(\sigma) \subseteq \operatorname{ran}(\sigma_1)$, and it follows that $\operatorname{dir\,lim}(w(\sigma)) \subseteq \widetilde{M}^{\sigma_1}$. To see this, notice that $\bar{\mathfrak{m}} = \operatorname{dir\,lim}((\pi^{\sigma_1})^{-1}(w(\sigma)) \subseteq M$, since every subset of $\bar{\rho} = (\pi^{\sigma_1})^{-1}(\rho^{\sigma_1})$ in $\mathfrak{m}$ is a member of M and there is a subset of $\bar{\rho}$ definable in M which is not a member of M.

Thus $\vec{\alpha}^{\sigma_1}$ is not an indiscernible sequence for $\vec{\gamma}^{\sigma_1}$ in $\widetilde{\mathcal{C}}^{\sigma_1}$. This contradicts the choice of $\vec{\alpha}^{\sigma_1}$ and $\vec{\gamma}^{\sigma_1}$, and hence completes the proof of lemma 4.24. $\dashv$

Continuation of the main proof. This completes the digression for non-countably closed precovering sets X, and we now return to the basic proof of the covering lemma. We want to look at the measures generated by the system $\mathcal{C}$ of indiscernibles, and in doing so it will be convenient to use the notation $\mathcal{U}(\alpha, \beta)$, which explicitly names the critical point of the measure, rather than the notation $\mathcal{U}_\gamma$. In doing so we will consistently adjust the notation described earlier by replacing γ with the pair (α,β): for example, we will write $s^{\mathcal{C}}(\alpha,\beta,\xi)$ instead of $s^{\mathcal{C}}(\gamma,\xi)$, and we will say that $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha},\vec{\beta})$ instead of $\vec{\gamma}$.

For $\beta < o^{\widetilde{M}}(\alpha)$ define $C^+_{\alpha,\beta} = \bigcup\left\{\,\widetilde{\mathcal{C}}(\alpha,\beta') : \beta \le \beta' < o^{\widetilde{M}}(\alpha)\,\right\}$. If $C^+_{\alpha,\beta}$ is cofinal in α then $\widetilde{\mathcal{C}}$ generates an ultrafilter $\mathcal{U}^*(\alpha,\beta)$ on α as follows:

4.26 Definition. Define the relation $\in_\beta$ as follows:

$$\nu \in_\beta x \iff \begin{cases} \nu \in x & \text{if } \nu \in \widetilde{\mathcal{C}}(\alpha,\beta) \\ x \cap \nu \in \mathcal{U}(\nu, \mathrm{Coh}_{\beta,\beta'}(\nu)) & \text{if } \nu \in C^+_{\alpha,\beta} - \widetilde{\mathcal{C}}(\alpha,\beta). \end{cases}$$

Then $\mathcal{U}^*(\alpha,\beta)$ is the set of $x \subseteq \alpha$ such that $\nu \in_\beta x$ for all sufficiently large $\nu \in C^+_{\alpha,\beta}$.

We are now ready to specify the second and third of the lower bounds on ρ:

$$\rho > \sup\left\{\, \alpha < \kappa : \exists\beta\ \big((\alpha,\beta) \in \mathrm{dom}(\mathcal{U}^*) \text{ and } \mathcal{U}(\alpha,\beta)\big) \ne \mathcal{U}^*(\alpha,\beta)\,\right\} \tag{B}$$

Condition (B) holds for all sufficiently large $\rho < \kappa$ since Lemma 4.21 implies that the equality on the right hand side of (B) holds for all but finitely many ordinals α.

$$\rho > \sup\bigcup\left\{\, \mathcal{C}(\kappa,\beta) : \mathcal{U}^*(\kappa,\beta) \text{ is not defined}\,\right\}. \tag{C}$$

To see that the right-hand side of condition (C) is smaller than κ, suppose to the contrary that there is a cofinal set $C \subseteq \kappa$ such that for each $\nu \in C$ there is β_ν such that $\nu \in \mathcal{C}(\kappa,\beta_\nu)$ and $\mathcal{U}^*(\kappa,\beta_\nu)$ is not defined. By taking a subsequence if necessary, we can assume that $(\,\beta_\nu : \nu \in C\,)$ is not decreasing, but this implies that $\mathcal{U}^*(\kappa,\beta_\nu)$ is defined for each $\nu \in C$.

If $\mathcal{U}^*(\kappa,\beta)$ is defined then the fact that X is countably closed implies that $\mathrm{ult}(K,\mathcal{U}^*(\kappa,\beta))$ is well founded, and hence $\mathcal{U}^*(\kappa,\beta) \in K$. Define a function τ by

$$\begin{aligned} \tau(\beta) &= \beta && \text{if } \beta < \kappa \\ \mathcal{U}(\kappa,\tau(\beta)) &= \mathcal{U}^*(\kappa,\beta) && \text{if } \beta > \kappa \text{ and } \mathcal{U}^*(\kappa,\beta) \text{ is defined.} \end{aligned}$$

Finally, define the desired system $\mathcal{C}$ of indiscernibles for K by setting $\mathcal{C}{\restriction}\kappa = \widetilde{\mathcal{C}}{\restriction}\kappa$, and $\mathcal{C}(\kappa,\tau(\beta)) = \widetilde{\mathcal{C}}(\kappa,\beta)$. We now make our single use of the assumption that X is $\mathrm{cf}(\kappa)$-closed, that is, that $X = Y \cap K_{\check{\kappa}}$ for some $Y \prec H_\lambda$ with ${}^{\mathrm{cf}(\kappa)}Y \subseteq Y$

4.27 Claim. *If $\mathcal{U}^*(\kappa,\beta)$ is defined then $\mathcal{U}^*(\kappa,\beta) \in X$, and hence $\tau(\beta) \in X$.*

Proof. First suppose that $\mathcal{C}(\kappa,\beta)$ is cofinal in κ, and let $C \subseteq \mathcal{C}(\kappa,\beta)$ be cofinal in κ with $\operatorname{otp}(C) = \operatorname{cf}(\kappa)$. Then $C \in Y$, so $\mathcal{U}(\kappa,\beta) \in Y$ and hence $\mathcal{U}(\kappa,\beta) \in X = Y \cap K_{\breve\kappa}$.

If $\mathcal{C}(\kappa,\beta)$ is not cofinal in κ then $C^{+}_{\kappa,\beta} - \mathcal{C}(\kappa,\beta)$ is cofinal in κ, since $\mathcal{U}^*(\kappa,\beta)$ is defined. In this case pick a cofinal set $C \subseteq C^{+}_{\kappa,\beta} - \mathcal{C}(\kappa,\beta)$ with order type $\operatorname{cf}(\kappa)$. For each $\nu \in C$ let β_ν be the ordinal such that $\nu \in \widetilde{\mathcal{C}}(\kappa,\beta_\nu)$. Then $\mathcal{U}(\kappa,\beta)$ is definable from C together with the sequence $\big(\operatorname{Coh}_{\beta,\beta_\nu}(\nu) : \nu \in C\,\big)$. Again, both of these sequences are subsets of $\kappa \cap Y$, and it follows they are members of Y, so that $\mathcal{U}^*(\kappa,\beta) \in Y$ and hence $\mathcal{U}^*(\kappa,\beta) \in X$. ⊣

4.28 Remark. If $o(\kappa) < \kappa^{+K}$ then the assumption that X is $\operatorname{cf}(\kappa)$-closed is unnecessary, for in that case there is a partition of κ into disjoint sets $(\,A_\beta : \beta < o(\kappa)\,)$ such that $A_\beta \in \mathcal{U}(\kappa,\beta)$ for each $\beta < o(\kappa)$. If $\vec{A} \in Y$ then there is $\eta < \kappa$ so that $\mathcal{C}(\kappa,\beta) - A_\beta \subseteq \eta$ for all $\beta < o(\kappa)$. If $\nu \in \mathcal{C}(\kappa,\beta) - \eta$ then β is definable from $\vec{A}$ as the unique ordinal β such that $\nu \in A_\beta$, so claim 4.27 holds for all $X = Y \cap K_{\breve\kappa}$ with $\vec{A} \in Y$.

It is easy to see from the construction that $\mathcal{C}$ is a sequence of indiscernibles for K. Thus $\mathcal{C}$ satisfies clauses 1 and 2 of theorem 4.15.

4.29 Claim. *There is a function $g \in K$ such that $X \subseteq g``(X \cap \kappa)$.*

Proof. Apply the proof of the covering lemma to the full set $X \prec_1 K_{\breve\kappa}$ (rather than to $X \cap K_\kappa$). Notice that, as in the proof of lemma 4.4, there are no measurable cardinals in the interval $(\kappa, \breve\kappa]$ and hence any indiscernibles which come up in the construction must be smaller than κ. It follows, just as in the proof of the covering lemma for L, that there is a function $g \in K$ such that $X = g``(X \cap \kappa)$. ⊣

We now put the fourth lower bound on ρ:

$$\rho > \sup\{\,\nu : \exists\beta\,(\nu \in \mathcal{C}(\kappa,\beta) \wedge \beta \notin g``(X \cap \nu))\,\}\,. \tag{D}$$

The following claim justifies this bound:

4.30 Claim. *There is an ordinal $\eta < \kappa$ such that $\beta \in g``(X \cap \nu)$ whenever $\eta < \nu \in \mathcal{C}(\kappa,\beta)$.*

Proof. For each pair $\xi, \xi' < \kappa$, define $x(\xi,\xi')$ to be the least set (in the order of construction of K) in $\mathcal{U}(\kappa, g(\xi)) - \mathcal{U}(\kappa, g(\xi'))$. Now define $t\colon \kappa \to \kappa$ by letting $t(\nu)$ be the least ordinal $\xi < \nu$ such that $\nu \in \bigcap_{\xi' < \nu} x(\xi,\xi')$, leaving $t(\nu)$ undefined if there is no such ordinal ξ. Notice that $t \in K$, and if $\beta \in \operatorname{ran}(g)$ and $\xi = \min(g^{-1}(\beta))$ then $\{\,\nu < \kappa : t(\nu) = \xi\,\} \in \mathcal{U}(\kappa,\beta)$.

Since $\mathcal{C}$ is a system of indiscernibles, there is an ordinal $\eta < \kappa$ so that for any β such that $(\kappa,\beta) \in \operatorname{dom}\mathcal{U}$ and any $\nu \in \mathcal{C}(\kappa,\beta) - \eta$ we have that

$t(\nu)$ is defined, and $\{\, \nu' : t(\nu') = t(\nu)\,\} \in \mathcal{U}(\kappa,\beta)$. It follows that for any $\nu \in \mathcal{C}(\kappa,\beta) - \eta$ we have $t(\nu) = \min(g^{-1}(\beta))$ so $gt(\nu) = \beta$.

Furthermore $\xi = t(\nu) \in X$. To see this, recall lemma 3.6 which asserts that g is a union of functions in X. Let $f \subseteq g$ be a member of X such that $f(\xi) = \beta$. Then $t(\nu) = \xi = \min(f^{-1}[\beta]) \in X$ since $\beta \in X$. $\dashv$

This completes the proof of the first three clauses of theorem 4.15. Towards the proof of clause 4, suppose that $\nu \in \mathcal{C}(\alpha,\beta_0)$ and let β_1 be the largest ordinal such that ν is an accumulation point in X for (α,β_1). Then $\bigcup_{\beta_1 \le \beta < o(\alpha)} \mathcal{C}(\alpha,\beta)$ is bounded in $\nu \cap X$, say by $\xi \in X \cap \nu$. If $\beta_0 \ge \beta_1$ then $\nu = s(\alpha,\beta_0,\xi) = s_*(\alpha,\beta_0,\xi)$, so we can suppose that $\beta_0 < \beta_1$. We claim that there are only finitely many accumulation points in X for (α,β_1) in the interval (ξ,ν), so that $\nu = a(\alpha,\beta_1,\xi')$ for some ξ' in $[\xi,\nu) \cap X$. If, to the contrary, there are infinitely many such accumulation points then let ν' be the least member of the interval $(\xi,\nu]$ which is a limit of accumulation points for (α,β_1). Then $\operatorname{cf}(\nu') = \omega$, so $\nu' \in X$ and any measure on ν' which can be generated by a countable sequence of members of $\bigcup_{\beta < o(\alpha)} \mathcal{C}(\alpha,\beta) \cap \nu'$ is a member of X. Since ν' is an accumulation point for (α,β_1), it follows that $\xi < \nu' \in \bigcup_{\beta_1 \le \beta < o(\alpha)} \mathcal{C}(\alpha,\beta)$, contradicting the choice of ξ.

This contradiction completes the proof of clause 4, except for the last sentence which states that $\nu = s(\alpha,\beta_0,\xi_0) = s_*(\alpha,\beta_0,\xi_0)$ whenever ν is a limit point of X. We will prove this here for the case $\operatorname{cf}(\alpha) = \omega$. The proof of the full statement will be defered until after the proof of clause 5, on which its proof depends.

4.31 Claim. *Clause 4 holds for all α with $\operatorname{cf}(\alpha) = \omega$.*

Proof. Suppose, to the contrary, that there is α with $\operatorname{cf}(\alpha) = \omega$ such that $\alpha \in \mathcal{C}(\lambda,\beta)$ for some ordinals $\lambda,\beta \in h^X``\alpha$, and $\alpha = \sup_n \nu_n$ where $\nu_n \in \mathcal{C}(\lambda,\beta_n)$ for some $\beta_n \ge \beta$. Define $U \subseteq \mathcal{P}(\alpha)$, using the notation of definition 4.26, by $x \in U$ iff $\alpha_n \in_\beta x$. Then $U \in X$ since X is countably closed, so $U = \mathcal{U}(\alpha,\gamma)$ for some $\gamma \in X$. Hence $\gamma = \operatorname{Coh}_{\beta',\beta}(\alpha)$ for some $\beta' < \beta$ in $h^X``\alpha$. Now let $x$ be the least set in $\mathcal{U}(\lambda,\beta) - \mathcal{U}(\lambda,\beta')$. Then $x \in g``\alpha$, where g is the function given by claim 4.29. We have $\alpha_n \in_\beta x$ for sufficiently large $n < \omega$, so $x \in U$. But $x \in g``(X \cap \alpha)$, so we must have $\alpha \notin_{\beta'} x$; that is, $x \notin \mathcal{U}(\alpha, \operatorname{Coh}_{\beta,\beta'}(\alpha)) = \mathcal{U}(\alpha,\gamma) = U$. The contradiction completes the proof of the claim. $\dashv$

Notice that claim 4.31 implies that any increasing ω-sequence $\vec{\nu}$ of indiscernibles from $\mathcal{C}$ is an indiscernible sequence for some $\vec{\alpha},\vec{\beta}$. To see this, suppose that $\nu_n \in \mathcal{C}(\alpha_n,\beta_n)$. If $\alpha_n \le \sup_n \nu_n$ for each n, then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha},\vec{\beta})$. Otherwise $\alpha_n = \alpha$ is constant for sufficiently large n with $\alpha_n > \alpha' = \sup_n \nu_n$, and claim 4.31 implies that $\alpha' \in \mathcal{C}(\alpha,\beta)$ for some $\beta < o(\alpha)$ such that $\beta_n > \beta$ for all suffiently large $n < \omega$. Then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}', \vec{\beta}')$ where $\alpha'_n = \alpha'$ and $\beta'_n = \mathrm{Coh}_{\beta_n,\beta}(\alpha')$ for all n such that $\alpha_n = \alpha$.

The proof of clause 5 relies on lemma 4.21, stating that if $\vec{\nu}$ and $(\vec{\alpha}, \vec{\beta})$ are sequences in X then $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in $\mathcal{C} = \mathcal{C}^X$ if and only if $\vec{\nu}$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in the general sense of definition 4.23, and similarly for X'. Suppose that clause 5 is false for the function $s^{\mathcal{C}}$. Then we can assume, without loss of generality, that there are infinite sequences $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$ in $X \cap X'$ such that for each $n < \omega$ we have (i) $\nu'_n = s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$ exists, (ii) $\xi_{n+1} > \nu'_n$, and (iii) $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ either does not exist or is strictly larger than $s^{\mathcal{C}'}(\alpha_n, \beta_n, \xi_n)$. Then $\vec{\nu}'$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$, and since $Y \prec H_\lambda$ it follows that Y satisfies that there is an indiscernible sequence $\vec{\nu}$ for $(\vec{\alpha}, \vec{\beta})$ such that $\nu_n > \xi_n$ for all n. Thus, by lemma 4.21, $s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$ exists for all but finitely many $n < \omega$, so we can set $\nu_n = s^{\mathcal{C}}(\alpha_n, \beta_n, \xi_n)$. By the choice of $\vec{\alpha}, \vec{\beta}$ and $\vec{\xi}$, we must have $\vec{\nu} >^* \vec{\nu}'$, but then again Y satisfies that there is an indiscernible sequence ν'' for $(\vec{\alpha}, \vec{\gamma})$ such that $\vec{\xi} <^* \vec{\nu}'' <^* \vec{\nu}$, so that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta})$ in $\mathcal{C}$, which contradicts the choice of $\vec{\nu}$.

The proof of clause 5 for the function $s^{\mathcal{C}}_*$ is similar, except that $\vec{\nu}'$ is an indiscernible sequence for some $(\vec{\alpha}, \vec{\beta}')$ with $\vec{\beta}' \geq^* \vec{\beta}$, instead of for $(\vec{\alpha}, \vec{\beta})$ itself.

The proof that clause 5 holds for the function $a^{\mathcal{C}}$ is similar but slightly more complicated. We say that $\vec{\nu}$ is an accumulation point sequence for $(\vec{\alpha}, \vec{\gamma})$ if for all sequences $\vec{\gamma}' <^* \vec{\gamma}$ and $\vec{\nu}' <^* \vec{\nu}$ there are sequences $\vec{\nu}''$ and $\vec{\beta}''$ with $\vec{\nu}' <^* \vec{\nu}'' <^* \vec{\nu}$ and $\vec{\beta}' \leq \vec{\beta}''$ such that $\vec{\nu}''$ is an indiscernible sequence for $(\vec{\alpha}, \vec{\beta}'')$. By using the elementarity of Y and the fact that being an indiscernible sequence is absolute between Y and V, it follows that being an accumulation point sequence is also absolute between Y and V. The rest of clause 5 follows as for the functions $s^{\mathcal{C}}$ and $s^{\mathcal{C}}_*$.

This completes the proof of clause 5, and we now return to the proof of the final sentence of clause 4, which states that if ν is any limit point of X such that $\nu \in \mathcal{C}(\alpha, \beta)$ for some $\beta < o(\alpha)$ then $\nu = s(\alpha, \beta, \xi)$ for some $\xi \in X \cap \nu$. Let Z be the set of ordinals $\nu \in X \cap \lim(X)$ for which there is no α, β and ξ in X such that $\nu = s(\alpha, \beta, \xi) = s_*(\alpha, \beta, \xi)$.

We specify the last lower bound on ρ:

$$\rho \geq \sup(Z). \tag{E}$$

This is justified by the following claim:

4.32 Claim. *The set Z is finite.*

Sketch of proof. Now we show that Z is finite. Suppose to the contrary that $\vec{\nu}$ is an increasing ω-sequence of members of Z. Then $\alpha = \sup_n \nu_n \notin Z$ by claim 4.31, and it follows that $\vec{\nu}$ is an indiscernible sequence in $\mathcal{C}$ for some pair $(\vec{\alpha}, \vec{\beta})$ with $\alpha_n \leq \alpha$. Since $\nu_n \in Z$, $\nu_n = a^{\mathcal{C}}(\alpha_n, \beta'_n, \xi_n)$ for some β'_n with $\beta_n < \beta'_n < o(\alpha_n)$ and $\xi_n \in X \cap \nu_n$. Now proceed as in the proof of claim 4.31 for each of the ordinals ν_n. Let $C_n = \bigcup \{ \mathcal{C}(\alpha_n, \beta) : \beta_n \leq \beta < \beta'_n \}$. Then C_n is cofinal in ν_n, so the set C_n generates a measure U_n on ν_n by

$$U_n = \{ x \subseteq \nu_n : \{ \nu \in C_n : \nu \in_{\beta_n} x \} \text{ is unbounded in } \nu_n \}.$$

Now pick a set $X' = Y' \cap K_{\breve{\kappa}}$ as in the hypothesis of theorem 4.15 so that $C_n \in Y'$ and $\vec{\beta}_n \in Y'$ for each $n < \omega$. Then the argument of claim 4.31 shows that it cannot be not true that $\nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and at the same time $\nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for unboundedly many $\nu \in C_n$. But by clause 5, for sufficiently large $n < \omega$ we have $\nu_n \in \mathcal{C}^{X}(\alpha_n, \beta_n) \implies \nu_n \in \mathcal{C}^{X'}(\alpha_n, \beta_n)$ and $\nu \in \mathcal{C}^{X}(\alpha_n, \beta_{n,\nu}) \implies \nu \in \mathcal{C}^{X'}(\alpha_n, \beta_{n,\nu})$ for all $\nu \in C_n$. This contradiction completes the proof of claim 4.32. $\dashv$

This completes the proof of the last sentence of clause (4), which is the end of the proof of theorem 4.15, the covering lemma for sequences of measures.

4.3. The Covering Lemma for Extenders

This subsection is unevenly divided into three parts. The largest part concerns the covering lemma up to $0^{\P}$, which is understood nearly as well as that for sequences of measures. A smaller part covers the covering lemma for the Steel core model, for which little is known beyond the weak covering lemma, and the final part describes what is known beyond this. No proofs are given. See [23] or chapter $\boxed{\text{steel}}$ for definitions and basic properties of extenders.

Up to a strong cardinal

This subsection covers the covering lemma when $o(\kappa) > \kappa^{++}$ but $0^{\P}$ does not exist; that is, when the core model contains extenders, but not overlapping extenders. More information may be found in [21].

As remarked in observation 7, the extension of the covering lemma to this region involves two significant changes: one which easy and largely notational, and one which is rather surprising. We will begin with the notational considerations, which come into play as soon as extenders are present.

The first observation is that since there are no overlapping extenders, the notations $\mathcal{E}(\alpha, \beta)$ and $o(\alpha)$ are still meaningful: $\mathcal{E}(\alpha, \beta)$ is the βth full extender on α, and $o(\alpha)$ is the order type of the set of full extenders on α. Some care is required in the use of this notation: it is not true, as it is for sequences of measures, that $E = \mathcal{E}(\alpha, \beta)$ implies $o^{i^E(\mathcal{E})}(\alpha) = \beta$. For an example of this, let $E = \mathcal{E}(\alpha, \beta)$ where $\beta \geq \alpha^{++}$, and let $U = E_\alpha$ be the associated ultrafilter, that is, $x \in U$ if and only if $\alpha \in i^E(x)$. Then

$U = \mathcal{E}(\alpha, \beta')$ for some $\beta' \leq \alpha^{++L[i^U(\mathcal{E})]}$, but $\{\, \nu : o(\nu) > \nu^{++} \,\} \in U$. Thus $o^{i^U(\mathcal{E})}(\alpha) > \alpha^{++L[i^U(\mathcal{E})]} \geq \beta'$.

Because of this, it is not strictly true that comparisons of these models use only linear iterations; however the tree iterations which they do use have a particularly simple form: there is a single trunk with no side branches of length more than one, and furthermore each extender used in the iteration tree is a member of (though not necessarily on the extender sequence of) the model to which it is applied. These simple trees can be modified to obtain linear iterations (*c.f.* [1, 21]) or they can be handled directly without using the stronger techniques required for larger extenders (*c.f.* [46]).

Before we can state a covering lemma for models with extenders, we need to develop some notation for dealing with indiscernibles for extenders. Consider for contrast the more familiar case of indiscernibles for measures. If U is an ultrafilter on κ in some model M, and $i = i^U : M \to \mathrm{ult}(M, U)$ is the canonical embedding, then κ is an indiscernible for $i(U)$ in the sense that

$$\forall x \in i``\mathcal{P}(\kappa) \ \left(\kappa \in x \iff i^{-1}(x) \in U \iff x \in i(U)\right),$$

and κ generates $\mathrm{ult}(M, U)$ in the sense that

$$\mathrm{ult}(M, U) = \left\{\, i^U(f)(\kappa) : f \in M \,\right\}.$$

Now let E be a (κ, λ)-extender, and let $i = i^E : M \to \mathrm{ult}(M, E)$ be the canonical embedding. In this case the role previously played by the ordinal κ is played by the interval $[\kappa, \lambda)$: if we write E_a for the ultrafilter corresponding to $a \in [\lambda]^{<\omega}$ then

$$\forall x \in i``\mathcal{P}(\kappa^{|a|}) \ \left(a \in x \iff i^{-1}(x) \in E_a \iff x \in i(E_a)\right),$$

and

$$\mathrm{ult}(M, E) = \left\{\, i(f)(a) : f \in M \wedge a \in [\kappa, \lambda)^{<\omega} \,\right\}.$$

4.33 Definition. If E is a (κ, λ)-extender then the critical point κ of i^E is called a *principle indiscernible* for $i(E)$, and an ordinal $\alpha \in [\kappa, \lambda)$ is called the *indiscernible for $i(E)_{i(\alpha)}$ belonging to κ.*

These definitions extend in a natural way to indiscernibles arising from iterated ultrapowers using extenders. Such a system of indiscernibles for a model $M = L[\mathcal{E}]$ may be written as a pair $(\mathcal{C}, b)$ of functions: If $\mathcal{E}_\gamma$ is a (κ, λ)-extender then $\mathcal{C}_\gamma$ is defined to be the set of principle indiscernibles for $\mathcal{E}_\gamma$, and if $\alpha \in \mathcal{C}_\gamma$ then $b(\gamma, \alpha, a)$ is defined to be the indiscernible for $(\mathcal{E}(\gamma))_a$ belonging to α.

Thus if $i_{0,\theta} : M_0 \to M_\theta = M$ is an iterated ultrapower then the system $(\vec{\mathcal{C}}, b)$ of indiscernibles for M_θ generated by $i_{0,\theta}$ is defined as follows:

1. $\alpha \in \mathcal{C}_\gamma$ if and only if there are $\nu < \nu' \le \theta$ such that $\alpha = \operatorname{crit}(i_{\nu,\nu'})$ and $\mathcal{E}_\gamma = \mathcal{E}_\gamma^{M_{\nu'}} = i_{\nu,\nu'}(E_\nu)$ where E_ν is the extender such that $M_{\nu+1} = \operatorname{ult}(M_\nu, E_\nu)$.
2. If $\alpha \in \mathcal{C}(\gamma)$, with ν and ν' as in clause 1, then $b(\gamma, \alpha, \eta)$ is defined if and only if $\eta \in i``[\alpha, \lambda)$ where E_ν is a (α, λ)-extender. In this case $b(\gamma, \alpha, \eta) = i^{-1}_{\nu,\nu'}(\eta)$.

In order to obtain an abstract definition of a system of indiscernibles, without any assumption that the system came from an iterated ultrapower, we modify clauses 4.11(2a,b) of the definition 4.11 of a system of indiscernibles for sequences of measures as follows. The finite sequence $\vec{a}$ comes from definition 4.11.

2a'. If $\alpha \in \mathcal{C}(\gamma)$, with $\vec{a} \cap [\alpha, \gamma) = \varnothing$, and $\bar{b} = b(\gamma, \alpha, b)$ then $\bar{b} \in x \iff x \cap V_\gamma \in (\mathcal{E}(\gamma))_b$.

2b'. If $\alpha < \alpha'$ are in $\mathcal{C}(\gamma)$ and $\vec{a} \cap [\alpha, \alpha'] = \varnothing$, and if $\bar{b} = b(\gamma, \alpha, b)$ and $\bar{b}' = b(\gamma, \alpha', b)$ then $\bar{b} \in x \iff \bar{b}' \in x$.

Some obvious changes need to be made to the definition of a h-coherent system of indiscernibles, and the definition of $X = h``(\rho; \mathcal{C})$ needs to be modified to $h``(\rho; \mathcal{C}, b)$.

4.34 Theorem (Covering for short extenders). *Assume that $n < \omega$ and that there is no inner model M such that $\{\, \alpha < \kappa : o^M(\alpha) = \alpha^{+n} \,\}$ is unbounded in κ for any cardinal κ. Let κ be a cardinal of K, set $\lambda = \kappa^{+n}$, and suppose $X = Y \cap K_\lambda$ where $Y \prec H_{\lambda^+}$ and ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$. Then there is a pair $(\mathcal{C}, b)$, a function $h \in K$, and an ordinal $\rho < \kappa$ such that*

1. *The pair $(\mathcal{C}, b)$ is a h-coherent system of indiscernibles for K.*
2. $\operatorname{dom}(\mathcal{C}) \cup \operatorname{dom}(b) \subseteq X$, *and* $\operatorname{ran}(b) \cup \bigcup \operatorname{ran}(\mathcal{C}) \subseteq X$.
3. *For all $\nu \in X - \rho$, one of the following four conditions hold:*
   (a) *$\nu \in h``(X \cap \nu)$.*
   (b) *$\nu = s^{\mathcal{C}}(\gamma, \xi)$ for some $\xi \in X \cap \nu$ and $\gamma \in h``(X \cap \nu)$.*
   (c) *$\nu = a^{\mathcal{C},X}(\gamma, \xi)$ for some $\xi \in X \cap \nu$ and $\gamma \in h``(X \cap \nu)$. Furthermore, this clause never holds if ν is a limit point of X.*
   (d) *$\nu = b(\gamma, \alpha, a)$ for some $\alpha \in X \cap \nu$ and $\gamma, a \in h``\alpha$.*
4. *If X' is another set satisfying the hypothesis of the theorem, and $\mathcal{C}'$, b', X' satisfy clauses (1-4), then there is a finite set $\vec{d}$ of ordinals such*

that if $\xi, \gamma \in X \cap X'$ with $[\xi, \gamma) \cap \vec{d} = \varnothing$ then

$$s^{\mathcal{C}}(\gamma, \xi) \leq s^{\mathcal{C}'}(\gamma, \xi)$$
$$a^{\mathcal{C},X}(\gamma, \xi) \leq a^{\mathcal{C}',X'}(\gamma, \xi)$$
$$b(\gamma, \alpha, \nu) = b'(\gamma, \alpha, \nu).$$

In particular, the left side of the above relations is defined whenever the right side is defined.

Longer extenders require the second, and more interesting, modification to the covering lemma. Again we contrast indiscernibles for extenders with those for measures. Suppose that M is a model of set theory and $C \subseteq \kappa$ is a ω-sequence of indiscernibles for M, in the sense that $U = \{ x \subseteq \kappa : C - x \text{ is finite} \}$ is a normal M-ultrafilter on κ. Then the added hypothesis ${}^{\omega}M \subseteq M$ implies that $C \in M$, so that $U \in M$ and hence $U \in K^M$.

Now if (C, b) is similarly a system of indiscernibles for a M-extender E, then the situation is more complicated. Again, C is an ω-sequence of indiscernibles which generates the normal measure $U = E_\kappa$ associated with E. In fact all of the ultrafilters E_a are members of M, since E_a is generated by the ω-sequence $(\, b(\nu, \alpha, a) : \nu \in C\,)$. It is not clear, however, that these ultrafilters E_a can be reassembled in M to obtain the extender E, and in fact Gitik showed in [19] (see $\boxed{\text{gitik}}$) that this reassembly is not always possible. He also showed that it is possible under the stronger hypothesis of theorem 4.34, namely that $\{ \alpha < \kappa : o(\alpha) > \alpha^{+n} \}$ is bounded in κ. In addition he discovered a game which does provide the desired reassembly, provided that it is applied to a sequence (C, b) of indiscernibles such that the length of C has uncountable cofinality. Hence a version of the covering lemma can be obtained for these longer extenders by systematically replacing ω with ω_1 [21]:

4.35 Theorem (Covering up to $0^{\P}$). *Assume that $0^{\P}$ does not exist. Let κ be a cardinal of K with $\operatorname{cf}(\kappa) > \omega$, and set $\lambda = o(\kappa)^{+K}$. Then if $\kappa \not\subseteq X = Y \cap K_\lambda$, where $Y \prec H_\lambda$ and ${}^{\operatorname{cf}(\kappa)}Y \subseteq Y$, then there is a pair $(\mathcal{C}, b)$ such that the conclusion of theorem 4.34 holds, except that clause 3c is modified as follows, where we write $a_\iota^{\mathcal{C},X}(\gamma, \xi)$ for the ιth accumulation point for E_γ above ξ:*

(3c′) $\nu = a_\iota^{\mathcal{C},X}(\gamma, \xi)$ for some $\xi \in X \cap \nu$, some $\gamma \in h``(X \cap \nu)$, and some $\iota < \omega_1$.

Furthermore clause (3c′) *does not hold for any limit point ν of X with* $\operatorname{cf}(\nu) > \omega$.

For details, see [21], which shows that these results are strong enough to give the correct lower bound for the consistency strength of a failure of the singular cardinal hypothesis at a cardinal of cofinality greater than ω.

Up to a Woodin Cardinal

The following form of the weak covering lemma is proved in [37] and [36]:

4.36 Theorem. *Suppose that there is no inner model with a Woodin cardinal, and that the Steel core model K exists. Then $(\lambda^+)^K = \lambda^+$ for every singular cardinal λ.*

The proof involves several technical difficulties which either do not occur or are easily dealt with below $0^\P$, but it closely parallels the earlier proofs. Like the first part of the proof of theorem 4.15 it gives, for any suitable precovering set X, a mouse $\widetilde{M}$, a system $\widetilde{\mathcal{C}}$ of indiscernibles for $\widetilde{M}$, and an ordinal $\rho < \kappa$ such that $X = h^{\widetilde{M}}``(X \cap \rho; \widetilde{\mathcal{C}})$. However the system $\widetilde{\mathcal{C}}$ of indiscernibles comes from an iteration tree, not a linear iteration, and no known analysis of such indiscernibles yields any useful information. The proof of lemma 4.36 sidesteps this problem: like the proof of the covering lemma 4.4 for sequences of measures, it relies on the observation that there are no measures, and hence no indiscernibles, in the interval $(\lambda, \lambda^{+K}]$.

Theorem 4.36 is actually weaker than it appears at first: its hypothesis that the Steel core model exists has no parallel in the covering lemmas for smaller cardinals. Recall that the proof of the full covering lemma for sequences of measures involved first proving the weak covering lemma for the model K^{c} constructed using countably complete measures, and then defining the true core model K as a elementary submodel of K. There are at least two problems in extending this procedure past $0^\P$. The most important of these the fact that countable completeness is not, so far as is known, sufficient to ensure iterability of extender sequences beyond a strong cardinal. Steel, in his construction of K^{c}, replaces countable completeness with a stronger notions, which he calls *countable certification*. However the proof of the weak covering lemma, as given for sequences of measures, does not work for K^{c} as defined from countably certified measures. Instead Steel defines K^{c} by using a weak large cardinal hypothesis in V, which provides the certification needed to prove that K^{c} satisfies a form of the weak covering lemma which is slightly weaker than the countably closed weak covering property, definition 3.38, but is sufficiently strong to support the definition of K and the proof of the full covering lemma.

The second problem involves the proof that the iteration from the basic construction in the proof of the core model drops immediately, that is, that $1 \in D$. In the case of the covering lemma for extenders below $0^\P$ this argument splits. The argument used to show that the weak covering lemma holds for countably complete cardinals λ is similar to that for sequences of measures but requires an extra assumption that $o(\alpha) < \lambda$ for all $\alpha < \lambda$. The proof of the full covering lemma, on the other hand, uses a different proof relying on the weak covering lemma; it does not show that the iteration drops, but instead shows that even when the iteration does not drop there is still a Skolem function $g^X \in K$ (derived, for a suitable set X, from an extender $\mathcal{E}_\gamma$ of length κ and critical point less than $\inf(\kappa - X)$) such that $X = g^X``(X \cap \rho^X; \mathcal{C}^X)$ for some $\rho^X < \lambda$. Beyond $0^\P$ the notion of $o(\alpha)$ is not meaningful, so only the second argument, which requires the weak covering lemma, is usable.

The only other result known using the ideas of the covering lemma is theorem 1.16, asserting that that any Jónsson cardinal κ is Ramsey in K. This proof avoids a measurable cardinal at κ, since if κ were measurable then it would be Ramsey, and it avoids smaller measurable cardinals by selecting the sets of indiscernibles needed to witness that κ is Ramsey to contain only nonmeasurable cardinals.

Beyond a Woodin cardinal

As was pointed out earlier, not even the weak covering lemma is valid for a model containing a Woodin cardinal δ: Woodin has defined a notion of forcing, the nonstationary tower forcing [49], such that the cardinal δ is still Woodin in the generic extension $L[\mathcal{E}][G]$ but there are cofinally many singular cardinals $\lambda < \delta$ such that $\lambda^{+L[\mathcal{E}]} < \lambda^{+L[\mathcal{E}][G]}$. Indeed it seems is conjectured that every sufficiently large successor cardinal less than δ is collapsed by this forcing.

It is possible that this situation is analogous to that of Prikry forcing at a measurable cardinal, in that one could hope hope for an analogue of the Dodd-Jensen lemma stating that any failure of the weak covering lemma is achieved by some variant of stationary tower forcing. Some very weak results in this direction are proved in [35], but there are many more questions than theorems. One difficulty is that the stationary tower forcing, unlike Prikry forcing, has a number of variants; furthermore there are other forcings, notably Woodin's "all sets generic" forcing, which require a Woodin cardinal in the universe and which may be relevant to this question.

More promising developments deal with core models which do not contain Woodin cardinals, but which are large in the sense that they have inner models with Woodin cardinals. The best result so far is due to Schimmerling and Woodin, in [39]:

4.37 Theorem. *Suppose that $\mathcal{E}$ is a good extender sequence and the model $W = L[\mathcal{E}, x]$ is sufficiently iterable. Then either there is an amenable ultrafilter U on W with $crit(W) > \mathrm{rank}(x)$ such that $\mathrm{ult}(W, U)$ is well founded, or else W has the weak covering property above $\mathrm{rank}(x)$.*

Using this and unpublished work of Woodin they prove the following theorem on the existence of a core model:

4.38 Theorem. *Suppose that there is an inaccessible cardinal and $L(\mathbb{R})$ does not satisfy the axiom of determinacy. Then one of the following is true:*

1. *The Steel core model exists, and satisfies the weak covering lemma.*
2. *There is a set x, and a good sequence $\mathcal{E}$ satisfying* $\operatorname{crit}(\mathcal{E}_\gamma) > \operatorname{rank}(x)$ *for all $\gamma \in \operatorname{dom}(\mathcal{E})$, such that the model $W = L[\mathcal{E}, x]$ is iterable, does not contain any measurable cardinals above* $\operatorname{rank}(x)$*, and has the weak covering property at all cardinals larger than* $\operatorname{rank}(x)$.

Notice that a model $L[\mathcal{E}']$ will satisfy clause (2) whenever $\mathcal{E}'$ is not a proper class, even though $L[\mathcal{E}']$ may contain many Woodin cardinals; however in order to satisfy clause (2) we must take $x = \mathcal{E}'$ and let the sequence $\mathcal{E}$ of the lemma be empty.

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