Frege systems for extensible modal logics

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Abstract

By a well-known result of Cook and Reckhow [4, 12], all Frege systems for the Classical Propositional Calculus (*CPC*) are polynomially equivalent. Mints and Kojevnikov [11] have recently shown p-equivalence of Frege systems for the Intuitionistic Propositional Calculus (*IPC*) in the standard language, building on a description of admissible rules of *IPC* by Iemhoff [8]. We prove a similar result for an infinite family of normal modal logics, including $K4$, GL, $S4$, and $S4Grz$.

1 Introduction

The basic topic of proof complexity is to study the efficiency of proof systems for logical systems, either absolute (lower and upper bounds on lengths of proofs) or relative (simulation or relative speed-up of proof systems). Frege proof systems, in which formulas are derived using a finite set of schematic inference rules (as in the usual "textbook" calculi), are among the most natural systems to study. The main interest in proof complexity is devoted to proof systems for the classical propositional logic (*CPC*), due to its relationship to central problems of computational complexity: as shown in [4], there exists a polynomially bounded proof system for *CPC* if and only if $NP = coNP$. Existence of lower bounds on lengths of proofs in Frege systems is an important open problem in this area; sofar we have good information only on restricted fragments of Frege systems, such as the bounded depth systems. Much less is known about the proof complexity of non-classical logics; among the most interesting results are the feasible disjunction property and feasible interpolation for intuitionistic logic [1, 2] and several modal logics [5].

In *CPC*, the notion of a Frege system is very robust: by Reckhow [12], all classical Frege systems are polynomially equivalent, regardless of the particular set of rules of the system, or

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the choice of the language (set of basic connectives). The situation is much more complicated for non-classical logics. Reckhow's proof of language independence fails for intuitionistic and modal logics[1], and it is not clear whether the expected answer should be positive or negative.

Even if we consider only Frege systems in a fixed language, a straightforward polynomial simulation does not work, due to presence of nontrivial *admissible rules.* A rule

$$\varphi_1, \dots, \varphi_n \vdash \psi$$

is admissible in a logic L, provided for every substitution σ, if L contains the formulas $\sigma\varphi_1$, $\dots$, $\sigma\varphi_n$, then it also contains $\sigma\psi$. Every rule which is valid in L (i.e., $\varphi_1, \dots, \varphi_n \vDash \psi$) is also admissible in L. It is not hard to see that the classical logic is *structurally complete*: every admissible rule is valid. On the other hand, nonclassical logics often admit invalid rules. For example, the Kreisel-Putnam rule

$$\neg\varphi \to \psi \vee \chi \vdash (\neg\varphi \to \psi) \vee (\neg\varphi \to \chi)$$

is admissible in the intuitionistic logic (*IPC*), although it is not intuitionistically valid. Similarly, in many modal logics (like $K4$ or GL) the rule

$$\Box\varphi \vdash \varphi$$

is admissible but invalid. Rules of a (sound and implicationally complete) Frege system for a logic L need not be valid: they only need to preserve the set of theorems of L, i.e., to be admissible. Consequently, a rule of a Frege system for a structually incomplete logic L may be nonderivable in another Frege system for L.

Admissibility in modal and superintuitionistic logics was studied in depth by Rybakov in the 80's and 90's, see [13]. He showed that the problem of recognizing admissible rules is decidable for a large class of logics, and provided semantical criteria for admissibility. Ghilardi [6, 7] found a characterization of admissible rules in terms of projective formulas, connecting admissibility to the unification problem for Heyting and modal algebras. Based on this result, Iemhoff [8] proved completeness of an explicit basis of admissible rules for *IPC*. In a similar spirit, bases of admissible rules for some modal logics were constructed by Jeřábek [9].

Mints and Kojevnikov [11] have shown that rules from the basis of [8] can be polynomially simulated in the natural deduction system for *IPC*, using an efficient variant of Kleene's slash [10, 5], thereby establishing polynomial equivalence of Frege systems for *IPC* in the standard language $\{\to, \wedge, \vee, \bot\}$.

In the present paper we will generalize the result of Mints and Kojevnikov to a family of normal modal logics, using the bases of admissible rules for these logics from [9]. We use propositional valuations as the modal analogue of Kleene's slash; unlike the intuitionistic case, we avoid translating the proofs back and forth to natural deduction, we work directly with Frege systems. This change considerably simplifies the argument, even for the few modal logics which are known to have a decent natural deduction proof system.

[1] We cannot "balance" formulas in a Frege proof to logarithmic (or even sublinear) depth, as there exist formulas of size $O(n)$ which are not equivalent to any formula of depth n.

Modal logics covered by our result include $K4$, GL, $S4$, their extensions by the Grzegorczyk or McKinsey axioms, and other logics, like Zeman's $S4.1.4$. In fact, the result applies to the infinite family of all (finitely axiomatizable) *extensible logics* as introduced in [9]. To achieve this level of generality, we provide a description of extensible logics using Zakharyaschev's *canonical formulas* [14]. We also show that Frege systems for extensible logics enjoy the feasible disjunction property, which generalizes the results of [5].

2 Preliminaries

Formulas of the *basic modal language* are constructed from propositional variables and the connectives $\to$, $\bot$, $\Box$. Other connectives are treated as abbreviations; apart from the usual propositional connectives $\wedge$, $\vee$, $\neg$, $\equiv$, we will use $\Diamond\varphi := \neg\Box\neg\varphi$, $\boxdot\varphi := \varphi \wedge \Box\varphi$, and $\diamonddot\varphi := \neg\boxdot\neg\varphi = \varphi \vee \Diamond\varphi$. A set L of formulas is a *normal modal logic*, if L contains all propositional tautologies and the schema

$$\text{(K)} \qquad \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi),$$

and L is closed under substitution, Modus Ponens (MP) and Necessitation (Nec):

$$\text{(MP)} \qquad \varphi, \varphi \to \psi \vdash \psi$$

$$\text{(Nec)} \qquad \varphi \vdash \Box\varphi$$

The minimal normal modal logic is called K. We denote by $L \oplus \varphi$ the smallest normal logic containing L and φ. The logics most often used in this paper are $K4 := K \oplus (4)$, $GL := K \oplus (GL) = K4 \oplus (GL)$, and $S4 := K4 \oplus (T)$, where

$$\text{(4)} \qquad \Box\varphi \to \Box\Box\varphi,$$

$$\text{(GL)} \qquad \Box(\Box\varphi \to \varphi) \to \Box\varphi,$$

$$\text{(T)} \qquad \Box\varphi \to \varphi.$$

An *inference system* F is a set of schematic inference rules. A finite sequence $\pi = \varphi_0, \dots, \varphi_n$ of formulas is an *F-proof* of a formula α from assumptions $\beta_1, \dots, \beta_k$, if $\varphi_n = \alpha$, and every φ_i is equal to some β_j, or is inferred by an instance of an F-rule from some of the formulas φ_j, $j < i$. We write $\beta_1, \dots, \beta_k \vdash_F \alpha$ if there is an F-proof of α from $\vec{\beta}$, and say that the rule $\vec{\beta} \vdash \alpha$ is *derivable* in F. Rules without assumptions are called *axioms*.

A *finite* inference system F is a *Frege system* for a normal logic L, if F is

- sound: $\vdash_F \varphi$ only if $\varphi \in L$,
- complete: $\vdash_F \varphi$ if $\varphi \in L$,
- implicationally complete: $\varphi, \varphi \to \psi \vdash_F \psi$.

Notice that a complete inference system is sound iff all rules of F are L-admissible.

The general concept of a proof system was introduced by Cook and Reckhow [4]: a *proof system* for a logic L is a polynomial-time computable function P such that $\mathrm{rng}(P) = L$. A proof system P *polynomially simulates* (p-simulates, in short) a proof system Q, if there exists a polynomial-time function f such that $Q = P \circ f$. We write $Q \leq_p P$ if P p-simulates Q. Proof systems P and Q are *polynomially equivalent*, if $P \leq_p Q$ and $Q \leq_p P$. A Frege system F (or in general, an inference system with a polynomial-time set of rules) fits the definition of a proof system if we put

$$P(\pi) = \begin{cases} \varphi, & \text{if } \pi \text{ is an } F\text{-proof of } \varphi, \\ \top, & \text{if } \pi \text{ is not an } F\text{-proof.} \end{cases}$$

The concept of admissibility can be naturally extended to *multiple-conclusion rules*. A multiple-conclusion rule

$$\varphi_1, \dots, \varphi_n \vdash \psi_1, \dots, \psi_m$$

is L-*admissible*, if the following holds for all substitutions σ: if $\sigma\varphi_i \in L$ for every i, then there exists j such that $\sigma\psi_j \in L$. For example, a logic L is consistent iff it admits the rule

$$\bot \vdash$$

L has the *disjunction property*, if it admits the rule

(DP) $$\Box\varphi_1 \vee \dots \vee \Box\varphi_n \vdash \varphi_1, \dots, \varphi_n$$

for every $n \in \omega$. (The empty disjunction is defined as $\bot$, and the empty conjunction is $\top$.)

A set B of (single-conclusion) rules is a *basis* of L-admissible rules, if L admits all rules from B, and every L-admissible rule is derivable in the inference system consisting of B, and the *postulated* inference rules of L (i.e., (MP), (Nec), axioms of K, and additional axioms of L, if any). A similar concept for multiple-conclusion rules may be introduced as follows. A set A of multiple-conclusion rules is an *AR-system* over L, if A is closed under substitution, cut, and weakening, and contains postulated rules of inference of L. The set of all L-admissible multiple-conclusion rules is an AR-system over L, denoted by A_L. A set B is a *basis* of L-admissible multiple-conclusion rules, if A_L is the smallest AR-system over L which contains B.

Following [9], we define the multiple-conclusion rules

($A^\bullet$) $$\Box\varphi \to \bigvee_{i<n} \Box\psi_i \vdash \{\boxdot\varphi \to \psi_i;\, i < n\}$$

(A°) $$\bigwedge_{j<m} (\varphi_j \equiv \Box\varphi_j) \to \bigvee_{i<n} \Box\psi_i \vdash \{\boxdot \bigwedge_{j<m} \varphi_j \to \psi_i;\, i < n\}$$

and their single-conclusion variants

($\widehat{A}^\bullet$) $$\Box(\Box\varphi \to \bigvee_{i<n} \Box\psi_i) \vee \Box\chi \vdash \bigvee_{i<n} \boxdot(\boxdot\varphi \to \psi_i) \vee \chi$$

($\widehat{A}^\circ$) $$\Box(\bigwedge_{j<m} (\varphi_j \equiv \Box\varphi_j) \to \bigvee_{i<n} \Box\psi_i) \vee \Box\chi \vdash \bigvee_{i<n} \boxdot(\boxdot \bigwedge_{j<m} \varphi_j \to \psi_i) \vee \chi$$

where $n, m \in \omega$. Their importance comes from the following theorem.

Theorem 2.1 ([9]) *Let A be one of*

- $K4 + A^\bullet + A^\circ$,
- $S4 + A^\circ$,
- $GL + A^\bullet$.

If a normal modal logic L admits A, then A is a basis of L-admissible multiple-conclusion rules, and $\widehat{A}$ is a basis of L-admissible single-conclusion rules. $\Box$

A normal modal logic L which satisfies the assumptions of theorem 2.1 is called *extensible*[2]. The logics $K4$, $S4$, and GL are extensible. Other examples of extensible logics include $S4Grz := S4 \oplus (Grz)$, $S4.1 := S4 \oplus (.1)$, $K4Grz := K4 \oplus (Grz)$, and $K4.1 := K4 \oplus (.1)$, where

$$\text{(Grz)} \qquad \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \Box\varphi,$$

$$\text{(.1)} \qquad \boxdot\Diamond\varphi \to \Diamond\!\!\!\cdot\,\Box\varphi.$$

(The system $S4Grz$ is often called just Grz.)

3 The structure of extensible logics

The transformation used in the proof of our main theorem in the next section is very sensitive to the syntactical form of an axiom system for the logics involved. As we apply the transformation to an infinite class of logics, we need some kind of a "normal form" for their axiomatization. To this end we provide a frame-theoretic characterization of extensible modal logics, which we then restate in terms of Zakharyaschev's canonical formulas. A reader interested only in $K4$, $S4$, or GL may safely skip this section.

We assume some degree of familiarity with Kripke semantics. For more background on general frame semantics and canonical formulas, consult Chagrov and Zakharyaschev [3]; here we only briefly mention the basic definitions to fix the notation.

Definition 3.1 A *Kripke frame* is a pair $\langle K, < \rangle$, where K is a nonempty set, and $<$ is a *transitive* binary relation on K. We denote by $\leq$ the reflexive closure of $<$.

A *general frame* is a triple $\langle K, <, V \rangle$, where $\langle K, < \rangle$ is a Kripke frame, and V is a set of subsets of K which is closed under (binary) intersection, complement, and the operation

$$\Box A = \{x \in K; \forall y > x\, y \in A\}.$$

A Kripke frame $\langle K, < \rangle$ is identified with the general frame $\langle K, <, \mathcal{P}(K) \rangle$. A valuation $\Vdash$ is *admissible* in $\langle K, <, V \rangle$, if $\Vdash (p) = \{x; x \Vdash p\} \in V$ for every variable p. A formula φ is *valid* in $\langle K, <, V \rangle$, if it is satisfied by all admissible valuations.

A general frame $\langle K, <, V \rangle$ is *descriptive* if

[2] The notion of an extensible modal logic was defined in [9] in a different way. It follows from the results of section 3 that a logic is extensible in the sense of [9] iff it is extensible and has the finite model property.

(*i*) $\forall A \in V\,(x \in A \Rightarrow y \in A) \Rightarrow x = y$,

(*ii*) $\forall A \in V\,(x \in \Box A \Rightarrow y \in A) \Rightarrow x < y$,

(*iii*) every subset of V with the finite intersection property has a nonempty intersection.

A Kripke model $\langle K, <, \Vdash\rangle$ *induces* a general frame $\langle K, <, V\rangle$ by $V = \{\Vdash(\varphi);\ \varphi \text{ a formula}\}$. $C_{L,\kappa}$ denotes the descriptive frame induced by the canonical model of L in κ variables.

Definition 3.2 Let $\langle F, <, V\rangle$ be a general frame, and $Y \subseteq K$. A node $x \in F$ is an *irreflexive tight predecessor* of Y, if

$$z > x \quad \text{iff} \quad \exists y \in Y\, z \geq y$$

for every $z \in F$. A node x is a *reflexive tight predecessor* of Y, if

$$z > x \quad \text{iff} \quad z = x \vee \exists y \in Y\, z \geq y$$

for every z. The frame F is $\bullet$-*extensible*, if every finite $Y \subseteq F$ has an irreflexive tight predecessor, and it is $\circ$-*extensible*, if every Y has a reflexive tight predecessor.

Lemma 3.3 *Let $* \in \{\bullet, \circ\}$, L a normal extension of $K4$, and κ any cardinal number. If L admits A^*, then $C_{L,\kappa}$ is $*$-extensible.*

Proof: If x is a set of formulas, let $\Box x := \{\Box\varphi;\ \varphi \in x\}$, $\neg x := \{\neg\varphi;\ \varphi \in x\}$, $x^{\Box} := \{\varphi;\ \Box\varphi \in x\}$, and $x^{\boxdot} := \{\varphi;\ \boxdot\varphi \in x\}$. If x and y are maximal L-consistent sets (L-MCS), we have $x < y$ iff $x^{\Box} \subseteq y$ by definition of the canonical model. It is easy to see that

$$x \leq y \quad \text{iff} \quad x^{\boxdot} \subseteq y.$$

Assume that L admits $A^\bullet$, and let $y_1, \dots, y_k$ be L-MCS. Put $a := (\bigcap_i y_i)^{\boxdot}$, and let b be the complement of a. We claim that $\Box a \cup \neg\Box b$ is L-consistent. If not, there are $\vec{\alpha} \in a$ and $\vec{\beta} \in b$ such that

$$\bigwedge_i \Box\alpha_i \to \bigvee_j \Box\beta_j \in L,$$

thus $\bigwedge_i \boxdot\alpha_i \to \beta_j \in L$ for some j by $A^\bullet$, contradicting $\beta_j \in b$.

Let x be a MCS extending $\Box a \cup \neg\Box b$, we will verify that x is an irreflexive t.p. of $\{y_1, \dots, y_k\}$. Clearly $x^{\Box} \subseteq a \subseteq y_i$, thus $x < z$ whenever $y_i \leq z$. Let z be a MCS such that $z \geq y_i$ for no i. Fix formulas φ_i such that $\boxdot\varphi_i \in y_i$, and $\varphi_i \notin z$. Then $\bigvee_i \boxdot\varphi_i \in a$, thus $\Box \bigvee_i \boxdot\varphi_i \in x$. However $\bigvee_i \boxdot\varphi_i \notin z$, thus $x \not< z$.

The proof for the reflexive case is analogous, taking $x \supseteq \{\varphi \equiv \Box\varphi;\ \varphi \in a\} \cup \neg\Box b$. $\Box$

Definition 3.4 The symbol $\sum_i F_i$ denotes the disjoint sum of general frames F_i. Let $* \in \{\bullet, \circ\}$. If $\langle F, <, V\rangle$ is a general frame, let $\langle F^*, <, V^*\rangle$ be the frame obtained from F by adjoining a new root r, such that r is reflexive if $* = \circ$ and irreflexive if $* = \bullet$, and $V^* = \{A \subseteq V^*;\ A \cap F \in V\}$. A class C of rooted general frames is $*$-*extensible* if $(\sum_{i<k} F_i)^* \in C$ for any finite sequence of frames $F_i \in C$, $i < k$.

Theorem 3.5 *Let $* \in \{\bullet, \circ\}$, and L a normal extension of $K4$. The following are equivalent.*

(i) *L admits A^*.*

(ii) *All canonical frames $C_{L,\kappa}$ are $*$-extensible.*

(iii) *The class of all rooted descriptive L-frames is $*$-extensible.*

(iv) *L is sound and complete wrt a $*$-extensible class C of general frames, closed under formation of rooted generated subframes.*

Proof: $(i) \to (ii)$ is lemma 3.3, and $(iii) \to (iv)$ is trivial.

$(iv) \to (i)$: assume that $\boxdot \bigwedge_i \varphi_i \to \psi_j \notin L$ for any $j < k$. Fix rooted models $\langle F_j, r_j, \Vdash_j \rangle$ such that $F_j \in C$, and $r_j \Vdash_j \boxdot \bigwedge_i \varphi_i \wedge \neg\psi_j$. Put $F = (\sum_j F_j)^* \in C$, and let $\Vdash$ be a valuation in F which agrees with $\Vdash_j$ on F_j, and is arbitrary in the root r of F. Clearly $r \nVdash \bigvee_j \Box\psi_j$. Moreover $r \Vdash \bigwedge_i \Box\varphi_i$ if $* = \bullet$, and $r \Vdash \bigwedge_i(\varphi_i \equiv \Box\varphi_i)$ if $* = \circ$, thus L does not prove $\bigwedge_i \Box\varphi_i \to \bigvee_j \Box\psi_j$ or $\bigwedge_i(\varphi_i \equiv \Box\varphi_i) \to \bigvee_j \Box\psi_j$ respectively.

$(ii) \to (iii)$: let $\langle F_i, r_i \rangle$ be rooted descriptive L-frames, $i < k$. We have to show that F^* is an L-frame, where $F = \sum_i F_i$.

As F is a descriptive L-frame, it is (isomorphic to) a generated subframe of a canonical frame $C_{L,\kappa}$ for some cardinal κ (cf. [3]). Let $x_i \in C_{L,\kappa}$ be the roots of F_i, and let $x \in C_{L,\kappa}$ be a (reflexive or irreflexive, as appropriate) tight predecessor of $\{x_i;\, i < k\}$. If x is distinct from all x_i, the subframe generated by x is isomorphic to F^*, thus F^* is an L-frame. If $x = x_i$ for some i, we must have $k = 1$. We have just established that $(F_0 + F_0)^*$ is an L-frame, and $F^* = F_0^*$ is a p-morphic image of $(F_0 + F_0)^*$, thus F^* is an L-frame as well. $\Box$

We remark that condition *(iii)* in theorem 3.5 can be generalized to nondescriptive frames, see theorem 3.11.

Definition 3.6 Let $\langle K, <, V \rangle$ be a general frame, and $\langle F, < \rangle$ a finite Kripke frame. A partial mapping f from K onto F is a *subreduction* of K to F, if for every $x, y \in K$ and $u \in F$,

(i) $x < y$ and $x, y \in \mathrm{dom}(f)$ implies $f(x) < f(y)$,

(ii) if $f(x) < u$, there exists $y \in \mathrm{dom}(f)$ such that $x < y$ and $f(y) = u$,

(iii) $f^{-1}(u) \in V$.

For any $X \subseteq K$, let $X{\uparrow} = \{y;\, \exists x \in X\, x \leq y\}$. We will abbreviate $\{x\}{\uparrow}$ as $x{\uparrow}$. A *domain* is an upwards closed subset $d \subseteq F$. A subreduction f satisfies the *closed domain condition* (*CDC*) for a domain d, if there is no $x \in \mathrm{dom}(f){\uparrow} \smallsetminus \mathrm{dom}(f)$ such that $f(x{\uparrow}) = d$. If D is a set of domains, f satisfies CDC for D if it satisfies CDC for every $d \in D$.

Definition 3.7 Let $\langle F, < \rangle$ be a finite Kripke frame with root $0 \in F$, and D a set of domains in F. The *canonical formula* $\alpha(F, D)$ in variables $\{p_i;\, i \in F\}$ is defined as

$$\bigwedge_{i \neq j} \boxdot(p_i \vee p_j) \wedge \bigwedge_{i<j} \boxdot(\Box p_j \to p_i) \wedge \bigwedge_{i \not< j} \boxdot(p_i \vee \Box p_j) \wedge \bigwedge_{d \in D} \boxdot(\bigwedge_i p_i \wedge \bigwedge_{i \notin d} \Box p_i \to \bigvee_{i \in d} \Box p_i) \to p_0$$

where indices i, j range over elements of F.

Lemma 3.8 (Zakharyaschev [14]) *A general frame* $\langle K, <, V\rangle$ *refutes* $\alpha(F, D)$ *if and only if there is a subreduction of* K *to* F *satisfying CDC for* D. □

Theorem 3.9 (Zakharyaschev [14]) *For every formula* φ*, there is a finite sequence of canonical formulas* $\alpha(F_i, D_i)$*,* $i < k$*, such that*

$$K4 \oplus \varphi = K4 \oplus \bigoplus_{i<k} \alpha(F_i, D_i). \qquad \square$$

Remark 3.10 We have departed from the original Zakharyaschev's presentation of canonical formulas in several details. Most importantly, we allow domains to be empty; a subreduction satisfies CDC for the empty domain iff it is cofinal, thus Zakharyaschev's $\alpha(F, D, \bot)$ is our $\alpha(F, D \cup \{\emptyset\})$.

We are ready for the main result of this section.

Theorem 3.11 *Let* L *be a consistent normal extension of* $K4$*. The following are equivalent.*

(i) *L is extensible.*

(ii) *L can be represented as*

$$L = L_0 \oplus \bigoplus_{i\in I} \alpha(F_i, D_i),$$

where L_0 *is* $K4$*,* $S4$*, or* GL*, and the root of each* F_i *belongs to a proper cluster.*

(iii) *For every general* L*-frame* K*, the frame* K^* *validates* L*, whenever* $* \in \{\bullet, \circ\}$ *is such that* $* = \bullet$ *if* $L \supseteq GL$*, and* $* = \circ$ *if* $L \supseteq S4$*.*

Proof: (iii) → (i) follows from theorem 3.5, as a disjoint union of L-frames is an L-frame.

(ii) → (iii): let K be a general L-frame, and $*$ as in (iii). Clearly K^* is an L_0-frame. Let f be a subreduction of K^* to F_i which satisfies CDC for D_i. As the root of F_i belongs to a proper cluster, there is an x distinct from the root of K^* such that $f(x)$ is the root of F_i. Then $f \restriction K$ is a subreduction of K to F_i with CDC for D_i, a contradiction. Therefore K^* is an L-frame.

(i) → (ii): the core of the argument is the following property.

Claim 1 *Suppose* $\alpha(F, D) \in L$*. Let* F_i $(i < k)$ *be all rooted subframes of* F *generated by immediate successors of the root of* F*, and put* $D_i = \{d \in D;\ d \subseteq F_i\}$*. Assume further that one of the following holds:*

(i) *the root of* F *is irreflexive, and* L *admits* $A^\bullet$*,*

(ii) *the root of* F *is a simple reflexive cluster, and* L *admits* A°*.*

Then there exists $i < k$ *such that* $\alpha(F_i, D_i) \in L$*.*

Proof: Assume that for every $i < k$, $\alpha(F_i, D_i) \notin L$. For each i, let K_i be a descriptive L-frame, and f_i a subreduction from K_i to F_i with CDC for D_i. We may assume that K_i is rooted, and the root of K_i is in the domain of f_i. Let $* = \bullet$ or $\circ$ according to whether (i) or (ii) holds, and let $K = (\sum_i K_i)^*$. As L admits A^*, K is an L-frame by theorem 3.5. Let f be the partial mapping from K to F which extends $\bigcup_i f_i$, and maps the root r of K to the root of F. It is easy to see that f is a subreduction.

To show that f satisfies CDC for D, assume $x \in K \smallsetminus \mathrm{dom}(f)$ and $f(x\uparrow) = d \in D$. As $x \neq r \in \mathrm{dom}(f)$, we have $x \in K_i$ for some $i < k$. Then $d = f_i(x\uparrow) \subseteq F_i$, thus x witnesses that f_i violates CDC for D_i, a contradiction. Consequently $\alpha(F, D) \notin L$. □ (Claim 1)

Choose L_0 in the obvious way, and put

$$L' = L_0 \oplus \bigoplus \{\alpha(F, D);\ \text{root cluster of } F \text{ is proper},\ \alpha(F, D) \in L\}.$$

Clearly $L' \subseteq L$. The other inclusion $L \subseteq L'$ is by theorem 3.9 equivalent to

$$\alpha(F, D) \in L \Rightarrow \alpha(F, D) \in L',$$

which follows from claim 1 by induction on the depth of F. □

Corollary 3.12 *Every extensible logic is a union of a non-decreasing sequence of finitely axiomatizable extensible logics.* □

Theorem 3.11 implies that the only extensible extension of GL is GL itself (this was already noted in [9]). On the other hand, the intervals $[K4, K4Grz]$ and $[S4, S4Grz]$ each contain 2^ω extensible logics. Infinitely many of them are finitely axiomatizable (witness e.g. $S4 \oplus \alpha(C_n, \emptyset)$, where C_n is the n-element cluster, $n \geq 2$).

Example 3.13 Some well-known extensible logics can be represented as follows:

$$\begin{aligned} S4Grz &= S4 \oplus \alpha(C_2, \emptyset), \\ S4.1 &= S4 \oplus \alpha(C_2, \{\emptyset\}), \\ K4Grz &= K4 \oplus \alpha(C_2, \emptyset), \\ K4.1 &= K4 \oplus \alpha(C_2, \{\emptyset\}), \end{aligned}$$

where C_2 is the 2-element cluster.

Example 3.14 Dummett's logic

$$Dum = S4 \oplus \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Diamond\Box\varphi \to \varphi)$$

is not extensible. We have $Dum = S4 \oplus \alpha(F_1, \emptyset) \oplus \alpha(F_2, \emptyset)$, where F_1 and F_2 are the frames depicted in figure 1 (cf. [3]). By the proof of theorem 3.11, any extensible logic which contains $\alpha(F_2, \emptyset)$ also proves $\alpha(C_2, \emptyset) = (Grz)$, but $Dum \subsetneq S4Grz$.

On the other hand, the closely related logic (called $S4.1.4$ by Zeman [15])

$$S4 \oplus \Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to (\Box\Diamond\Box\varphi \to \varphi) = S4 \oplus \alpha(F_1, \emptyset)$$

is extensible.

Figure 1: forbidden subframes for *Dum*

4 Equivalence of Frege systems for extensible logics

In this section, we are going prove our main result (theorem 4.8): Frege systems for any extensible logic are p-equivalent. We begin with a few simple observations.

Lemma 4.1 ([12]) *Let F and G be inference systems, such that G is finite, and all rules of G are derivable in F. Then $G \leq_p F$.*

Proof: For each rule $\varrho \in G$, fix a derivation π_ϱ of ϱ in F. Given a G-proof π, construct an equivalent F-proof by replacing each application of $\varrho \in G$ in π with the appropriate substitution instance of π_ϱ. □

Definition 4.2 Let $L = K \oplus \varphi_1 \oplus \cdots \oplus \varphi_k$ be a finitely axiomatizable normal modal logic. The *standard Frege system* F_{std} for L consists of the rules (MP) and (Nec), a finite axiomatization of *CPC*, the Kripke axiom (K), and the axioms $\varphi_1, \ldots, \varphi_k$. (Notice that all standard Frege systems for the same logic are p-equivalent by lemma 4.1.)

Lemma 4.3 *Let L be a finitely axiomatizable normal extension of $K4$. Then any Frege system F for L p-simulates F_{std}.*

Proof: Let F_0 be the Frege system consisting of Modus Ponens, and axioms φ, $\Box\varphi$, for any axiom φ of F_{std} (including explicitely the axiom (4)). We have $F \geq_p F_0$ by lemma 4.1, it thus suffices to show $F_0 \geq_p F_{std}$.

Given an F_{std}-proof $\pi = \varphi_0, \ldots, \varphi_n$ of a formula $\varphi = \varphi_n$, we construct the sequence $\varphi_0, \Box\varphi_0, \ldots, \varphi_n, \Box\varphi_n$, and complete it to an F_0-proof by inserting instances of (K) and (4): when a formula φ was inferred in π by (MP) from φ_j and $\varphi_k = (\varphi_j \to \varphi_i)$, we include the axiom

$$\Box(\varphi_j \to \varphi_i) \to (\Box\varphi_j \to \Box\varphi_i),$$

and derive $\Box\varphi_i$ from $\Box\varphi_j$ and $\Box(\varphi_j \to \varphi_i)$ by two applications of (MP). When $\varphi_i = \Box\varphi_j$ was inferred from φ_j by (Nec), we use in a similar fashion the axiom

$$\Box\varphi_j \to \Box\Box\varphi_j$$

to derive $\Box\Box\varphi_j$. □

Definition 4.4 Let L be a finitely axiomatizable extensible logic. For definiteness, we assume that F_{std} for L is given by its representation

$$L = L_0 \oplus \bigoplus_{i<k} \alpha(F_i, D_i)$$

from theorem 3.11. The inference system F_{adm} consists of F_{std}, and the infinitely many rules

- $\widehat{A}^\bullet$, if $L_0 \neq S4$,
- $\widehat{A}^\circ$, if $L_0 \neq GL$.

Corollary 4.5 *Let L be a finitely axiomatizable extensible logic. Then F_{adm} p-simulates any Frege system for L.*

Proof: Use theorem 2.1 and lemma 4.1. □

Lemma 4.6 *Let L be a extensible logic. There exists a Frege system for L if and only if L is finitely axiomatizable.*

Proof: The "if" direction is trivial. Let F be a finite set of Frege rules which is complete for L. By theorem 2.1, we may assume that $F = F_{std} \cup F'$, where F_{std} is the standard Frege system of a finitely axiomatized logic $L' \subseteq L$, and F' consists of instances of $\widehat{A}^\bullet$ or $\widehat{A}^\circ$. By corollary 3.12, we may assume that L' is extensible. Then L' admits the rules from F', thus $L = L'$ is finitely axiomatizable. □

Definition 4.7 A *propositional valuation* is an assignment of truth values $0, 1$ to modal formulas, which respects Boolean connectives. To stress the analogy with the intuitionistic case, we will denote propositional valuations by the slash symbol $|$, and we will write $|\varphi$ instead of $|(\varphi) = 1$.

Theorem 4.8 *Let L be a extensible modal logic. Then any two Frege systems for L (in the basic modal language) are polynomially equivalent.*

Proof: By lemma 4.6, we may assume that L is finitely axiomatizable. By lemma 4.3 and corollary 4.5, it suffices to show $F_{adm} \leq_p F_{std}$.

We define an auxiliary Frege system F_1, which consists of F_{std}, the relativized necessitation rule

$$\text{(RNec)} \qquad \boxdot\varphi \to \psi \vdash \boxdot\varphi \to \Box\psi,$$

and finitely many propositional rules. (We do not list them explicitely, we will simply use freely propositional reasoning in the rest of the proof; the reader can easily verify that a finite list of extra rules is sufficient to support the argument.) As $F_1 \leq_p F_{std}$ by lemma 4.1, it is sufficient to show $F_{adm} \leq_p F_1$.

Assume we are given an F_{adm}-proof π of a formula Φ. Let $Sub(\pi)$ be the set of subformulas of all formulas from π,

$$S = Sub(\pi) \cup \{\varphi \to \psi;\ \varphi, \psi \in Sub(\pi)\},$$

and let $P(\varphi)$ denote the property

$$\exists\Pi \subseteq S\ \Pi \text{ is an } F_1\text{-proof of } \varphi.$$

Claim 1 *If $L_0 \neq S4$, $P(\Box\chi \to \bigvee_i \Box\omega_i)$, and $\boxdot\chi \in Sub(\pi)$, then $P(\boxdot\chi \to \omega_i)$ for some i.*

Proof: Define a propositional valuation $|$ by

$$|\Box\varphi \quad \text{iff} \quad P(\boxdot\chi \to \varphi).$$

As $P(\boxdot\chi \to \chi)$, i.e., $|\Box\chi$, it suffices to show that $P(\varphi)$ implies $|\varphi$, which we verify by induction on the length of an F_1-proof $\Pi \subseteq S$ of φ.

The steps for propositional rules are trivial.

(K): if $P(\boxdot\chi \to (\varphi \to \psi))$ and $P(\boxdot\chi \to \varphi)$, we get $P(\boxdot\chi \to \psi)$ by propositional reasoning.

(Nec): $P(\varphi)$ and $P(\Box\varphi)$ (hence $\varphi \in Sub(\pi)$) imply $P(\boxdot\chi \to \varphi)$ by propositional reasoning.

(RNec): assume $P(\boxdot\varphi \to \psi)$, and $|\boxdot\varphi$. Then $P(\boxdot\chi \to \varphi)$, thus $P(\boxdot\chi \to \Box\varphi)$ by (RNec), and $P(\boxdot\chi \to \psi)$ by propositional reasoning, which means $|\Box\psi$.

(4): assume $P(\Box\varphi \to \Box\Box\varphi)$ and $|\Box\varphi$, i.e., $P(\boxdot\chi \to \varphi)$. Then $P(\boxdot\chi \to \Box\varphi)$ by (RNec), thus $|\Box\Box\varphi$.

(GL): assume $P(\Box(\Box\varphi \to \varphi) \to \Box\varphi)$ and $|\Box(\Box\varphi \to \varphi)$, i.e., $P(\boxdot\chi \to (\Box\varphi \to \varphi))$. Then $P(\boxdot\chi \to \Box(\Box\varphi \to \varphi))$ by (RNec) and $P(\boxdot\chi \to \varphi)$ by propositional reasoning.

$\alpha(F, D)$: we have $\varphi = \tau \to \varphi_0$, where τ has the form

$$\boxdot(\Box\varphi_0 \to \varphi_0) \wedge \bigwedge_i \boxdot\psi_i$$

as the root of F is reflexive. Assume $P(\varphi)$ and $|\tau$. Then $P(\boxdot\chi \to (\Box\varphi_0 \to \varphi_0))$, and $P(\boxdot\chi \to \psi_i)$, thus $P(\boxdot\chi \to \tau)$ by (RNec) and propositional reasoning. This propositionally implies $P(\boxdot\chi \to \varphi_0)$, i.e., $|\Box\varphi_0$. As we have $|(\Box\varphi_0 \to \varphi_0)$ from $|\tau$, we obtain $|\varphi_0$. $\Box$ (Claim 1)

Claim 2 *Assume $L_0 \neq GL$, $P(\bigwedge_j(\chi_j \equiv \Box\chi_j) \to \bigvee_i \Box\omega_i)$, and $\boxdot \bigwedge_j \chi_j \in Sub(\pi)$. Then $P(\boxdot \bigwedge_j \chi_j \to \omega_i)$ for some i.*

Proof: We define a propositional valuation $|$ as

$$|\Box\varphi \quad \text{iff} \quad P(\boxdot\chi \to \varphi) \wedge |\varphi$$

by induction on the complexity of φ, where $\chi := \bigwedge_j \chi_j$. As $P(\boxdot\chi \to \chi_j)$ by propositional reasoning, we have $|(\bigwedge_j(\chi_j \equiv \Box\chi_j))$. Therefore it suffices to verify that $P(\varphi)$ implies $|\varphi$, which we show again by induction on the length of proof.

The steps for propositional rules and the axiom (T) are trivial.

(K): if $P(\boxdot\chi \to (\varphi \to \psi))$ and $P(\boxdot\chi \to \varphi)$, we get $P(\boxdot\chi \to \psi)$ by propositional reasoning, and $|(\varphi \to \psi)$, $|\varphi$ imply $|\psi$.

(Nec): $P(\varphi)$ and $P(\Box\varphi)$ imply $P(\boxdot\chi \to \varphi)$ by propositional reasoning, and we have $|\varphi$ by the induction hypothesis.

(RNec): assume $P(\boxdot\varphi \to \psi)$, and $|\boxdot\varphi$. Then $P(\boxdot\chi \to \varphi)$, thus $P(\boxdot\chi \to \Box\varphi)$ by (RNec), and $P(\boxdot\chi \to \psi)$ by propositional reasoning. We have $|(\boxdot\varphi \to \psi)$ by the induction hypothesis, which implies $|\psi$ and $|\Box\psi$.

(4): assume $P(\Box\varphi \to \Box\Box\varphi)$ and $|\Box\varphi$. Then $P(\boxdot\chi \to \varphi)$, and $P(\boxdot\chi \to \Box\varphi)$ by (RNec), thus $|\Box\Box\varphi$.

$\alpha(F, D)$: we have $\varphi = \tau \to \varphi_0$, where τ has the form

$$\boxdot(\Box\varphi_1 \to \varphi_0) \land \boxdot(\Box\varphi_0 \to \varphi_1) \land \boxdot(\varphi_0 \lor \varphi_1) \land \bigwedge_i \boxdot\psi_i,$$

as the root cluster of F is proper. Assume $P(\varphi)$ and $|\tau$. Then $P(\boxdot\chi \to (\Box\varphi_1 \to \varphi_0))$, $P(\boxdot\chi \to (\Box\varphi_0 \to \varphi_1))$, $P(\boxdot\chi \to \varphi_0 \lor \varphi_1)$, and $P(\boxdot\chi \to \psi_i)$. By (RNec) and propositional reasoning, we have $P(\boxdot\chi \to \varphi_0)$, $P(\boxdot\chi \to \Box\varphi_0)$, and $P(\boxdot\chi \to \varphi_1)$. We have also $|(\varphi_0 \lor \varphi_1)$. If $|\varphi_0$, we are done; otherwise $|\varphi_1$, thus $|\Box\varphi_1$, which implies $|\varphi_0$ as $|(\Box\varphi_1 \to \varphi_0)$. □ (Claim 2)

Claim 3 *If $P(\bigvee_i \Box\omega_i)$, then $P(\omega_i)$ for some i.*

Proof: By omitting $\boxdot\chi$ from the proof of claim 1 or claim 2, whichever is applicable. □ (Claim 3)

We resume the proof of the main theorem. We show by induction on the length of proof that

$$\varphi \in \pi \Rightarrow P(\varphi).$$

The induction steps for rules of F_{std} are straightforward, as $F_{std} \subseteq F_1$, and $\pi \subseteq S$. Consider an instance

$$\Box(\Box\varphi \to \bigvee_{i<n} \Box\psi_i) \lor \Box\chi \vdash \bigvee_{i<n} \boxdot(\boxdot\varphi \to \psi_i) \lor \chi$$

of $\widehat{A}^\bullet$. We have

$$P(\Box(\Box\varphi \to \bigvee_{i<n} \Box\psi_i) \lor \Box\chi)$$

by the induction hypothesis. By claim 3, $P(\chi)$ or

$$P(\Box\varphi \to \bigvee_{i<n} \Box\psi_i).$$

In the latter case, we get $P(\boxdot\varphi \to \psi_i)$ for some $i < n$ by claim 1. Using necessitation and propositional reasoning, we obtain

$$P(\bigvee_{i<n} \boxdot(\boxdot\varphi \to \psi_i) \lor \chi).$$

Instances of $\widehat{A}^\circ$ are handled similarly, using claim 2.

In particular, $P(\Phi)$ holds, i.e., there is an F_1-proof Π of Φ such that $\Pi \subseteq S$. The remaining task is to construct such Π from π in polynomial time, which can be easily accomplished by a standard algorithm. We iteratively compute the set P_d of formulas which have an F_1-proof $\Pi \subseteq S$ of depth at most d. On each iteration, we try to prove every formula from $S \smallsetminus P_d$ by a single application of an F_1-rule to formulas from P_d. We will reach Φ after at most $|S|$ iterations, thus the algorithm runs in polynomial time. □

The proof of theorem 4.8 (specifically, claim 3) implies another interesting result.

Definition 4.9 A proof system P has the *feasible disjunction property*, if there exists a polynomial-time algorithm which transforms a P-proof of $\bigvee_i \Box\varphi_i$ into a P-proof of one of the formulas φ_i.

Corollary 4.10 *Frege systems for any extensible modal logic have the feasible disjunction property.* $\Box$

This corollary generalizes the results of Ferrari et al. [5], who have shown the feasible disjunction property of the natural deduction system for $S4$, $S4.1$, and $S4Grz$, and of the Frege system for GL. The proofs are apparently based on a similar intuition; the main difference is that we do not use the complicated machinery of extraction calculi.

We also mention that the proof of the main result of Mints and Kojevnikov [11] can be simplified along the lines of our theorem 4.8. In the original proof, instances of Visser's rule

$$(V_n) \qquad \Big(\bigwedge_{i<n}(\alpha_i \to \beta_i) \to \alpha_n \vee \alpha_{n+1}\Big) \vee \chi \vdash \bigvee_{j\le n+1} \Big(\bigwedge_{i<n}(\alpha_i \to \beta_i) \to \alpha_j\Big) \vee \chi$$

are successively eliminated from a Frege proof by translating the proof to natural deduction, applying an efficient version of Kleene's slash à la [5], and translating the proof back to the Frege system. The basic steps in this transformation are polynomial-time, but a polynomial increase in length iterated polynomially many times may result in doubly exponential increase in general; thus Mints and Kojevnikov need to establish delicate tight bounds to show that the net effect is in fact only polynomial. We outline below how to eliminate the use of natural deduction from the argument.

We consider intuitionistic logic formulated in the language $\{\to, \vee, \wedge, \bot\}$, and we fix the intuitionistic standard Frege system F_{std} consisting of (MP), and the axioms

$$(A1) \qquad (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$$

$$(A2) \qquad \varphi \to (\psi \to \varphi)$$

$$(A3) \qquad \varphi_1 \wedge \varphi_2 \to \varphi_i$$

$$(A4) \qquad \varphi_1 \to (\varphi_2 \to \varphi_1 \wedge \varphi_2)$$

$$(A5) \qquad \varphi_i \to \varphi_1 \vee \varphi_2$$

$$(A6) \qquad (\varphi_1 \to \psi) \to ((\varphi_2 \to \psi) \to (\varphi_1 \vee \varphi_2 \to \psi))$$

$$(A7) \qquad \bot \to \varphi$$

where $i = 1, 2$. As the rules V_n form a basis of admissible rules for IPC by Iemhoff [8], it suffices to show that F_{std} p-simulates $F_{adm} := F_{std} + \{V_n;\, n \in \omega\}$.

First we need an analogy to definition 4.7. If $P(\varphi)$ is any property of intuitionistic formulas, a *P-slash* is a (classical) valuation $|\varphi$ of intuitionistic formulas which satisfies the conditions

$$\begin{aligned}|(\varphi \to \psi) &\Leftrightarrow (\|\varphi \to |\psi),\\ |(\varphi \wedge \psi) &\Leftrightarrow (|\varphi \wedge |\psi),\\ |(\varphi \vee \psi) &\Leftrightarrow (\|\varphi \vee \|\psi),\\ |\bot &\Leftrightarrow \bot,\end{aligned}$$

where $\|\varphi$ is defined by

$$\|\varphi \Leftrightarrow (P(\varphi) \wedge |\varphi).$$

For example, Kleene's slash is a $\vdash$-slash, i.e., a P-slash for the property $P(\varphi)$ iff "φ is provable".

In the modal case, propositional valuations automatically satisfy propositional axioms. In the intuitionistic case, we have the following substitute.

Lemma 4.11 *Let $|$ be a P-slash for an arbitrary P. Then $|\varphi$ holds for all instances of axioms* (A2)–(A7).

Proof: Consider for example (A6) (which is actually the most complicated case). Unwinding the definition reveals that we have to show that $\|(\varphi_1 \to \psi)$, $\|(\varphi_2 \to \psi)$ and $\|(\varphi_1 \vee \varphi_2)$ imply $|\psi$. Since $|(\varphi_1 \vee \varphi_2)$, we have $\|\varphi_i$ for some i, thus $|\psi$ follows from $|(\varphi_i \to \psi)$. □

The next lemma is an analogue of claim 3 in theorem 4.8.

Lemma 4.12 *Let π be an F_{std}-proof of $\varphi_1 \vee \cdots \vee \varphi_k$. Then the closure of π under* (MP) *contains one of the formulas φ_i.*

Proof: Let Π be the closure of π under (MP), let $P(\varphi)$ denote the property $\varphi \in \Pi$, and let $|$ be a P-slash. By the definition of $|$, it suffices to show that $|\varphi$ holds for every formula $\varphi \in \pi$, and we prove this by induction on the length of proof.

Axioms (A2)–(A7) are handled by lemma 4.11. To see that $|(A1)$ holds, assume $\|(\varphi \to (\psi \to \chi))$, $\|(\varphi \to \psi)$, and $\|\varphi$, we will show $|\chi$. Since $\|\varphi$, the other assumptions imply $|(\psi \to \chi)$ and $|\psi$. Moreover $P(\varphi)$ and $P(\varphi \to \psi)$ imply $P(\psi)$ since Π is closed under (MP), thus $\|\psi$, and $|\chi$.

Assume that ψ was derived by (MP) from φ and $\varphi \to \psi$. We have $|\varphi$ and $|(\varphi \to \psi)$ from the induction hypothesis, and $P(\varphi)$ as $\varphi \in \pi \subseteq \Pi$, thus $\|\varphi$, and $|\psi$. □

Now we can prove the Mints–Kojevnikov theorem.

Theorem 4.13 ([11]) *All intuitionistic Frege systems in the language $\{\to, \wedge, \vee, \bot\}$ are polynomially equivalent.*

Proof: Let π be an F_{adm}-proof of a formula Φ, we want to construct an F_{std}-proof of Φ. We may assume that π contains F_{std}-subproofs of formulas $\alpha \to \alpha$, for every α appearing in an instance of (V_n) used in π. Let S be the set of all subformulas of formulas from π, and let R

be the set of all formulas of the form

$$\begin{array}{ll} \varphi & (\varphi \to \psi) \to (\varphi \to \chi) \\ \varphi \to \psi & (\varphi \to \psi) \to (\omega \to (\varphi \to \psi)) \\ \varphi \to (\psi \to \chi) & (\varphi \to (\psi \to \chi)) \to (\omega \to (\varphi \to (\psi \to \chi))) \\ \omega \to (\varphi \to (\psi \to \chi)) & (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) \end{array}$$

where $\varphi, \psi, \chi, \omega \in S$. The point of this definition is that R has the following properties: R is closed under subformulas (hence under (MP)), contains π, and satisfies

- $((\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))) \in R$ if and only if $\varphi, \psi, \chi \in S$,
- if $\varphi \to \psi \in R$ and $\alpha \in S$, then $\varphi \to (\alpha \to \varphi) \in R$.

We define

$$P(\varphi) \Leftrightarrow \exists \Pi \subseteq R \; \Pi \text{ is an } F_{std}\text{-proof of } \varphi,$$
$$P_\alpha(\varphi) \Leftrightarrow P(\alpha \to \varphi),$$

and let $\alpha|\varphi$ be a P_α-slash.

Claim 1 *If $\alpha \in S$, and $P(\varphi)$, then $\alpha|\varphi$.*

Proof (sketch): Let $\Pi \subseteq R$ be an F_{std}-proof of φ, we prove $\alpha|\varphi$ by induction on the length of Π. The steps for axioms (A2)–(A7) follow from lemma 4.11. The steps for (A1) and (MP) can be shown by an easy manipulation of the slashes, using the above mentioned closure properties of R. □ (Claim 1)

As in theorem 4.8, it suffices to demonstrate that Φ has an F_{std}-proof using only formulas from R, i.e., $P(\Phi)$. We show $P(\varphi)$ for every $\varphi \in \pi$ by induction on the length of the subproof of φ. The only non-trivial case is the induction step for (V_n). Consider an instance

$$(\alpha \to \alpha_n \vee \alpha_{n+1}) \vee \chi \vdash \bigvee_{j \leq n+1} (\alpha \to \alpha_j) \vee \chi$$

of (V_n) in π, where $\alpha = \bigwedge_{i<n}(\alpha_i \to \beta_i)$. By the induction hypothesis, we have $P((\alpha \to \alpha_n \vee \alpha_{n+1}) \vee \chi)$. Since R is closed under (MP), lemma 4.12 implies $P(\chi)$ or $P(\alpha \to \alpha_n \vee \alpha_{n+1})$. In the latter case, we use claim 1 to get $\alpha|(\alpha \to \alpha_n \vee \alpha_{n+1})$, i.e., $\alpha\|\alpha \to \alpha|(\alpha_n \vee \alpha_{n+1})$. Since $P_\alpha(\alpha)$, it is easy to see that $\alpha\|\alpha_i$ for some $i \leq n+1$, thus $P(\alpha \to \alpha_i)$.

In each case, we obtained $P(\omega)$ where ω is one of the disjuncts in the conclusion of (V_n). We can easily extend the proof of ω to a proof of the whole disjunction, because $\omega_i \to \omega_1 \vee \omega_2$ is in R whenever $\omega_1 \vee \omega_2 \in S$. □

5 A few remarks

Due to the difficulties mentioned in the introduction, we stated the main theorem only for Frege systems in the basic modal language. Nevertheless, we actually do have some degree of freedom in the choice of the language. The proof of theorem 4.8 (with minor modifications) still works if we replace $\{\to, \bot\}$ with any complete set of Boolean connectives, and we may likewise replace (or combine) $\Box$ with $\Diamond$, or with the strict implication $\varphi \Rightarrow \psi := \Box(\varphi \to \psi)$. (We do not know whether we can take an *arbitrary* complete set of definable connectives as basic.)

If we consider Frege systems F_1 and F_2 formulated in different languages B_1 and B_2, we only know how to p-simulate them in the trivial case where the languages are polynomially translatable to each other: that is, if every connective from B_1 is definable by a B_2-formula with at most one occurence of each variable, and vice versa.

As in the classical logic, we may also define modal *Extended Frege* proof systems [4], either by introducing the extension rule, or by allowing modal circuits instead of formulas in proofs. We can obtain easily a modification of our main theorem: all Extended Frege systems for a given extensible modal logic are polynomially equivalent. In this case, we do not need any restrictions on the languages of the proof systems.

We have only considered extensible logics, it is an interesting question for which other modal (or intermediate) logics the p-equivalence of Frege systems holds. We mention that there is a trivial affirmative answer for all extensions of $S4.3$ (in particular, for $S5$). As shown by Rybakov [13], admissible rules in such logics have a basis consisting of the single rule

$$\Diamond\varphi, \Diamond\neg\varphi \vdash \bot.$$

However, F_{std} extended by this rule is p-equivalent (in fact, identical) to F_{std}: the extra rule cannot appear in any Frege proof, because its conclusion is inconsistent. On the other hand, there are many important modal logics for which a description of admissible rules is known, yet it is not clear how to modify our methods to establish p-equivalence of their Frege systems; examples include $S4.2$ and $K4.3$. Extended Frege systems may be easier to analyze than Frege systems; for example, it is not hard to show that all EF systems for $GL.3$ are p-equivalent, whereas the corresponding problem for Frege systems is open.

To generalize the question in other way, we may reformulate our results as follows: if $A = A^\bullet$ or $A = A^\circ$, then A is feasibly admissible in any logic which admits A. Are there other natural sets of rules which share this "automatic feasibility" property?

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