A note on the substructural hierarchy

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Abstract

We prove that all axiomatic extensions of the full Lambek calculus with exchange can be axiomatized by formulas on the $\mathcal{N}_3$ level of the substructural hierarchy.

1 Introduction

A standard technique for reduction of the complexity of propositional formulas (nesting depth of connectives) in proof complexity and other branches of logic is to introduce *extension variables*: we name each subformula by a new propositional variable, and include appropriate clauses forcing the variables to be equivalent to the original formulas. This idea may have been independently discovered multiple times; in the context of classical logic, extension variables appear in the work of Tseitin [5]. Extension variables are systematically used by Rybakov [4] for the purpose of reducing the formula complexity of nonclassical consequence relations. It may not be immediately obvious that the method also applies to axioms of substructural logics without contraction, but as we will see, this can be done with just a little care.

The context we are specifically interested in is the *substructural hierarchy* introduced by Ciabattoni, Galatos, and Terui [1, 2], stratifying formulas of the full Lambek calculus (**FL**) into classes $\mathcal{P}_k$ and $\mathcal{N}_k$, $k \in \omega$, based on alternation of polarities of connectives. As shown in [1, 2], $\mathcal{N}_2$-axiomatized extensions of **FL** can be equivalently expressed by structural rules in the sequent calculus, and similarly, $\mathcal{P}_3$ axioms (with certain restrictions) can be expressed by structural hypersequent rules; moreover, analyticity (subformula property) of the resulting calculi can be characterized algebraically by closure under a certain kind of completion.

We are going to prove that—at least when the base logic is commutative ($\mathbf{FL_e}$)—all remaining axiomatic extensions already appear at the lowest level of the hierarchy not covered by their results, namely $\mathcal{N}_3$.

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2 Preliminaries

We refer the reader to Galatos et al. [3] for comprehensive information on **FL** and its extensions, however we include a few words below to clarify our terminology and notation.

The language of $\mathbf{FL_e}$ consists of propositional formulas generated from a countable set of variables $p_0, p_1, \dots$ using the connectives $\to, \cdot, \wedge, \vee, 0, 1$. We might also include the lattice constants $\bot, \top$; none of our results depend on their presence or absence (actually, our arguments only rely on the availability of $\to, \wedge, 1$, and $\to$ alone suffices over $\mathbf{FL_{ei}}$). We abbreviate

$$(\varphi \leftrightarrow \psi) = (\varphi \to \psi) \wedge (\psi \to \varphi),$$
$$\prod_{i<n} \varphi_i = \varphi_0 \cdot \varphi_1 \cdots \varphi_{n-1},$$
$$\varphi^n = \prod_{i<n} \varphi,$$

with the understanding that the empty product is 1. We write $\psi \subseteq \varphi$ if ψ is a subformula of φ; usually, we will need to count multiple occurrences of ψ in φ as distinct subformulas.

We employ the notational convention that $\to$ and $\leftrightarrow$ bind weaker than other connectives, so that for instance,

$$\prod_{i<3} \varphi_i \to \psi = \varphi_0 \cdot \varphi_1 \cdot \varphi_2 \to \psi = (\varphi_0 \cdot \varphi_1 \cdot \varphi_2) \to \psi.$$

The logic $\mathbf{FL_e}$ can be naturally presented by a sequent calculus, but it will be more convenient for our purposes to define it using a Hilbert-style calculus: it is axiomatized by a handful of axiom schemata listed in [3, Fig. 2.9], and the two rules

$$\varphi, \varphi \to \psi \; / \; \psi, \tag{1}$$
$$\varphi \; / \; \varphi \wedge 1. \tag{2}$$

If X is a set of formulas, $\mathbf{FL_e} + X$ denotes the extension of $\mathbf{FL_e}$ with substitution instances of formulas from X as additional axioms. If $L = \mathbf{FL_e} + X$, and $\Gamma \cup \{\varphi\}$ is a set of formulas, we write $\Gamma \vdash_L \varphi$ if φ has a derivation in the calculus of L from a set of premises included in Γ. We will identify L with its consequence relation $\vdash_L$. Logics of the form $\mathbf{FL_e} + X$ are called *axiomatic extensions* of $\mathbf{FL_e}$. (In general, an *extension* of $\mathbf{FL_e}$ is a Tarski-style consequence relation that contains $\vdash_{\mathbf{FL_e}}$ and is closed under substitution. However, we are not interested in non-axiomatic extensions in this paper.)

Let $\mathcal{PCRL}$ denote the variety of *pointed commutative residuated lattices*: i.e., structures $\langle L, \to, \cdot, 1, \wedge, \vee, 0\rangle$ such that $\langle L, \cdot, 1\rangle$ is a commutative monoid, $\langle L, \wedge, \vee\rangle$ is a lattice, and

$$x \le y \to z \quad \text{iff} \quad x \cdot y \le z$$

for all $x, y, z \in L$, where $x \le y$ denotes the lattice order $x \wedge y = x$.

The logic $\mathbf{FL_e}$ is *algebraizable* wrt $\mathcal{PCRL}$:

$$\varphi_1, \dots, \varphi_n \vdash_{\mathbf{FL_e}} \varphi_0 \quad \text{iff} \quad 1 \wedge \varphi_1 \approx 1, \dots, 1 \wedge \varphi_n \approx 1 \models_{\mathcal{PCRL}} 1 \wedge \varphi_0 \approx 1,$$
$$\varphi_1 \approx \psi_1, \dots, \varphi_n \approx \psi_n \models_{\mathcal{PCRL}} \varphi_0 \approx \psi_0 \quad \text{iff} \quad \varphi_1 \leftrightarrow \psi_1, \dots, \varphi_n \leftrightarrow \psi_n \vdash_{\mathbf{FL_e}} \varphi_0 \leftrightarrow \psi_0,$$
$$1 \wedge \varphi \leftrightarrow 1 \vdash_{\mathbf{FL_e}} \varphi,$$
$$1 \wedge (\varphi \leftrightarrow \psi) \approx 1 \models_{\mathcal{PCRL}} \varphi \approx \psi.$$

In particular, $\mathbf{FL_e}$ is an *equivalential* logic with equivalence connective $\leftrightarrow$:

Lemma 2.1 *For any formulas $\varphi, \psi, \chi, \varphi', \psi'$, we have*

$$\vdash_{\mathbf{FL_e}} \varphi \leftrightarrow \varphi, \tag{3}$$
$$\varphi \leftrightarrow \psi, \varphi \leftrightarrow \chi \vdash_{\mathbf{FL_e}} \psi \leftrightarrow \chi, \tag{4}$$
$$\varphi, \varphi \leftrightarrow \psi \vdash_{\mathbf{FL_e}} \psi, \tag{5}$$
$$\varphi \leftrightarrow \varphi', \psi \leftrightarrow \psi' \vdash_{\mathbf{FL_e}} (\varphi \circ \psi) \leftrightarrow (\varphi' \circ \psi') \tag{6}$$

where $\circ \in \{\to, \cdot, \wedge, \vee\}$. □

We will also use the *local deduction theorem* for $\mathbf{FL_e}$ [3, Cor. 2.15]. We include a short proof for convenience.

Lemma 2.2 *Let L be an axiomatic extension of $\mathbf{FL_e}$. If $\Gamma, \varphi \vdash_L \psi$, then*

$$\Gamma \vdash_L (\varphi \wedge 1)^n \to \psi \tag{7}$$

for some $n \in \omega$. If π is a (tree-like) L-derivation of ψ from $\Gamma \cup \{\varphi\}$, we may take for n the number of times the premise φ is used in π.

Proof: By induction on the length of π. If ψ is an axiom of L, or $\psi \in \Gamma$, we can derive $1 \to \psi$ from ψ, and $1 = (\varphi \wedge 1)^0$ by definition. If $\psi = \varphi$, we have $\vdash_L \varphi \wedge 1 \to \varphi$.

If ψ is derived from χ and $\chi \to \psi$ by (1), we have

$$\begin{aligned} &\Gamma \vdash_L (\varphi \wedge 1)^n \to \chi, \\ &\Gamma \vdash_L (\varphi \wedge 1)^m \to (\chi \to \psi) \end{aligned} \tag{8}$$

by the induction hypothesis, which implies

$$\Gamma \vdash_L (\varphi \wedge 1)^{n+m} \to \psi$$

using

$$\vdash_{\mathbf{FL_e}} \alpha \cdot (\alpha \to \beta) \to \beta,$$
$$\alpha \to \beta, \alpha' \to \beta' \vdash_{\mathbf{FL_e}} \alpha \cdot \beta \to \alpha' \cdot \beta'. \tag{9}$$

Finally, if $\psi = \chi \wedge 1$ is derived from χ by (2), the induction hypothesis gives (8). Using $\vdash_{\mathbf{FL_e}} \varphi \wedge 1 \to 1$, $\vdash_{\mathbf{FL_e}} 1^n \to 1$, and (9), we also have

$$\vdash_{\mathbf{FL_e}} (\varphi \wedge 1)^n \to 1,$$

which together with (8) yields $\Gamma \vdash_L (\varphi \wedge 1)^n \to \chi \wedge 1$. □

3 Substructural hierarchy

The substructural hierarchy introduced in [1, 2] consists of sets of formulas $\mathcal{P}_k$ and $\mathcal{N}_k$ for $k \in \omega$, generated by the closure conditions below.

Definition 3.1 $\mathcal{P}_k$ and $\mathcal{N}_k$ are the smallest sets of formulas with the following properties:

- $\mathcal{P}_0 = \mathcal{N}_0$ is the set of propositional variables.
- $\mathcal{P}_k \cup \mathcal{N}_k \subseteq \mathcal{P}_{k+1} \cap \mathcal{N}_{k+1}$.
- If $\varphi, \psi \in \mathcal{P}_{k+1}$, then $\varphi \cdot \psi$, $\varphi \vee \psi$, 1, and $\bot$ are also in $\mathcal{P}_{k+1}$.
- If $\varphi, \psi \in \mathcal{N}_{k+1}$, then $\varphi \wedge \psi$, 0, and $\top$ are also in $\mathcal{N}_{k+1}$.
- If $\varphi \in \mathcal{P}_{k+1}$ and $\psi \in \mathcal{N}_{k+1}$, then $\varphi \to \psi$ is in $\mathcal{N}_{k+1}$.

The two groups of connectives[1] implicit in the definition arise from the sequent calculus formulation of $\mathbf{FL_e}$: the left introduction rules for $\cdot, \vee, 1, \bot$, and the right introduction rules for $\to, \wedge, 0, \top$, are invertible.

Our main result shows that for the purpose of classification of axioms over $\mathbf{FL_e}$, the hierarchy collapses to $\mathcal{N}_3$.

Theorem 3.2 *Every axiomatic extension of* $\mathbf{FL_e}$ *is axiomatizable by* $\mathcal{N}_3$ *formulas.*

Proof: Fix an axiom φ; we will construct an $\mathcal{N}_3$ formula φ' such that $\mathbf{FL_e} + \varphi = \mathbf{FL_e} + \varphi'$.

For each occurrence of a subformula $\psi \subseteq \varphi$, we consider a fresh propositional variable p_ψ, and an associated extension axiom

$$E_\psi = \begin{cases} p_\psi \leftrightarrow \psi & \text{if } \psi \text{ is a variable or a constant,} \\ p_\psi \leftrightarrow (p_{\psi_0} \circ p_{\psi_1}) & \text{if } \psi = \psi_0 \circ \psi_1,\ \circ \in \{\to, \cdot, \wedge, \vee\}. \end{cases}$$

Notice that being an equivalence between a variable and a $\mathcal{P}_1$ or $\mathcal{N}_1$ formula, $E_\psi \in \mathcal{N}_2$.

First, we claim that

$$\{E_\chi : \chi \subseteq \psi\} \vdash_{\mathbf{FL_e}} p_\psi \leftrightarrow \psi, \qquad \psi \subseteq \varphi. \tag{10}$$

We prove this by induction on the complexity of ψ. If ψ is a variable or a constant, the right-hand side of (10) is just E_ψ. If $\psi = \psi_0 \circ \psi_1$, we have

$$\{E_\chi : \chi \subseteq \psi\} \vdash_{\mathbf{FL_e}} p_{\psi_0} \leftrightarrow \psi_0, p_{\psi_1} \leftrightarrow \psi_1 \vdash_{\mathbf{FL_e}} (p_{\psi_0} \circ p_{\psi_1}) \leftrightarrow \psi$$

by the induction hypothesis and (6), hence

$$\{E_\chi : \chi \subseteq \psi\} \vdash_{\mathbf{FL_e}} p_\psi \leftrightarrow \psi$$

[1] Following a terminology from linear logic, [1, 2] call these the positive and negative connectives, respectively, which is what the letters $\mathcal{P}$ and $\mathcal{N}$ stand for. We avoid these terms here for danger of confusion with the conventional notion of positive and negative occurrences of subformulas (Definition 3.4).

using (4) and the definition of E_ψ.

Taking $\psi = \varphi$ in (10), we obtain

$$\{E_\psi : \psi \subseteq \varphi\} \vdash_{\mathbf{FL_e}} \varphi \to p_\varphi.$$

By the deduction theorem (Lemma 2.2), we can fix $n \in \omega$ such that

$$\vdash_{\mathbf{FL_e}} \prod_{\psi \subseteq \varphi} (E_\psi \wedge 1)^n \to (\varphi \to p_\varphi). \tag{11}$$

Let us now define

$$\varphi' = \prod_{\psi \subseteq \varphi} (E_\psi \wedge 1)^n \to p_\varphi.$$

Since $E_\psi \wedge 1$ is $\mathcal{N}_2$, the product is $\mathcal{P}_3$, and $\varphi' \in \mathcal{N}_3$ as required.

We can rewrite (11) as $\vdash_{\mathbf{FL_e}} \varphi \to \varphi'$, and a fortiori $\vdash_{\mathbf{FL_e}+\varphi} \varphi'$. On the other hand, let σ denote the substitution

$$\sigma(p_\psi) = \psi.$$

Since $\sigma(E_\psi) = (\psi \leftrightarrow \psi)$ is provable in $\mathbf{FL_e}$, we have

$$\vdash_{\mathbf{FL_e}} \sigma\Bigl(\prod_{\psi \subseteq \varphi} (E_\psi \wedge 1)^n\Bigr) \leftrightarrow \prod_{\psi \subseteq \varphi} 1^n$$

which is equivalent to 1, thus $\sigma(\varphi')$ is equivalent to $\sigma(p_\varphi)$, i.e.,

$$\vdash_{\mathbf{FL_e}} \sigma(\varphi') \to \varphi.$$

This gives $\vdash_{\mathbf{FL_e}+\varphi'} \varphi$, hence $\mathbf{FL_e} + \varphi = \mathbf{FL_e} + \varphi'$. □

Remark 3.3 Let us stress that we restrict attention to *axiomatic* extensions of the base logic because that is the hard case; axiomatization of general extensions by *rules* of bounded complexity is straightforward. Indeed, it is easy to see that an arbitrary logic L (i.e., a structural consequence relation) extending **FL** is axiomatized over **FL** by rules of the form $\Gamma \;/\; p$ whose conclusion is a variable, and each formula in Γ is either a variable, or an equivalence between a variable and a formula containing only one connective; if L is finitary, Γ can be taken finite. The same holds for any (finitely) equivalential base logic in place of **FL**. In terms of the substructural hierarchy, this means that all extensions of **FL** are axiomatizable by rules with $\mathcal{N}_2$ premises, and $\mathcal{N}_0$ conclusions.

A concrete illustration of Theorem 3.2 is given later in Example 3.9. We have to postpone it for the following reason: the $\mathcal{N}_3$ axiom φ' constructed in the proof of Theorem 3.2 is not presented fully explicitly, as it depends on n. We can in principle compute n for a given φ as the proofs of Lemmas 2.1 and 2.2 are constructive, but in fact, we can do better: digging a bit deeper into the guts of the argument will reveal that we can just take $n = 1$ for all φ; moreover, we can shorten φ' somewhat by employing implications instead of equivalences, distinguishing between positively and negatively occurring subformulas of φ. We now present the details.

First, let us recall the concept of positive and negative occurrences.

Definition 3.4 An occurrence of a subformula ψ in φ is classified as *positive* or *negative* as follows.

- The occurrence of φ in itself is positive.
- For any positive (negative) occurrence of $\psi_0 \circ \psi_1$ in φ, where $\circ \in \{\cdot, \wedge, \vee\}$, the indicated occurrences of ψ_0 and ψ_1 in φ are also positive (negative, resp.).
- For any positive (negative) occurrence of $\psi_0 \to \psi_1$ in φ, the indicated occurrence of ψ_0 in φ is negative (positive, resp.), and the occurrence of ψ_1 is positive (negative, resp.).

Let us abbreviate

$$(\varphi \Rightarrow \psi) = (\varphi \to \psi) \wedge 1.$$

Lemma 3.5 $\mathbf{FL_e}$ *proves the schemata*

$$\varphi \Rightarrow \varphi, \tag{12}$$
$$(\varphi \Rightarrow \psi) \cdot (\psi \Rightarrow \chi) \to (\varphi \Rightarrow \chi), \tag{13}$$
$$(\varphi' \Rightarrow \varphi) \cdot (\psi \Rightarrow \psi') \to \big((\varphi \to \psi) \Rightarrow (\varphi' \to \psi')\big), \tag{14}$$
$$(\varphi \Rightarrow \varphi') \cdot (\psi \Rightarrow \psi') \to \big((\varphi \circ \psi) \Rightarrow (\varphi' \circ \psi')\big) \tag{15}$$

for $\circ \in \{\cdot, \wedge, \vee\}$.

Proof: Straightforward, using e.g. the algebraic semantics of $\mathbf{FL_e}$.

For instance, let us check (15) with $\circ = \wedge$. Let L be a residuated lattice, and $x, x', y, y' \in L$, we need to show

$$(x \Rightarrow x') \cdot (y \Rightarrow y') \leq (x \wedge y) \Rightarrow (x' \wedge y').$$

Clearly,

$$(x \Rightarrow x') \cdot (y \Rightarrow y') \leq 1 \cdot 1 = 1,$$

thus it suffices to show

$$(x \Rightarrow x') \cdot (y \Rightarrow y') \leq (x \wedge y) \to (x' \wedge y').$$

This follows from

$$(x \wedge y) \cdot (x \Rightarrow x') \cdot (y \Rightarrow y') \leq x \cdot (x \to x') \cdot 1 \leq x',$$

and the symmetric inequality for y'. □

Definition 3.6 Let φ be a formula. We will define an $\mathcal{N}_3$ formula φ^+ as follows.

If ψ is an occurrence of a variable[2] in φ, we consider p_ψ a shorthand for ψ. For any occurrence of a subformula $\psi \subseteq \varphi$ which is not a variable, we introduce a new variable p_ψ,

[2]The separate treatment of variables only serves the purpose of making φ^+ shorter, otherwise we could handle them more uniformly as in the proof of Theorem 3.2.

and put

$$E_\psi^+ = \begin{cases} \psi \Rightarrow p_\psi & \text{if } \psi \text{ is a constant and occurs positively in } \varphi, \\ p_\psi \Rightarrow \psi & \text{if } \psi \text{ is a constant and occurs negatively in } \varphi, \\ (p_{\psi_0} \circ p_{\psi_1}) \Rightarrow p_\psi & \text{if } \psi = \psi_0 \circ \psi_1 \text{ occurs positively in } \varphi, \\ p_\psi \Rightarrow (p_{\psi_0} \circ p_{\psi_1}) & \text{if } \psi = \psi_0 \circ \psi_1 \text{ occurs negatively in } \varphi, \end{cases}$$

where $\circ \in \{\to, \cdot, \wedge, \vee\}$. Finally,

$$\varphi^+ = \prod_{\psi \subseteq' \varphi} E_\psi^+ \to p_\varphi,$$

where $\psi \subseteq' \varphi$ means that $\psi \subseteq \varphi$ and ψ is not a variable.

We observe that E_ψ^+ is $\mathcal{N}_2$, and φ^+ is $\mathcal{N}_3$.

Remark 3.7 When there are multiple occurrences of the same formula ψ in φ, each gets its own variable p_ψ according to the given definition. This is not really essential, but what matters is that φ^+ includes one E_ψ^+ for every occurrence.

Theorem 3.8 *For any formula φ, the $\mathcal{N}_3$ formula φ^+ satisfies $\mathbf{FL_e} + \varphi = \mathbf{FL_e} + \varphi^+$.*

More precisely, $\mathbf{FL_e}$ proves

$$\varphi \to \varphi^+, \tag{16}$$

$$\sigma(\varphi^+) \to \varphi, \tag{17}$$

where σ denotes the substitution $\sigma(p_\psi) = \psi$.

Proof: The same argument as in the proof of Theorem 3.2 shows (17).

As for (16), we prove by induction on the complexity of $\psi \subseteq \varphi$ that

$$\vdash_{\mathbf{FL_e}} \prod_{\chi \subseteq' \psi} E_\chi^+ \to (\psi \Rightarrow p_\psi) \tag{18}$$

if the occurrence of ψ in φ is positive, and

$$\vdash_{\mathbf{FL_e}} \prod_{\chi \subseteq' \psi} E_\chi^+ \to (p_\psi \Rightarrow \psi) \tag{19}$$

if it is negative.

The claim is immediate from the definition if ψ is a constant or a variable.

Let $\psi = \psi_0 \to \psi_1$. If ψ occurs positively, we have

$$\prod_{\chi \subseteq' \psi} E_\chi^+ = \Big(\prod_{\chi \subseteq' \psi_0} E_\chi^+\Big) \cdot \Big(\prod_{\chi \subseteq' \psi_1} E_\chi^+\Big) \cdot \big((p_{\psi_0} \to p_{\psi_1}) \Rightarrow p_\psi\big),$$

and the induction hypothesis gives

$$\vdash_{\mathbf{FL_e}} \prod_{\chi \subseteq' \psi_0} E_\chi^+ \to (p_{\psi_0} \Rightarrow \psi_0),$$

$$\vdash_{\mathbf{FL_e}} \prod_{\chi \subseteq' \psi_1} E_\chi^+ \to (\psi_1 \Rightarrow p_{\psi_1}),$$

thus

$$\vdash_{\mathbf{FL_e}} \prod_{\chi\subseteq'\psi} E^+_\chi \to (p_{\psi_0} \Rightarrow \psi_0)\cdot(\psi_1 \Rightarrow p_{\psi_1})\cdot\big((p_{\psi_0} \to p_{\psi_1}) \Rightarrow p_\psi\big).$$

Using (14), this implies

$$\vdash_{\mathbf{FL_e}} \prod_{\chi\subseteq'\psi} E^+_\chi \to \big((\psi_0 \to \psi_1) \Rightarrow (p_{\psi_0} \to p_{\psi_1})\big)\cdot\big((p_{\psi_0} \to p_{\psi_1}) \Rightarrow p_\psi\big),$$

hence

$$\vdash_{\mathbf{FL_e}} \prod_{\chi\subseteq'\psi} E^+_\chi \to \big((\psi_0 \to \psi_1) \Rightarrow p_\psi\big)$$

by (13).

If ψ occurs negatively in φ, the induction hypothesis and the definition of E^+_ψ give

$$\vdash_{\mathbf{FL_e}} \prod_{\chi\subseteq'\psi} E^+_\chi \to (\psi_0 \Rightarrow p_{\psi_0})\cdot(p_{\psi_1} \Rightarrow \psi_1)\cdot\big(p_\psi \Rightarrow (p_{\psi_0} \to p_{\psi_1})\big),$$

which implies

$$\vdash_{\mathbf{FL_e}} \prod_{\chi\subseteq'\psi} E^+_\chi \to \big(p_\psi \Rightarrow (\psi_0 \to \psi_1)\big)$$

in a similar way using (13) and (14).

If $\psi = \psi_0 \circ \psi_1$ with $\circ \in \{\cdot, \wedge, \vee\}$, we proceed analogously with (15) in place of (14).

Taking $\psi = \varphi$ in (18) gives

$$\vdash_{\mathbf{FL_e}} \prod_{\psi\subseteq'\varphi} E^+_\psi \to (\varphi \to p_\varphi)$$

using the definition of $\Rightarrow$, thus $\vdash_{\mathbf{FL_e}} \varphi \to \varphi^+$. $\square$

Example 3.9 Let φ be Cintula's product axiom (cf. [3, p. 114])

$$\big((r \to 0) \to 0\big) \to \big[(r \to r\cdot q) \to q\cdot((q \to 0) \to 0)\big],$$

which is ostensibly $\mathcal{N}_4$. Then φ^+ is

$$\begin{aligned}
&\big[(0 \Rightarrow p_{0,0})\cdot\big((r \to p_{0,0}) \Rightarrow p_{\neg r}\big)\cdot(p_{0,1} \Rightarrow 0)\cdot\big(p_{\neg\neg r} \Rightarrow (p_{\neg r} \to p_{0,1})\big)\\
&\quad\cdot(p_{r\cdot q} \Rightarrow r\cdot q)\cdot\big(p_{r\to r\cdot q} \Rightarrow (r \to p_{r\cdot q})\big)\cdot(p_{0,2} \Rightarrow 0)\cdot\big(p_{\neg q} \Rightarrow (q \to p_{0,2})\big)\\
&\quad\cdot(0 \Rightarrow p_{0,3})\cdot\big((p_{\neg q} \to p_{0,3}) \Rightarrow p_{\neg\neg q}\big)\cdot(q\cdot p_{\neg\neg q} \Rightarrow p_{q\cdot\neg\neg q})\\
&\quad\cdot\big((p_{r\to r\cdot q} \to p_{q\cdot\neg\neg q}) \Rightarrow p_{(r\to r\cdot q)\to q\cdot\neg\neg q}\big)\cdot\big((p_{\neg\neg r} \to p_{(r\to r\cdot q)\to q\cdot\neg\neg q}) \Rightarrow p_{\neg\neg r\to((r\to r\cdot q)\to q\cdot\neg\neg q)}\big)\big]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\to p_{\neg\neg r\to((r\to r\cdot q)\to q\cdot\neg\neg q)},
\end{aligned}$$

where we used the abbreviation $\neg\alpha = (\alpha \to 0)$ in the subscripts, and the four extension variables corresponding to occurrences of 0 were disambiguated by extra subscripts $0, \dots, 3$. We could have actually used just a single variable p_0, cf. Remark 3.7. It turns out that since φ contains no lattice connectives, it would also suffice to use plain $\to$ rather than $\Rightarrow$.

$$
\begin{tikzcd}
& \mathcal{N}_3 & \\
& \mathcal{P}_3 \arrow[u] & \\
\mathcal{P}_2 \arrow[ur] & & \mathcal{N}_2 \arrow[ul] \\
& \mathcal{N}_1 \arrow[ul] \arrow[ur] & \\
& \mathcal{P}_0 = \mathcal{N}_0 \equiv \mathcal{P}_1 \arrow[u] &
\end{tikzcd}
$$

Figure 1: The substructural hierarchy

In contrast, the corresponding formula φ' from Theorem 3.2 is

$$
\begin{aligned}
&\big[(p_{r,0} \Leftrightarrow r) \cdot (p_{0,0} \Leftrightarrow 0) \cdot \big(p_{\neg r} \Leftrightarrow (p_{r,0} \to p_{0,0})\big) \cdot (p_{0,1} \Leftrightarrow 0) \cdot \big(p_{\neg\neg r} \Leftrightarrow (p_{\neg r} \to p_{0,1})\big) \\
&\quad \cdot (p_{r,2} \Leftrightarrow r) \cdot (p_{q,0} \Leftrightarrow q) \cdot (p_{r\cdot q} \Leftrightarrow p_{r,2} \cdot p_{q,0}) \cdot (p_{r,1} \Leftrightarrow r) \cdot \big(p_{r\to r\cdot q} \Leftrightarrow (p_{r,1} \to p_{r\cdot q})\big) \\
&\quad \cdot (p_{q,2} \Leftrightarrow q) \cdot (p_{0,2} \Leftrightarrow 0) \cdot \big(p_{\neg q} \Leftrightarrow (p_{q,2} \to p_{0,2})\big) \\
&\quad \cdot (p_{0,3} \Leftrightarrow 0) \cdot \big(p_{\neg\neg q} \Leftrightarrow (p_{\neg q} \to p_{0,3})\big) \cdot (p_{q,1} \Leftrightarrow q) \cdot (p_{q\cdot\neg\neg q} \Leftrightarrow p_{q,1} \cdot p_{\neg\neg q}) \\
&\quad \cdot \big(p_{(r\to r\cdot q)\to q\cdot\neg\neg q} \Leftrightarrow (p_{r\to r\cdot q} \to p_{q\cdot\neg\neg q})\big) \cdot \big(p_{\neg\neg r\to((r\to r\cdot q)\to q\cdot\neg\neg q)} \Leftrightarrow (p_{\neg\neg r} \to p_{(r\to r\cdot q)\to q\cdot\neg\neg q})\big)\big] \\
&\hspace{28em} \to p_{\neg\neg r\to((r\to r\cdot q)\to q\cdot\neg\neg q)},
\end{aligned}
$$

where $\alpha \Leftrightarrow \beta$ stands for $(\alpha \leftrightarrow \beta) \wedge 1$. Here we use the fact that we can take $n = 1$ in (11), which can be proved in a similar way as Theorem 3.8.

For ease of reference in the next remark, we state a normal form for $\mathcal{P}_k$ and $\mathcal{N}_k$ formulas proved in [2, Lemma 3.3]. Recall that the empty product is 1; likewise, empty disjunctions and (lattice) conjunctions are defined as $\bot$ and $\top$, respectively.

Lemma 3.10 *Let $k \geq 0$.*

(i) Any $\mathcal{P}_{k+1}$ formula is equivalent over $\mathbf{FL_e}$ *to a disjunction of products of $\mathcal{N}_k$ formulas.*

(ii) Any $\mathcal{N}_{k+1}$ formula is equivalent over $\mathbf{FL_e}$ *to $\bigwedge_{i<n}(\alpha_i \to \beta_i)$, where each α_i is a product of $\mathcal{N}_k$ formulas, and each β_i is a $\mathcal{P}_k$ formula or 0.* □

Remark 3.11 Figure 1 shows what is left of the substructural hierarchy over $\mathbf{FL_e}$. Concerning $\mathcal{P}_1 \equiv \mathcal{P}_0$, any $\varphi(p_1, \dots, p_n) \in \mathcal{P}_1$ can be written as a disjunction of products of variables by Lemma 3.10 (i). If one of the products is empty, φ is provable in $\mathbf{FL_e}$; otherwise $\varphi(p \wedge 1, \dots, p \wedge 1)$ implies p. Thus, the only $\mathcal{P}_1$-axiomatizable logics are $\mathbf{FL_e}$ itself and the inconsistent logic.

The hierarchy is not going to collapse any further, as all remaining inclusions are strict:

An example of a nontrivial $\mathcal{N}_1$ axiom is left weakening $p \to (q \to p)$.

By [2, Cor. 7.7], the $\mathcal{P}_2$ linearity axiom $(p \to q) \vee (q \to p)$ is not $\mathcal{N}_2$-axiomatizable. The same holds for the law of excluded middle $p \vee (p \to 0)$.

The right weakening axiom $0 \to p$ is $\mathcal{N}_2$, but it is not $\mathcal{P}_2$-axiomatizable over Johansson's logic ($\mathbf{FL_{eci}}$). Assuming otherwise, it would be axiomatizable by disjunctions of $\mathcal{N}_1$ axioms over $\mathbf{FL_{eci}}$ by Lemma 3.10 (i), using $\cdot = \wedge$. Since $\mathbf{FL_{eci}} + (0 \to p) = \mathbf{IPC}$ has the disjunction property, we could replace each disjunction with one of its disjuncts, hence the logic would be actually $\mathcal{N}_1$-axiomatizable. By (ii), we could axiomatize it by a set of axioms of the form $\alpha \to \beta$, where α is a product of variables, and β is a variable or 0. However, such an axiom is valid in $\mathbf{IPC}$ only when β is a variable occurring in α, in which case it is already provable in $\mathbf{FL_{eci}}$, hence this is impossible.

Finally, a proper superintuitionistic logic with the disjunction property, such as $\mathbf{KP} = \mathbf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$, is not $\mathcal{P}_3$-axiomatizable over $\mathbf{IPC}$. Assuming otherwise, the same argument as above would imply the logic is in fact $\mathcal{N}_2$-axiomatizable. However, as shown in [2], any $\mathcal{N}_2$ axiom is either provable or contradictory over $\mathbf{IPC}$.

4 Conclusion

We have seen that over $\mathbf{FL_e}$, arbitrary axioms can be unwinded to deductively equivalent $\mathcal{N}_3$ axioms, hence the substructural hierarchy collapses. This entails some ramifications for the program of algebraic proof theory: the optimist may say that now it suffices to extend the structure theory for $\mathcal{N}_2$ and $\mathcal{P}_3$ logics just one step to $\mathcal{N}_3$ to deal with arbitrary extensions of $\mathbf{FL_e}$, while the pessimist may point out that this sounds too good to be feasible, and it rather means that the class $\mathcal{N}_3$ as a whole is already intractable to informative analysis, and might need further subclassification.

Our arguments relied on commutativity, which raises the question what happens if we drop this assumption:

Problem 4.1 *Are all axiomatic extensions of* $\mathbf{FL}$ $\mathcal{N}_k$*-axiomatizable for some fixed* k*?*

We mention that while the basic structure of the proof of Theorem 3.2—which essentially uses only the equivalentiality of the logic and the deduction theorem—applies to $\mathbf{FL}$ as well, this does not yield the desired reduction in formula complexity. The problem is that the form of deduction theorem valid for $\mathbf{FL}$ has $\varphi \wedge 1$ in (7) replaced with iterated conjugates $\gamma_1(\gamma_2(\ldots(\gamma_m(\varphi))\ldots))$, where each $\gamma_i(x)$ is $(\alpha_i \backslash (x \cdot \alpha_i)) \wedge 1$ or $((\alpha_i \cdot x)/\alpha_i) \wedge 1$ for some formulas α_i. Even if we disregard the complexity of α_i itself (which we can't), each conjugate strictly raises the level in the substructural hierarchy, hence the resulting formula may have unbounded complexity.

The low-level proof of Theorem 3.8 does not work in the noncommutative setting either. The argument relies on exchange through repeated use of Lemma 3.5; it is unclear whether one can choose an ordering of the factors in the definition of φ^+ and directions of the relevant residua in a consistent way so that everything cancels out as intended.

We thus leave Problem 4.1 open.

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References

[1] Agata Ciabattoni, Nikolaos Galatos, and Kazushige Terui, *From axioms to analytic rules in nonclassical logics*, in: Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science, 2008, pp. 229–240.

[2] ———, *Algebraic proof theory for substructural logics: cut-elimination and completions*, Annals of Pure and Applied Logic 163 (2012), no. 3, pp. 266–290.

[3] Nikolaos Galatos, Peter Jipsen, Tomasz Kowalski, and Hiroakira Ono, *Residuated lattices: An algebraic glimpse at substructural logics*, Studies in Logic and the Foundations of Mathematics vol. 151, Elsevier, Amsterdam, 2007.

[4] Vladimir V. Rybakov, *Admissibility of logical inference rules*, Studies in Logic and the Foundations of Mathematics vol. 136, Elsevier, 1997.

[5] Grigorii S. Tseitin, *On the complexity of derivation in propositional calculus*, in: Studies in Constructive Mathematics and Mathematical Logic, Part 2 (A. O. Slisenko, ed.), Seminars in mathematics, V. A. Steklov Mathematical Institute, Leningrad, 1970, pp. 115–125, translated from: Zapiski Nauchnykh Seminarov LOMI 8 (1968), pp. 234–259.