Logics with directed unification

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Unification and propositional logics

Equational unification

Θ: a background equational theory (or a variety of algebras)

Basic Θ-unification problem:
Given a set of equations $\Gamma = \{t_1 \approx s_1, \dots, t_n \approx s_n\}$, is there a substitution σ (a Θ-unifier of Γ) s.t.

$$\sigma(t_1) =_\Theta \sigma(s_1), \dots, \sigma(t_n) =_\Theta \sigma(s_n)?$$

What is the structure of Θ-unifiers?

Preorder of unifiers

Substitutions σ, τ are equivalent, written $\sigma =_\Theta \tau$, if $\sigma(x) =_\Theta \tau(x)$ for every x

σ is more general than τ, written $\tau \le_\Theta \sigma$, if $\upsilon \circ \sigma =_\Theta \tau$ for some υ

$\le_\Theta$ is a preorder on the set $U_\Theta(\Gamma)$ of unifiers of Γ

Complete set of unifiers: a cofinal subset $C \subseteq U_\Theta(\Gamma)$ (every $\tau \in U_\Theta(\Gamma)$ is less general than some $\sigma \in C$)

Minimal c. s. of u.: no proper subset of C is complete

Equivalently: C consists of pairwise incomparable maximal unifiers

Classification of unification problems

If Γ has a minimal complete set of unifiers C, it is of

- type 1 (unitary) if $|C| = 1$ (most general unifier (mgu))
- type ω (finitary) if C is finite, $|C| > 1$
- type ∞ (infinitary) if C is infinite

Otherwise (= the set of all maximal unifiers is not cofinal):

- type 0 (nullary)

Unification type of Θ is the maximal (=worst) type among unifiable Θ-unification problems Γ, where

$$1 < \omega < \infty < 0$$

Propositional logics

Propositional logic L:

Language: formulas built from atoms (variables) $\{x_n : n \in \omega\}$ using a fixed set of connectives of finite arity

Consequence relation: a relation $\Gamma \vdash_L \varphi$ between sets of formulas and formulas such that

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Delta \vdash_L \varphi$
- $\Gamma, \Delta \vdash_L \varphi$ and $\forall \psi \in \Delta\, \Gamma \vdash_L \psi$ imply $\Gamma \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution σ

Algebraizable logics

A logic L is finitely algebraizable wrt a class K of algebras if there is a finite set $E(x, y)$ of formulas and a finite set $T(x)$ of equations such that

- $\Gamma \vdash_L \varphi \Leftrightarrow T(\Gamma) \vDash_K T(\varphi)$
- $\Delta \vDash_K t \approx s \Leftrightarrow E(\Delta) \vdash_L E(t, s)$
- $x \dashv\vdash_L E(T(x))$
- $x \approx y \;{=\!\!\mid\!\vDash}_K T(E(x, y))$

Example (modal logic, . . .):
$T(x) = \{x \approx 1\}$, $E(x, y) = \{x \leftrightarrow y\}$

Unification in propositional logics

If L is a logic algebraizable wrt a (quasi)variety K, we can express K-unification in terms of L:

An L-unifier of a formula φ is σ such that $\vdash_L \sigma(\varphi)$

Then we have:

- L-unifier of φ = K-unifier of $T(\varphi)$
- K-unifier of $t \approx s$ = L-unifier of $E(t, s)$
- $\sigma =_L \tau$ iff $\vdash_L E(\sigma(x), \tau(x))$ for every x
  $\Rightarrow$ express accordingly $\leq_L$, $U_L(\Gamma)$, unification types, ...

Equivalential logics

More generally, unification theory makes sense for equivalential logics L:

Set of formulas $E(x,y)$ s.t.

- $\vdash_L E(x,x)$
- $E(x,y), \varphi(x) \vdash_L \varphi(y)$ for each φ (may have other variables)

Then define:

- L-unifier of Γ is σ s.t. $\vdash_L \sigma(\Gamma)$
- $\sigma =_L \tau$ iff $\vdash_L E(\sigma(x), \tau(x))$ for each x
- this induces $\langle U_L(\Gamma), \leq_L \rangle$ as before

Unification with parameters

Elementary unification vs. unification with free constants:

Distinguish two kinds of atoms:

- variables $\{x_n : n \in \omega\}$
- constants (parameters) $\{p_n : n \in \omega\}$

Substitutions only modify variables, we require $\sigma(p_n) = p_n$

Adapt accordingly the other notions:

- L-unifier
- $=_L, \leq_L, \dots$

Directed unification

Directed unification

Common situation (modal logics, . . .):

- we prove unification is at most finitary
- we wish to distinguish type 1 from type ω

Directed (aka filtering) unification:

$\langle U_L(\Gamma), \leq_L \rangle$ is a directed preorder for each Γ

$$\forall \sigma_0, \sigma_1 \in U_L(\Gamma)\ \exists \sigma \in U_L(\Gamma)\ (\sigma_0 \leq_L \sigma\ \&\ \sigma_1 \leq_L \sigma)$$

Directedness and unification type

Observe:

- Γ has mgu $\Rightarrow U_L(\Gamma)$ is directed
- Γ has ≥ 2 maximal unifiers $\Rightarrow U_L(\Gamma)$ is not directed

Corollary: If L does not have type 0, then

- directed unification $\Rightarrow$ type 1
- nondirected unification $\Rightarrow$ type ω or ∞

Transitive modal logics

Theorem [Ghilardi & Sacchetti '04]:
A normal modal logic $L \supseteq \mathbf{K4}$ has directed unification iff L extends

$$\mathbf{K4.2} := \mathbf{K4} + \Diamond\!\!\!\cdot\, \boxdot\varphi \to \boxdot\Diamond\!\!\!\cdot\,\varphi$$

(We write $\boxdot\varphi = \varphi \land \Box\varphi$, $\Diamond\!\!\!\cdot\,\varphi = \neg\boxdot\neg\varphi = \varphi \lor \Diamond\varphi$.)

- sophisticated argument involving algebra, category theory, and topological frames
- specific to transitive modal logics
- given only for elementary unification (no free constants)

It turns out this has a simple syntactic proof (next slide . . .)

Elementary proof

$\Rightarrow$ Let σ be a unifier of $\boxdot x \vee \boxdot\neg x$ more general than $x/\top$, $x/\bot$. Put $\alpha = \sigma(x)$, fix σ_i s.t. $\vdash_L \sigma_1(\alpha), \neg\sigma_0(\alpha)$. Define

$$\tau(x_j) = \big(y \wedge \sigma_1(x_j)\big) \vee \big(\neg y \wedge \sigma_0(x_j)\big)$$

for each variable x_j in α. We have

$$\vdash_L \boxdot y \to \bigwedge_j \boxdot\big(\tau(x_j) \leftrightarrow \sigma_1(x_j)\big) \to \tau(\alpha)$$

$$\vdash_L \boxdot\neg\tau(\alpha) \to \boxdot\neg\boxdot y$$

and similarly, $\vdash_L \boxdot\tau(\alpha) \to \boxdot\neg\boxdot\neg y$. Since $\vdash_L \boxdot\alpha \vee \boxdot\neg\alpha$, we obtain $\vdash_L \boxdot\Diamond\neg y \vee \boxdot\Diamond y$.

Elementary proof (cont'd)

$\Leftarrow$ Let σ_0, σ_1 be unifiers of φ. Define

$$\sigma(x_j) = \bigl(\boxdot\Diamond y \wedge \sigma_0(x_j)\bigr) \vee \bigl(\neg\boxdot\Diamond y \wedge \sigma_1(x_j)\bigr).$$

Clearly, $\sigma_0 \le_L \sigma$ via $y/\top$, and $\sigma_1 \le_L \sigma$ via $y/\bot$. Also,

$$\vdash_L \boxdot\Diamond y \to \bigwedge_j \boxdot\bigl(\sigma(x_j) \leftrightarrow \sigma_0(x_j)\bigr) \to \sigma(\varphi)$$

$$\vdash_L \boxdot\neg\boxdot\Diamond y \to \bigwedge_j \boxdot\bigl(\sigma(x_j) \leftrightarrow \sigma_1(x_j)\bigr) \to \sigma(\varphi)$$

Since $\vdash_{\mathbf{K4.2}} \boxdot\Diamond y \vee \boxdot\neg\boxdot\Diamond y$, we obtain $\vdash_L \sigma(\varphi)$.

Comments

- L has directed unification $\Leftrightarrow$
  there is a unifier of $\boxdot x \vee \boxdot \neg x$ more general than $x/\top$, $x/\bot$
  (IOW, $\exists \alpha$ s.t. $\vdash_L \boxdot \alpha \vee \boxdot \neg \alpha$, and α and $\neg \alpha$ are unifiable)
- L has directed elementary unification
  $\Leftrightarrow$ L has directed unification with constants
- The proof applies to larger classes of logics:

Example: Let L be an n-transitive multimodal logic ($\Box \varphi := \Box_1 \varphi \wedge \cdots \wedge \Box_k \varphi$ satisfies $\vdash_L \Box^{\leq n} \varphi \to \Box^{n+1} \varphi$). TFAE:

(1) L has directed unification

(2) $\exists \alpha$ s.t. $\vdash_L \Box^{\leq n} \alpha \vee \Box^{\leq n} \neg \alpha$, and α and $\neg \alpha$ are unifiable

(3) $\vdash_L \Diamond^{\leq n} \Box^{\leq n} x \to \Box^{\leq n} \Diamond^{\leq n} x$

Generalization

By disentangling the roles of various subformulas used in the proof, we can make it work for logics L satisfying a handful of more abstract properties.

Assumption 0: L is equivalential wrt a set $E(x,y)$ of formulas

Example: $E(x,y) = x \leftrightarrow y$

Assumption 1: There is a finite set $D(x,y)$ of formulas that behaves as a deductive disjunction:

$$\Gamma, D(\varphi,\psi) \vdash_L \chi \;\Leftrightarrow\; \begin{cases} \Gamma, \varphi \vdash_L \chi \\ \Gamma, \psi \vdash_L \chi \end{cases}$$

Example: $D(x,y) = \Box^{\le n} x \lor \Box^{\le n} y$

Switch and box formulas

Assumption 2: There are unifiable formulas $C_0(x)$ and $C_1(x)$, and a switch formula $S(x, y_0, y_1)$:

$$C_e(x) \vdash_L E(S(x, y_0, y_1), y_e)$$

(Actually, the unifiability of C_0, C_1 follows from assumption 3)

Example $C_1(x) = x$, $C_0(x) = \neg x$, $S(x, y_0, y_1) = (x \wedge y_1) \vee (\neg x \wedge y_0)$

Assumption 3: There is a formula $B(x)$ such that

$$\Gamma \vdash_L \varphi \;\Rightarrow\; \Gamma \vdash_L C_1(B(\varphi)) \quad \text{(i.e., } x \vdash_L C_1(B(x))\text{)}$$

$$\Gamma, \varphi \vdash_L \bot \;\Rightarrow\; \Gamma \vdash_L C_0(B(\varphi))$$

Here: $\Delta \vdash_L \bot$ shorthand for $\forall \psi\, \Delta \vdash_L \psi$ (i.e., Δ is inconsistent)

Example: $B(x) = \Box^{\leq n} x$

General characterization

Theorem [J.]:
For a logic L satisfying assumptions $0, 1, 2, 3$ above, TFAE:

(1) L has directed unification

(2) $\exists\alpha$ s.t. $\vdash_L D(C_0(\alpha), C_1(\alpha))$, and $C_0(\alpha), C_1(\alpha)$ are unifiable

(3) $\vdash_L D(C_0(B(C_0(x))), C_0(B(C_1(x))))$

Comments:

- Assumptions $0, 1, 2$ suffice for (1) $\Leftrightarrow$ (2)
- Also applies to unification with constants
- If E, D, S, C_0, C_1 without free constants:
  L has directed elementary unification $\Leftrightarrow$
  L has directed unification with constants

Less abstract statement

Corollary:
Let $L \supseteq \mathbf{FL_o} \restriction \{\to, \wedge, \vee, 0, 1\}$ (possibly with larger language) be equivalential wrt $E(x, y) = (x \to y) \wedge (y \to x)$, and have the deduction-detachment theorem in the form

$$\Gamma, \varphi \vdash_L \psi \quad \text{iff} \quad \Gamma \vdash_L \Delta\varphi \to \psi$$

for some formula $\Delta(x)$. TFAE:

(1) L has directed unification

(2) $\exists \alpha$ s.t. $\vdash_L \Delta\alpha \vee \Delta\neg\alpha$, and $\alpha, \neg\alpha$ are unifiable

(3) $\vdash_L \Delta\neg\Delta x \vee \Delta\neg\Delta\neg x$

Proof: Take $D(x, y) = \Delta x \vee \Delta y$, $C_1(x) = x$, $C_0(x) = \neg x$, $S(x, y_0, y_1) = (1 \wedge x \to y_1) \wedge (1 \wedge \neg x \to y_0)$, $B(x) = \Delta x$

Applications

Examples:

- n-transitive multimodal logics: $\Delta x = \Box^{\leq n} x$ (we've seen that already)
- n-contractive (= $\vdash_L x^n \to x^{n+1}$) simple axiomatic extensions of $\mathbf{FL_{ew}}$:
  - take $\Delta x = x^n$
  - L has directed unification $\Leftrightarrow \vdash_L \left(\neg x^n\right)^n \vee \left(\neg(\neg x)^n\right)^n$
  - $n = 1$: $L \supseteq \mathbf{IPC}$ has directed unification $\Leftrightarrow L \supseteq \mathbf{KC}$

Thank you for attention!

References

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