Admissible rules and their complexity

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Outline of the talks

1. Logics and admissibility
2. Transitive modal logics
3. Toy model: logics of bounded depth
4. Projective formulas
5. Admissibility in clx logics
6. Problems and complexity classes
7. Complexity of derivability
8. Complexity of admissibility
Logics and admissibility

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Propositional logics

Propositional logic $L$:

**Language:** formulas built from atoms $x_0, x_1, x_2, \ldots$ using a fixed set of finitary connectives

**Consequence relation:** a relation $\Gamma \vdash_L \varphi$ between sets of formulas and formulas s.t.

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Delta \vdash_L \varphi$
- $\Gamma, \Delta \vdash_L \varphi$ and $\forall \psi \in \Delta \Gamma \vdash_L \psi$ imply $\Gamma \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution $\sigma$
Unifiers and admissible rules

\( \Gamma, \Delta: \) finite sets of formulas

**L-unifier of \( \Gamma \):** substitution \( \sigma \) s.t. \( \vdash_L \sigma(\varphi) \) for all \( \varphi \in \Gamma \)

**Single-conclusion rule:** \( \Gamma / \varphi \)

**Multiple-conclusion rule:** \( \Gamma / \Delta \)

- **\( \Gamma / \Delta \) is \( L \)-derivable (or valid)** if \( \Gamma \vdash_L \delta \) for some \( \delta \in \Delta \)
- **\( \Gamma / \Delta \) is \( L \)-admissible** (written as \( \Gamma \sim_L \Delta \)) if every \( L \)-unifier of \( \Gamma \) also unifies some \( \delta \in \Delta \)

**NB:** \( \Gamma \) is \( L \)-unifiable iff \( \Gamma \not\vdash_L \varnothing \)
Examples

- **CPC**: admissible = derivable (structural completeness)
- **IPC** and intermediate logics admit Kreisel–Putnam rule:

\[
\neg x \rightarrow y \lor z \vdash (\neg x \rightarrow y) \lor (\neg x \rightarrow z)
\]

- \(\Box x / x\) admissible in \(K, K4\), derivable in \(KT, S4\)
- **Löb’s rule** \(\Box x \rightarrow x / x\) admissible in \(K\), derivable in \(GL\)
- \(\Diamond x \land \Diamond \neg x / \bot\) admissible in all normal modal logics
- \(\bot \vdash_L \emptyset\) iff \(L\) is consistent
- \(L\) has the (modal) disjunction property iff

\[
\Box x_1 \lor \cdots \lor \Box x_n \vdash_L x_1, \ldots, x_n \quad (n \geq 0)
\]

- Rule of margins \(x \rightarrow \Box x / x, \neg x\) admissible in \(KT, KTB\)
Basic questions

What rules are $L$-admissible?

- NB: $\vdash_L$ forms a (multiple-conclusion) consequence relation
- Semantic characterization of $\vdash_L$ by a class of models (algebras, Kripke models, ...)
- Syntactic presentation of $\vdash_L$:
  - Basis of admissible rules = axiomatization of $\vdash_L$ over $\vdash_L$
  - Can we describe an explicit basis?
  - Are there finite bases? Independent bases?

How to check $\Gamma \vdash_L \Delta$?

- Is admissibility algorithmically decidable?
- What is its computational complexity?
Algebraizable logics

$L$ a logic, $K$ a class of algebras (quasivariety)

$L$ is (finitely) algebraizable wrt $K$ if there are

- formulas $E(x, y) = \{\varepsilon_1(x, y), \ldots, \varepsilon_n(x, y)\}$
- equations $T(x) = \{t_1(x) \approx s_1(x), \ldots, t_m(x) \approx s_m(x)\}$

such that

- $\Gamma \vdash_L \varphi \iff T(\Gamma) \vdash_K T(\varphi)$
- $\Sigma \vdash_K t \approx s \iff E(\Sigma) \vdash_L E(t, s)$
- $x \vdash_L E(T(x))$
- $x \approx y \vdash_K T(E(x, y))$

In modal logic: $T(x) = \{x \approx 1\}$, $E(x, y) = \{x \leftrightarrow y\}$,
$K$ is a variety of modal algebras
Elementary equational unification

$\Theta$: equational theory (or a class of algebras)

$\Sigma = \{ t_1 \approx s_1, \ldots, t_n \approx s_n \}$ finite set of equations

$\Theta$-unifier of $\Sigma$: a substitution $\sigma$ s.t.

$$\sigma(t_1) =_{\Theta} \sigma(s_1), \ldots, \sigma(t_n) =_{\Theta} \sigma(s_n)$$

$U_{\Theta}(\Sigma) = \text{set of } \Theta$-unifiers of $\Sigma$

If $L$ is a logic algebraizable wrt a quasivariety $K$:

- $L$-unifier of $\varphi$ = $K$-unifier of $T(\varphi)$
- $K$-unifier of $t \approx s$ = $L$-unifier of $E(t, s)$
Properties of unifiers

Preorder on substitutions:
\( \sigma \) more general than \( \tau \) (\( \sigma \preceq \Theta \tau \)) if \( \exists \upsilon \upsilon \circ \sigma = \Theta \tau \)

Complete set of unifiers (csu) of \( \Sigma \): \( S \subseteq U_{\Theta}(\Sigma) \) s.t.
\( \forall \tau \in U_{\Theta}(\Sigma) \exists \sigma \in S \ (\sigma \preceq \Theta \tau) \)

Most general unifier (mgu) of \( \Sigma \): \( \sigma \) s.t. \( \{\sigma\} \) csu

Basic questions:

▶ Is \( \Sigma \) unifiable?
▶ Does every \( \Sigma \) a finite csu? Or even mgu (if unifiable)?
▶ Is it decidable if \( \Sigma \) is unifiable? Can we compute a csu?
▶ What is the computational complexity?
Rules $\rightarrow$ algebraic clauses

$L$ logic algebraizable wrt a quasivariety $K$

For simplicity: assume $|E(x, y)| = |T(x)| = 1$

Clause: (universally quantified) disjunction of atomic ($\equiv$ equations) and negated atomic formulas

Quasi-identity: clause with 1 positive literal

Rule $\Gamma / \Delta$ translates to a clause $T(\Gamma / \Delta)$:

\[
\bigwedge_{\varphi \in \Gamma} T(\varphi) \rightarrow \bigvee_{\psi \in \Delta} T(\psi)
\]

$\Gamma / \Delta$ single-conclusion rule $\Rightarrow$ $T(\Gamma / \Delta)$ quasi-identity
Conversely: clause $C = \bigwedge_{i<n} t_i \approx t'_i \rightarrow \bigvee_{j<m} s_j \approx s'_j$

translates to a rule $E(C)$:

$$\{ E(t_i, t'_i) : i < n \} / \{ E(s_j, s'_j) : j < m \}$$

$C$ quasi-identity $\implies E(C)$ single-conclusion rule

$\blacktriangleright$ $(\Gamma / \Delta) \vdash_{L} E(T(\Gamma / \Delta))$

$\blacktriangleright$ $C \models_{K} T(E(C))$

(abusing the notation)
Admissible rules algebraically

Derivability:

- Single-concl. rules $\iff$ quasiequational theory of $K$
- Multiple-concl. rules $\iff$ clausal/universal theory of $K$

$\Gamma \trianglerighteq L \Delta \iff T(\Gamma/\Delta)$ holds in all $K$-algebras

Admissibility:

$\Gamma \bowtie L \Delta \iff T(\Gamma/\Delta)$ holds in free $K$-algebras

$\iff F_K(\omega) \models T(\Gamma/\Delta)$

$\iff F_K(n) \models T(\Gamma/\Delta)$ for all $n \in \omega$
In applications, propositional atoms model both “variables” and “constants”.

We don’t want substitution for constants.

Example (description logic):

1. $\forall \text{child}. (\neg \text{HasSon} \land \exists \text{spouse}. \top)$
2. $\forall \text{child}. \forall \text{child}. \neg \text{Male} \land \forall \text{child}. \text{Married}$
3. $\forall \text{child}. \forall \text{child}. \neg \text{Female} \land \forall \text{child}. \text{Married}$

**Good:** Unify 1 with 2 by $\text{HasSon} \leftrightarrow \exists \text{child}. \text{Male}$, $\text{Married} \leftrightarrow \exists \text{spouse}. \top$

**Bad:** Unify 2 with 3 by $\text{Male} \leftrightarrow \text{Female}$
Admissibility with parameters

In unification theory, it is customary to consider unification with unconstrained constants.

We consider setup with two kinds of atoms:

- **variables** $x_0, x_1, x_2, \cdots \in \text{Var}$ (countable infinite set)
- **parameters (constants)** $p_0, p_1, p_2, \cdots \in \text{Par}$ (countable, possibly finite)

Substitutions only modify variables, we require $\sigma(p_n) = p_n$.

Adapt accordingly other notions:

- $L$-unifier, $L$-admissible rule, . . .

**Exception**: logics are always assumed to be closed under substitution for parameters.
Parameters as signature expansion

Admissibility/unification with parameters in $L$ ⇔ plain admissibility/unification in $L^{\text{Par}}$:

- language expanded with nullary connectives $p \in \text{Par}$
- $\vdash_{L^{\text{Par}}} = \text{least consequence relation that contains } \vdash_L$

$L$ algebraizable wrt $K$ \implies $L^{\text{Par}}$ algebraizable wrt $K^{\text{Par}}$:

- arbitrary expansions of $K$-algebras with the new constants

$L$-admissibility with parameters ⇔ validity in free $K^{\text{Par}}$-algebras

NB: $|\text{Par}| = m \implies F_{K^{\text{Par}}}(n) \simeq F_{K}(n + m)$ with fixed valuation of $m$ generators
Transitive modal logics
Transitive modal logics

We consider axiomatic extensions of the logic $\textbf{K4}$:

- **Language**: Boolean connectives, $\Box$
- **Consequence relation**:
  - axioms of $\textbf{CPC}$
  - $\varphi, \varphi \rightarrow \psi \vdash \psi$
  - $\vdash \Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$
  - $\vdash \Box \varphi \rightarrow \Box \Box \varphi$
  - $\varphi \vdash \Box \varphi$

Algebraizable wrt the variety of $\textbf{K4}$-algebras:
Boolean algebras with operator $\Box$ satisfying $\Box 1 = 1$,
$\Box(a \land b) = \Box a \land \Box b$, $\Box a \leq \Box \Box a$
Frame semantics

Kripke frames: $\langle W, < \rangle$, $< \subseteq W \times W$ transitive

$\implies$ dual $\mathbf{K4}$-algebra $\langle \mathcal{P}(W), \Box \rangle$, $\Box X = W \setminus (W \setminus X)\downarrow$

General frames: $\langle W, <, A \rangle$, $A$ subalgebra of $\langle \mathcal{P}(W), \Box \rangle$

$\implies$ dual $\mathbf{K4}$-algebra $A$

Back: $\mathbf{K4}$-algebra $A \implies$ dual frame $\langle \text{St}(A), <, \text{CO}(\text{St}(A)) \rangle$

duals of $\mathbf{K4}$-algebras $\simeq$ descriptive frames

We will use frame semantics as it is more convenient, but the general algebraic theory still applies

Convention: frame = general frame,
but finite frame = finite Kripke frame
\[ \langle W, < \rangle \text{ transitive frame, } u, v \in W \]

- \(u\) reflexive \iff \(u < u\), otherwise irreflexive
- \(u \leq v\) \iff \(u < v\) or \(u = v\) preorder
- \(u \sim v\) \iff \(u \leq v\) and \(v \leq u\) equivalence relation

**equivalence classes = clusters:**
- reflexive/irreflexive
- proper: size \(\geq 2\) (\(\iff\) reflexive)
- \(\text{cl}(u) = \text{the cluster containing } u\)

- \(u \preccurlyeq v\) \iff \(u < v\) and \(v \not< u\) strict order
- \(X\downarrow = \{u : \exists v \in X u < v\}\), \(X\downarrow\downarrow = \{\ldots u \leq v\}\), \(X\uparrow, X\uparrow\uparrow\)

- \(W\) rooted if \(W = r\uparrow\) for some \(r \in W\)
- \(r\text{cl}(W) = \text{cl}(r)\) root cluster
Examples of transitive logics

<table>
<thead>
<tr>
<th>logic</th>
<th>axiom (on top of K4)</th>
<th>finite rooted frames</th>
</tr>
</thead>
<tbody>
<tr>
<td>S4</td>
<td>$\square x \rightarrow x$</td>
<td>reflexive</td>
</tr>
<tr>
<td>D4</td>
<td>$\Diamond \top$</td>
<td>final clusters reflexive</td>
</tr>
<tr>
<td>GL</td>
<td>$\square(\square x \rightarrow x) \rightarrow \square x$</td>
<td>irreflexive</td>
</tr>
<tr>
<td>K4Grz</td>
<td>$\square(\square(x \rightarrow \square x) \rightarrow x) \rightarrow \square x$</td>
<td>no proper clusters</td>
</tr>
<tr>
<td>K4.1</td>
<td>$\square \Diamond x \rightarrow \Diamond \square x$</td>
<td>no proper final clusters</td>
</tr>
<tr>
<td>K4.2</td>
<td>$\Diamond \square x \rightarrow \square \Diamond x$</td>
<td>unique final cluster</td>
</tr>
<tr>
<td>K4.3</td>
<td>$\square(\square x \rightarrow y) \lor \square(\square y \rightarrow x)$</td>
<td>linear (chain of clusters)</td>
</tr>
<tr>
<td>K4B</td>
<td>$x \rightarrow \square \Diamond x$</td>
<td>lone cluster</td>
</tr>
<tr>
<td>S5</td>
<td>$= S4 \oplus B$</td>
<td>lone reflexive cluster</td>
</tr>
</tbody>
</table>

and their various combinations

Shorthands: $\Diamond \varphi = \neg \square \neg \varphi$, $\Box \varphi = \varphi \land \Box \varphi$, $\Diamond \varphi = \neg \Box \neg \varphi$
Frame measures

A frame \( \langle W, <, A \rangle \) has various invariants in \( \mathbb{N} \cup \{\infty\} \):

- depth = maximal length of strict chains
- cluster size = maximal size of clusters
- width = maximal size of antichains in rooted subframes
- branching = maximal number of immediate successor clusters of any point

A logic \( L \) has depth (cl. size, width) \( \leq k \)

\[ \iff \] all descriptive \( L \)-frames have depth (cl. size, width) \( \leq k \)

\[ \iff \] \( L \supseteq \text{K4BD}_k (\text{K4BC}_k, \text{K4BW}_k) \)

Branching:
more complicated (directly works only for finite frames)
\( L \supseteq \text{K4BB}_k \)
Frames for rules

\[ M = \langle W, <, \models \rangle \] Kripke model:

1. \[ M \models \varphi \iff u \models \varphi \text{ for all } u \in W \]
2. \[ M \models \Gamma / \Delta \iff M \models \varphi \text{ for all } \varphi \in \Gamma \implies M \models \psi \text{ for some } \psi \in \Delta \]

\[ \langle W, <, A \rangle \] frame:

\[ W \models \Gamma / \Delta \iff \langle W, <, \models \rangle \models \Gamma / \Delta \text{ for all admissible } \models \]

Validity of rules preserved by \textit{p-morphic images}, but not by generated subframes

Only single-conclusion rules preserved by disjoint sums
Parametric frames

\textbf{K4-algebras are dual to frames}

\textbf{K4}^{\text{Par}}\text{-algebras are dual to parametric frames }\langle \mathcal{W}, <, A, \models_{\text{Par}} \rangle

- \langle \mathcal{W}, <, A \rangle \text{ frame}
- \models_{\text{Par}} \text{ fixed admissible valuation of parameters } p \in \text{Par}

Model based on } \langle \mathcal{W}, <, A, \models_{\text{Par}} \rangle: \langle \mathcal{W}, <, \models \rangle \text{ s.t.}

- \models \text{ admissible valuation in the frame } \langle \mathcal{W}, <, A \rangle
- \models \text{ extends } \models_{\text{Par}}
Canonical frames

Free \( L \)-algebras \( F_L(V) \) are dual to canonical \( L \)-frames \( C_L(V) \):

- **points:** maximal \( L \)-consistent subsets of \( \text{Form}(V) \)
- **\( X < Y \) \( \iff \) \( \forall \varphi (\Box \varphi \in X \Rightarrow \varphi \in Y) \)**
- **\( A = \) definable sets:** \( \{X : \varphi \in X\}, \varphi \in \text{Form}(V) \)

Free \( L^{\text{Par}} \)-algebras \( F_{L^{\text{Par}}}(V) \) are dual to canonical parametric frames \( C_L(\text{Par}, V) \):

- **underlying frame** \( C_L(\text{Par} \cup V) \)
- **\( X \models p \) \( \iff \) \( p \in X \)**
Universal frames of finite rank (1)

Canonical frames are too large
But: their top parts have an explicit description

Universal model $M_{K4}(V)$, $V \subseteq \text{Var}$ finite:

- start with empty model
- for each finite rooted model $F$ with $C = \text{rcl}(F)$: if
  - points of $C$ are distinguished by valuation of $V$,
  - $F \setminus C$ is a generated submodel of $M_{K4}(V)$, and
  - $\neg(F \setminus C$ is rooted, $\text{rcl}(F \setminus C)$ is reflexive, and includes a copy of $C$ wrt valuation $)$

then extend $M_{K4}(V)$ with a copy of $C$ below $F \setminus C$
(unless there already is one)
Universal frames of finite rank (2)

Characterization:
\( \mathcal{M}_{K4}(V) = \) unique model with valuation for \( V \) s.t.

- \( \mathcal{M}_{K4}(V) \) is locally finite
  (= rooted generated submodels are finite)
- each finite model with valuation for \( V \) has a unique \( p \)-morphism to \( \mathcal{M}_{K4}(V) \)

Universal frame \( \mathcal{U}_{K4}(V) = \) underlying frame of \( \mathcal{M}_{K4}(V) \)

\( P \subseteq \text{Par finite:} \)
Universal parametric frame \( \mathcal{U}_{K4}(P, V) = \) underlying frame of \( \mathcal{M}_{K4}(P \cup V) \) with its valuation of \( P \)
Universal frames of finite rank (3)

Generalization to $L \supseteq K4$ with finite model property (fmp):

$M_L(V) =$ the part of $M_{K4}(V)$ that's based on an $L$-frame

$\implies U_L(V), U_L(P, V)$

Properties:

- all finite subsets of $M_L(P, V)$ definable
- the dual of $U_L(P, V)$ is $F_{LP}(V)$
- $U_L(P, V)$ is the top part of $C_L(P, V)$:
  - $U_L(P, V)$ generated subframe of $C_L(P, V)$ (the points of finite depth)
  - all remaining points of $C_L(P, V)$ see points of $U_L(P, V)$ of arbitrarily large depth
- all $\neq \emptyset$ admissible subsets of $C_L(P, V)$ intersect $U_L(P, V)$
Admissibility using universal frames

\[ P \subseteq \text{Par finite}, \, \Gamma, \Delta \subseteq \text{Form(}P, \text{Var)} \text{ finite}, \, L \supseteq \text{K4 fmp} \]

Summary:

\[ \Gamma \vdash\bowtie L \Delta \iff \forall V \subseteq \text{Var finite}: \, F_{LP}(V) \models \Gamma / \Delta \]

\[ \iff \forall V \subseteq \text{Var finite}: \, C_L(P, V) \models \Gamma / \Delta \]

\[ \iff \forall V \subseteq \text{Var finite}: \, \langle U_L(P, V), <, D, \models_P \rangle \models \Gamma / \Delta \]

where \( D = \) subsets definable in \( M_L(P, V) \)

Typically:

Validity in \( U_L(P, V) \) is not difficult to characterize, but the restriction to \( D \) seriously complicates it
Toy model: logics of bounded depth

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If $L$ is a logic of bounded depth:

$\begin{align*}
\& C_L(P, V) = U_L(P, V) \\
\& C_L(P, V) \text{ is a finite frame}
\end{align*}$

$\implies$ admissibility easy to analyze

**Teaser:** Let $L$ be a logic of bounded depth. If

$\begin{align*}
\& \text{Par is finite, or} \\
\& \text{the set of finite } L\text{-frames is decidable,}
\end{align*}$

then $L$-unifiability is decidable.

**Proof:** $\Gamma \subseteq \text{Form}(P, \text{Var})$ is unifiable iff

\[ \exists \vdash \langle U_L(P, \emptyset), \vdash \rangle \vdash \Gamma. \]

We can compute $U_L(P, \emptyset)$. QED
...but some remain

The characterization

\[ \Gamma \models^L \Delta \iff U_L(P, V) \models \Gamma / \Delta \quad \forall V \text{ finite} \]

is not quite useful:

- \( U_L(P, V) \) are too rigidly specified
- \( U_L(P, V) \) are too large: \( \approx 2^{2^{\text{height}}} \) (height \( \approx \) depth of \( L \))
- we have no control over \( V \), anyway

\implies \text{need more convenient semantical description}
$L$-extensible models

$L$ logic of bounded depth, fix $P \subseteq \text{Par}$ finite

$F$ finite rooted parametric $L$-frame, $C = \text{rcl}(F)$:

- $F$ has **loosely separated root** if points of $C$ are distinguished by valuation of parameters
- $F$ has **separated root** if moreover $\neg (F \setminus C$ is rooted, $\text{rcl}(F \setminus C)$ is reflexive, and includes a copy of $C$ wrt valuation)

$W$ finite parametric $L$-frame:

- $W$ is **$L$-extensible** if $\forall F$ with a separated root: if $F \setminus \text{rcl}(F) \subseteq \cdot W$, it extends to $F \subseteq \cdot W$
- $W$ is **strongly $L$-extensible** if $\forall F$ with a loosely separated root . . .
Extensibility and canonical frames

Example: \( C_L(P, \emptyset) \) is the minimal \( L \)-extensible frame

More generally: \( C_L(P, V) \) is \( L \)-extensible for any finite \( V \subseteq \text{Var} \)

Converse: \( W \) \( L \)-extensible \( \iff \) p-morphic image of some \( C_L(P, V) \)

Corollary: If \( W \) \( L \)-extensible,

\[ \Gamma \vdash_L \Delta \implies W \models \Gamma / \Delta \]

for all \( \Gamma, \Delta \subseteq \text{Form}(P, \text{Var}) \)
Injectivity of extensible frames

$W$ finite parametric $L$-frame

$W$ is $L$-injective if $\forall$ finite par. $L$-frames $F_0 \subseteq F_1$: any p-morphism $F_0 \rightarrow W$ extends to a p-morphism $F_1 \rightarrow W$

Proposition: The following are equivalent:

- $W$ is $L$-extensible
- $W$ is $L$-injective
- $W$ is a retract of some $C_L(P, V)$: there are p-morphisms $f, g$ such that $f \circ g = \text{id}_W$
Connections among the properties

Proposition: The following are equivalent:

- $W$ is a p-morphic image of some $C_L(P, V)$
- $\Gamma \models_L \Delta \implies W \models \Gamma / \Delta$ for all $\Gamma, \Delta \subseteq \text{Form}(P, \text{Var})$

Warning: In general, $C_L(P, V)$ are not strongly $L$-extensible

strongly $L$-ext. $\implies$ $L$-ext. $\implies$ image of $C_L(P, V)$

Proposition: Any finite par. $L$-frame is a generated subframe of a strongly $L$-extensible frame

Corollary: Any $L$-extensible frame is a retract of a strongly $L$-extensible frame
Recall: $L$ logic of bounded depth, $P \subseteq \text{Par}$ finite

**Theorem:** For any $\Gamma, \Delta \subseteq \text{Form}(P, \text{Var})$, TFAE:

- $\Gamma \sim_L \Delta$
- $\Gamma / \Delta$ holds in all $L$-extensible frames
- $\Gamma / \Delta$ holds in all strongly $L$-extensible frames

$L$-extensible frames are structurally important

strongly $L$-extensible frames are *simpler* to define and a bit more robust to work with
Application

What to do next depends on the logic

Logics of bounded depth can still be quite wild

Tame subclass: logics of bounded depth and width

- finitely axiomatizable
- polynomial-size model property
- frames recognizable in polynomial time

Theorem: Let $L$ be a logic of bounded depth and width, $P \subseteq \text{Par}$ finite and $\Gamma, \Delta \subseteq \text{Form}(P, \text{Var})$ of size $n$.

If $\Gamma \not\vdash_L \Delta$, then $\Gamma / \Delta$ fails in a strongly $L$-extensible model of size at most $\text{poly}(n2^{2|P|})$.

In particular, $\vdash_L$ is decidable.
Addendum: smaller models

For fixed finite $P$, the models are polynomial-size, but in general doubly-exponential.

Let $\Sigma \subseteq \text{Form finite}$, closed under subformulas.

$\Sigma$-pruned $L$-extensible model: Like $L$-extensible, but when extending with a cluster $C$, allow it to shrink to a subset if satisfaction of $\Sigma$-formulas is preserved.

Theorem: Let $L$ be logic of bounded depth and width, $\Gamma \cup \Delta \subseteq \Sigma$. TFAE:

- $\Gamma \models_L \Delta$
- $\Gamma \vdash \Delta$ holds in $\Sigma$-pruned $L$-extensible models of size $2^{O(n^2)}$
Projective formulas

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Historical note

Projective formulas introduced by [Ghilardi’00]:

- semantical characterization of projective formulas
- existence of projective approximations for extensible logics
  \[\Rightarrow\] unification finitary
- parameter-free case only

We generalize it to the setup with parameters
Motivation

In the case of logics of bounded depth, we saw:

Admissibility closely connected to injective $L$-frames

These are dual to projective $L$-algebras

Finitely presented projective $L$-algebras are described by projective formulas:

Definition: $\varphi$ is $L$-projective if it has an $L$-unifier $\sigma$ s.t.

$$\varphi \vdash_L \sigma(\psi) \leftrightarrow \psi \quad \forall \psi \in \text{Form}$$

- it suffices to check $\psi \in \text{Var}$
- general algebraizable logics: $x \leftrightarrow y$ stands for $E(x, y)$
- $\sigma$ is a mgu of $\varphi$
Löwenheim substitutions

If $\sigma_1, \ldots, \sigma_m$ are substitutions s.t.

$$\varphi \vdash_L \sigma_i(\psi) \leftrightarrow \psi \quad \forall \psi \in \text{Form},$$

(*)

then this also holds for $\sigma_m \circ \cdots \circ \sigma_1$

$\implies$ build projective unifier \textit{inductively} by small steps

Löwenheim subtitutions satisfy (*)&

Fix $\varphi \in \text{Form}(P, V)$, where $P \subseteq \text{Par}$ and $V \subseteq \text{Var}$ finite

Let $F = \langle f_x : x \in V \rangle$, each $f_x : 2^P \to 2$ Boolean function of the parameters:

$$\theta_{\varphi, F}(x) = (\Box \varphi \land x) \lor (\neg \Box \varphi \land f_x(\vec{p}))$$

$\theta_{\varphi} = \text{composition of all } \theta_{\varphi, F} \text{ (in any order)}$
Theorem: Let $L \supseteq K4$ fmp, $\varphi \in \text{Form}(P, V)$. TFAE:

- $\varphi$ is projective
- $\theta_{\varphi}^N$ is a unifier of $\varphi$, where $N = (2^{|P|} + 1)|\varphi|$
- $\varphi$ has the model extension property:

Definition:

- $\text{Mod}_L = \text{finite rooted } L\text{-models}$
- $F, F' \in \text{Mod}_L$ are variants if they only differ in valuation of variables in root cluster
- $M \subseteq \text{Mod}_L$ has the model extension property if any $F \in \text{Mod}_L$ has a variant in $M$ whenever its proper rooted submodels belong to $M$
- $\varphi$ has m.e.p. iff $\text{Mod}_L(\varphi) = \{F \in \text{Mod}_L : F \models \varphi\}$ does
NB: projective formulas $\pi$ are admissibly saturated:

$$\pi \sim_L \Delta \iff \pi \vdash_L \Delta$$

$\Pi$ is a projective approximation of a formula $\varphi$ if

- $\Pi$ finite set of projective formulas
- $\varphi \sim_L \Pi$
- $\pi \vdash_L \varphi$ for each $\pi \in \Pi$

If $\varphi$ has a projective approximation $\Pi$:

- the set of proj. unifiers of $\pi \in \Pi$ is a finite csu of $\varphi$
- $\varphi \sim_L \Delta \iff \pi \vdash_L \Delta$ for all $\pi \in \Pi$

**Price:** existence of proj. apx. needs strong assumptions on $L$
Cluster-extensible logics

$L \supseteq K4 \text{ fmp, } n \in \omega, C \text{ finite cluster type:}$
irreflexive •, $k$-element reflexive ⊠

A finite rooted frame $F$ is of type $\langle C, n \rangle$ if

- $\text{rcl}(F)$ is of type $C$
- $\text{rcl}(F)$ has $n$ immediate successor clusters (≡ branching $n$)

$L \langle C, n \rangle$-extensible:
For each type-$\langle C, n \rangle$ frame $F$, if $F \setminus \text{rcl}(F)$
is an $L$-frame, then so is $F$

$L$ cluster-extensible (clx):
$\langle C, n \rangle$-extensible whenever it has some
type-$\langle C, n \rangle$ frame
Properties of clx logics

Examples: Any combinations of $K4, S4, GL, D4, K4Grz, K4.1, K4.3, K4B, S5, K4BB_k, K4BC_k$

Closed under joins and directed intersections (countable complete lattice)

Nonexamples: $K4.2, S4.2, \ldots$

Theorem: Every clx logic $L$

- is finitely axiomatizable
- has the exponential-size model property
- is $\forall\exists$-definable on finite frames
- is described by finitely many forbidden types $\langle C, n \rangle$
- is described by finitely many extension conditions: $\langle C, n \rangle$ where $n \in \omega \cup \{\infty\}$, $C$ cluster type or $\infty$
Theorem: $L$ clx logic $\Rightarrow$ every formula $\varphi$ has a projective approximation $\Pi$ s.t.

- each $\pi \in \Pi$ is a Boolean combination of subformulas of $\varphi$
- $|\Pi| \leq 2^{2^n}$, $|\pi| = O(n2^n)$ for each $\pi \in \Pi$

Corollary:

- each $\varphi$ has a finite csu
- we can compute it
- $L$-admissibility and $L$-unifiability are decidable
Admissibility in clx logics
Admissibility in transitive modal logics studied in depth by [Rybakov’97]

- semantical characterizations
- decidability results
- many results include parameters

We follow a different route, based on Ghilardi’s work on projectivity
Tight predecessors

\( P \subseteq \text{Par finite}, \ C \text{ finite cluster type}, \ n \in \omega \)

\( L \text{ clx logic, } W \text{ parametric } L\text{-frame:} \)

\( \mathbf{\check{\triangleright}} \ W \text{ is } \langle C, n \rangle\text{-extensible } \iff \)
\( \forall \ E \subseteq 2^P, \ 0 < |E| \leq |C| \)
\( \forall \ X = \{ w_i : i < n \} \subseteq W \)
\( \exists \text{ tight predecessor (tp)} \ T = \{ u_e : e \in E \} \subseteq W: \)

\( u_e \vDash P^e, \quad u_e \uparrow = \begin{cases} X \uparrow \\ X \uparrow \cup T \end{cases} \quad C = \bullet \)

\( C \text{ reflexive} \)

\( \mathbf{\check{\triangleright}} \ W \text{ is } L\text{-extensible if it is } \langle C, n \rangle\text{-extensible whenever } L \text{ is} \)
Extension rules

\( P \subseteq \text{Par} \) finite, \( C \) finite cluster type, \( n \in \omega \)

\( \langle C, n \rangle \)-extensible frames axiomatized by extension rules \( \text{Ext}_{P}^{C,n} \):

\( \blacklozenge \) \( C = \bullet \): for each \( e \in 2^{P} \),

\[ P^{e} \land \Box y \rightarrow \bigvee_{i < n} \Box x_{i} \]  
\( \bigg\{ \Box y \rightarrow x_{i} : i < n \bigg\} \)

\( \blacklozenge \) \( C = \bigodot k \): for each \( E \subseteq 2^{P} \) and \( e_{0} \in E \), where \( |E| \leq k \),

\[ \left[ P^{e_{0}} \land \Box \left( y \rightarrow \bigvee_{e \in E} \Box (P^{e} \rightarrow y) \right) \bigwedge_{e \in E} \Box \left( \Box (P^{e} \rightarrow \Box y) \rightarrow y \right) \right] \rightarrow \bigvee_{i < n} \Box x_{i} \]  
\( \bigg\{ \Box y \rightarrow x_{i} : i < n \bigg\} \)
Semantics of extension rules

**Theorem:** Let \( W \) be a parametric frame

- If \( W \) is \( \langle C, n \rangle \)-extensible, then \( W \models \text{Ext}^P_{C,n} \)
- If \( W \models \text{Ext}^P_{C,n} \), then \( W \) is \( \langle C, n \rangle \)-extensible, provided \( W \) is descriptive or Kripke

**Moreover:** If \( L \) has fmp, then \( L + \text{Ext}^P_{C,n} \) is complete wrt locally finite \( \langle C, n \rangle \)-extensible Kripke frames

**Theorem:** If \( L \supseteq \text{K4} \) has fmp, TFAE:

- \( L \) is \( \langle C, n \rangle \)-extensible
- \( \text{Ext}_{C,n} \) is \( L \)-admissible
Characterization of admissibility

**Theorem:** If $L$ is clx logic, TFAE:

- $\Gamma \vdash_L \Delta$
- $\Gamma / \Delta$ holds in all $L$-extensible parametric frames
- $\Gamma / \Delta$ holds in all locally finite $L$-extensible parametric Kripke frames
- $\Gamma / \Delta$ is derivable in $L + \{\text{Ext}_{C,n}^{\text{Par}} : L \text{ is } \langle C, n \rangle$-extensible\}

**NB:** To pass from a locally finite $L$-extensible countermodel to a definable valuation, use projective formulas
Corollary: If $L$ is clx, the rules $\text{Ext}_{C,n}$ are a basis of $L$-admissible rules

Variation:

- Explicit single-conclusion bases
- If Par is finite:
  - $L$ has a finite basis $\iff L$ has bounded branching
  - $L$ has explicit independent bases (mc or sc)
- If Par is infinite:
  - No consistent logic has a finite basis
  - Open problem: independent bases?
**Smaller models**

$L$-extensible frames are usually infinite

Let $\Sigma \subseteq \text{Form}$ finite, closed under subformulas

Finite models $L$-pseudoextensible wrt $\Sigma$:
Like $L$-extensible, but instead of tp’s, have
tight pseudopredecessors wrt $\Sigma$
$\approx$ behave as tp as concerns satisfaction of $\Sigma$-formulas

**Theorem:** If $L$ clx logic and $\Gamma, \Delta \subseteq \Sigma$, TFAE:

- $\Gamma \models_L \Delta$
- $\Gamma / \Delta$ holds in all $L$-pseudoextensible models wrt $\Sigma$
- $\Gamma / \Delta$ holds in all $L$-pseudoextensible models wrt $\Sigma$ of size $2^{O(n)}$
Variants of clx logics

Tweak the definition to cover other kinds of logics:

▶ Logics with a single top cluster (extensions of $\text{K4.2}$)
  ▶ Top-restricted cluster-extensible (tclx) logics: extension conditions only for frames with a single top cluster
  ▶ Examples: joins of $\text{K4.2}$ with clx logics

▶ Superintuitionistic logics
  ▶ Behave much like their largest modal companions (Blok–Esakia isomorphism)
  ▶ The only (t)clx si logics are $\text{IPC}$, $\text{T}_n$, $\text{KC}$, $\text{KC + T}_n$
    ($\text{NB: } \text{T}_1 = \text{LC}, \text{T}_0 = \text{CPC}$)
Problems and complexity classes

1. Logics and admissibility
2. Transitive modal logics
3. Toy model: logics of bounded depth
4. Projective formulas
5. Admissibility in clx logics
6. Problems and complexity classes
7. Complexity of derivability
8. Complexity of admissibility
Main questions

For a fixed logic $L$, what is the computational complexity of:

- Given $\Gamma, \Delta$, is $\Gamma / \Delta$ $L$-admissible?
- Is a given $\Gamma$ $L$-unifiable?

Here, $\Gamma$ and $\Delta$ may be sets of formulas

- without parameters
- with parameters
- with $O(1)$ parameters

Recall: unifiability is a special case of inadmissibility

\[ \Rightarrow \] typically, admissibility and unifiability captured by dual complexity classes

Ideally:

- lower bounds for unifiability
- upper bounds for admissibility
Common classes of languages:

- $P =$ deterministic polynomial time
- $NP =$ nondeterministic polynomial time
- $coX = \{\Sigma^* \setminus L : L \in X\}$, e.g. $coNP$
- $P^X =$ polynomial time with oracle from $X$, etc.
- polynomial hierarchy: $\Sigma^p_0 = \Delta^p_0 = \Pi^p_0 = P$, $\Sigma^p_k = NP^{\Sigma^p_{k-1}}$, $\Delta^p_k = P^{\Sigma^p_{k-1}}$, $\Pi^p_k = coNP^{\Sigma^p_{k-1}}$
- $PSPACE =$ polynomial space
- $EXP =$ deterministic exponential ($2^{nc}$) time
- $NEXP =$ nondeterministic exponential time
- exponential hierarchy:
  $\Sigma^{exp}_k = NEXP^{\Sigma^p_{k-1}}$, $\Delta^{exp}_k = EXP^{\Sigma^p_{k-1}}$, $\Pi^{exp}_k = coNEXP^{\Sigma^p_{k-1}}$
Alternating Turing machines

alternating Turing machine (ATM):

- multiple transitions from a given configuration ($\approx$ NTM)
- states labelled existential or universal
- acceptance defined inductively:
  - configuration in an $\exists$ state is accepting $\iff \exists$ transition to an accepting configuration
  - configuration in a $\forall$ state is accepting $\iff \forall$ transitions lead to accepting configurations
- alternation: go from $\exists$ state to $\forall$ state or vice versa
- $\Sigma^k_{\text{TIME}}(f(n))$: computable by ATM in time $f(n)$, start in $\exists$ state, make $\leq k - 1$ alternations
- $\Pi^k_{\text{TIME}}(f(n))$: start in $\forall$ state
  - $\Sigma^1_{\text{TIME}} = \text{NTIME}, \Pi^1_{\text{TIME}} = \text{coNTIME}$
Expressing classes with ATMs

- polynomial and exponential hierarchies:

\[
\Sigma^p_k = \Sigma^\text{k-TIME}(\text{poly}(n)) \quad \Pi^p_k = \Pi^\text{k-TIME}(\text{poly}(n)) \\
\Sigma^{\exp}_k = \Sigma^\text{k-TIME}(2^{\text{poly}(n)}) \quad \Pi^{\exp}_k = \Pi^\text{k-TIME}(2^{\text{poly}(n)})
\]

- \( \text{PSPACE} = \text{AP} \) (alternating polynomial time)

- \( \text{EXP} = \text{APSPACE} \) (alternating polynomial space)
Reductions and completeness

- **Y** (many-one) reducible to **X** if there is \( f : \Sigma^* \rightarrow \Sigma^* \) s.t.

\[
w \in Y \iff f(w) \in X,
\]

- \( f \) efficiently computable:
  - polynomial-time
  - logspace: in space \( O(\log n) \), excluding input tape (read-only) and output tape (write-only)

- **C** a class: **X** is **C**-hard if every \( Y \in C \) reduces to **X**
- **X** **C**-complete if \( X \in C \) and **C**-hard

Examples:

- **SAT** is **NP**-complete, **TAUT** (ie, **CPC**) is **coNP**-complete
- **QSAT** is complete for \( AP = \text{PSPACE} \)
Completeness in exponential hierarchy

**Theorem:** Fix $k \geq 1$. The set of true $\Sigma^2_k$ sentences

$$\exists X_1 \subseteq 2^n \forall X_2 \subseteq 2^n \ldots Q X_k \subseteq 2^n \overline{Q} t_1, \ldots, t_m \in 2^n \varphi$$

is a $\Sigma^\text{exp}_k$-complete problem, where

- $2 = \{0, 1\}$
- $Q = \exists$ for $k$ odd, $\forall$ for $k$ even
- $\overline{Q} =$ dual of $Q$
- $n$ given in unary
- $\varphi$ Boolean combination of atomic formulas $t_\alpha \in X_j$, $t_\alpha(i)$ ($i < n$ constant)
Complexity of derivability

1. Logics and admissibility
2. Transitive modal logics
3. Toy model: logics of bounded depth
4. Projective formulas
5. Admissibility in clx logics
6. Problems and complexity classes
7. Complexity of derivability
8. Complexity of admissibility
Derivability and tautologicity

Before admissibility, let’s consider a baseline problem:

▶ Given $\Gamma, \Delta$, is $\Gamma \vdash \Delta$ $L$-derivable?

In transitive logics, this is equivalent to $L$-tautologicity:

▶ Given $\varphi$, is $\vdash L \varphi$?

NB: Special case of $L$-admissibility, but also of $L$-unifiability with parameters:

$$\varphi \in \text{Form}(\text{Par}, \emptyset) \implies (\vdash L \varphi \iff \varphi \text{ unifiable})$$
coNP cases

Lower bound: By reduction from CPC, $L$-derivability is coNP-hard for any consistent $L$

Upper bound: $L$-derivability is in coNP if:

- $L$ has a polynomial-size model property
- finite $L$-frames are recognizable in P (or NP)

Corollary: $L$-derivability coNP-complete for:

- consistent linear (= width 1) clx logics
- consistent logics of bounded depth and width
Theorem [Ladner’77]
Derivability in $K, T, S4$ is PSPACE-complete
For any $K \subseteq L \subseteq S4$, it is PSPACE-hard

Upper bound:

$\approx$ explore proof tree/countermodel one branch a time
Can be adapted (bounded branching little tricky):

Theorem: Derivability in any (t)clx logic is in PSPACE

Lower bound:

- reduction from QSAT
- easily adapted to all logics with the disjunction property
  - reference?
  - superintuitionistic logics with DP: [Chagrov’85]
We give another generalization, using reduction from IPC.

Theorem: Derivability is PSPACE-hard for all logics $L \supseteq K4$ that are subframe-universal for trees.

- **subreduction**: $\approx \ p$-morphism from a subframe
- **weak subreduction**: ignore reflexivity
- **$L$ subframe-universal for trees** if $\forall$ finite tree $T$  
  $\exists$ weak subreduction from an $L$-frame onto $T$
- **cofinal subreduction**: $\text{dom}(f)^\uparrow \subseteq \text{dom}(f)^\downarrow$  
  $\implies$ cofinal weak subreduction,  
  cofinally subframe-universal for trees
Applications of the lower bound

**Theorem:** All logics $L \supseteq K4$ with the disjunction property are cofinally subframe-universal for trees

**Corollary:**

- Derivability in $L \supseteq K4$ with DP is PSPACE-hard
- Derivability in nonlinear clx logics is PSPACE-complete
- Derivability is also PSPACE-complete in nonlinear tclx logics: $K4.2, S4.2, \ldots$
Completeness results for complexity of clx logics:

<table>
<thead>
<tr>
<th>logic</th>
<th>$\not\models_L$</th>
<th>unifiability, $\not\models_L$</th>
<th>examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>branching</td>
<td>clust. size</td>
<td>parameters:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\not\models_L$</td>
<td>no†</td>
<td>$O(1)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>any</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0·0</td>
<td>$\Pi_2^p$</td>
<td>$S_5 \oplus \text{Alt}_k,\text{CPC}$</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>$\text{coNEXP}$</td>
<td>$S_5, \text{K4B}$</td>
</tr>
<tr>
<td>1</td>
<td>0·0</td>
<td>$\text{PSPACE}$</td>
<td>$\text{GL.3, LC}$</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>$\text{coNEXP}$</td>
<td>$\text{S4.3, K4.3}$</td>
</tr>
<tr>
<td>≥ 2</td>
<td>0·0</td>
<td>$\text{PSPACE}$</td>
<td>$\text{GL, S4Grz, IPC}$</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td>$\Sigma_2^\text{exp}$</td>
<td>$\text{K4, S4}$</td>
</tr>
</tbody>
</table>

† The parameter-free case is for $\not\models_L$ only.
Completeness results for complexity of $tclx$ logics:

<table>
<thead>
<tr>
<th>logic</th>
<th>unifiability, $\vdash L$</th>
<th>examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>cluster size: inner</td>
<td>$\vdash L$</td>
<td>parameters:</td>
</tr>
<tr>
<td>top</td>
<td></td>
<td>no$^\dagger$</td>
</tr>
</tbody>
</table>

| $< \infty$ | $< \infty$ | PSPACE | NEXP | GL.2, Grz.2, KC |
| $\infty$   | $\infty$   |        |      | $\Theta_2^{\exp}$ S4.1.4 $\oplus$ S4.2 |
| $\infty$   | $\infty$   |        |      | $\Sigma_2^{\exp}$ K4.2, S4.2 |

$^\dagger$ The parameter-free case is for $\vdash L$ only

**NB:** branching $\geq 2$ by definition
### Complexity results for logics of bounded depth and width:

<table>
<thead>
<tr>
<th>logic</th>
<th>$\mathcal{L}$</th>
<th>unif.</th>
<th>$\mathcal{L}$</th>
<th>notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>width</td>
<td>cluster size</td>
<td>sing.-c.</td>
<td>mult.-c.</td>
<td></td>
</tr>
<tr>
<td>$1$</td>
<td>$&lt; \infty$</td>
<td>$\Pi^b_{2d}$</td>
<td>$\text{coNEXP}$</td>
<td>depth $d$</td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>$\mathsf{NP}$</td>
<td>$\mathsf{NEXP}$</td>
<td>$\mathsf{DEXP}$</td>
<td>$\mathsf{BH}^\mathsf{exp}_4$</td>
</tr>
<tr>
<td></td>
<td>$&lt; \infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\infty$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

($\dagger$) also the complexity of unifiability and $\mathcal{L}$ with O(1) parameters
Upper bounds

Semantic characterization:
pseudoextensible / pruned extensible models (size $2^{\text{poly}(n)}$)

\[ \implies \text{inadmissibility in (t)clx or bd-dp-wd logics is in } \Sigma_{\exp}^2: \]

\[ \exists \text{model } \forall E \subseteq 2^P \ldots \]

Optimization in certain cases:

- bounded cluster size:
  \[ \forall E \subseteq 2^P, |E| \leq k \text{ becomes a poly-size quantifier} \]
- constant number of parameters: same reason
- width 1:
  - the model is an upside-down tree of clusters
  - an ATM can explore it while keeping only one branch
Basic tenet: Hardness of $L$-unifiability stems from finite patterns that occur as subframes in $L$-frames.

That is, study conditions of the form:

$\exists$ an $L$-frame that subreduces to $F \implies L$-unifiability $C$-hard

Example conditions:

- $L$ has unbounded depth:
  $L$-frames weakly subreduce to arbitrary finite chains

- $L$ has unbounded cluster size:
  $L$-frames subreduce to arbitrary reflexive clusters

- $L$ is nonlinear ($\approx$ width $\geq 2 = \text{branching} \geq 2$):
  an $L$-frame subreduces to a 2-prong fork
Lower bound technique

Reduce a $C$-complete problem to $L$-unifiability:

- **PSPACE**, $\Sigma^p_k/\Pi^p_k$: QSAT, $\Sigma_k/\Pi_k$-SAT
- $\Sigma^{\text{exp}}_k/\Pi^{\text{exp}}_k$: special $\Sigma^2_k/\Pi^2_k$-sentences as above

Encode quantifiers:

- $\exists$ simulated by variables, $\forall$ by parameters
- $t \in 2^n$: directly by $n$-tuple of atoms
- $\forall X \subseteq 2^n$: parameter assignments realized in a cluster
- $\exists X \subseteq 2^n$: single variable $x$
  - use antichains to enforce consistency:
  - $u \vDash \sigma(x)$ unaffected by a change of parameters in $v \not\equiv u$
\( \Sigma^\text{exp}_2 \) bounds

**Lower bound:**
If \( \forall n \) an \( L \)-frame subreduces to a rooted frame containing

- an \( n \)-element cluster, and
- an incomparable point

\[ \implies \text{\( L \)-unifiability is } \Sigma^\text{exp}_2 \text{-hard} \]

**Upper bound:**
\( L (t)\text{clx} \) or \( \text{bd-dp-wd logic} \implies \text{\( L \)-inadmissibility is in } \Sigma^\text{exp}_2 \)

**Examples:**
\( K4, S4, S4.1, S4.2, K4BB_2, S4BB_2BD_2, \ldots \)
NEXP bounds

Lower bound:
\[ L \text{ nonlinear} \implies L\text{-unifiability is NEXP-hard} \]

\((O(1) \text{ parameters? next slide})\)

Upper bounds:

- \( L (t)\text{clx logic of bounded cluster size, or a tabular logic} \implies L\text{-inadmissibility is in NEXP} \)
- \( L (t)\text{clx logic} \implies L\text{-inadmissibility with } O(1) \text{ parameters is in NEXP} \)

Examples:

GL, K4Grz, S4Grz, S4Grz.2, IPC, KC, \ldots

(± bounded branching)
NEXP lower bounds w/ $O(1)$ parameters

Need stronger hypothesis (cf. logics of bounded depth)

Theorem: The following problems are NEXP-hard

- $L$-unifiability with 2 parameters
  - if $L$ subframe-universal for trees
- $L$-unifiability with 1 parameter
  - if $L$ cofinally subframe-universal for trees
    (includes: logics with disjunction property)
  - if $L$ subframe-universal respecting reflexivity
- $L$-unifiability with 0 parameters
  - if $K4 \subseteq L \subseteq K4.2GrzBB_2$
- single-conclusion $L$-inadmissibility with 0 parameters
  - if $L$ has a certain weak extension property
  - this includes: nonlinear clx logics
coNEXP bounds

Lower bound:
$L$ unbounded cluster size
\[\Rightarrow \text{L-unifiability is coNEXP-hard}\]

Upper bound:
$L$ linear clx or bd-dp logic
\[\Rightarrow \text{L-inadmissibility is in coNEXP}\]

Examples: S5, K4.3, S4.3, ...
PSPACE bounds

Lower bound:
$L$ unbounded depth ⇒
$L$-unifiability with 2 parameters is PSPACE-hard

Corollary:
$L$-unifiability is PSPACE-hard
unless $L$ linear tabular logic

Upper bound:
$L$ linear clx logic of bounded cluster size
⇒ $L$-admissibility is in PSPACE

Examples:
GL.3, K4Grz.3, S4Grz.3, LC, ...
Recall: $L$-unifiability $\text{PSPACE}$-hard unless $L$ linear tabular

Remaining case:
$L$ linear tabular logic of depth $d \Rightarrow$
$L$-unification and $L$-inadmissibility are $\Pi_{2d}^P$-complete

Examples:
$\text{CPC}$, $G_{d+1}$, $\text{S5} \oplus \text{Alt}_k$, $\text{K4} \oplus \square \bot$, ...

$L$ unbounded depth $\Rightarrow$ $\text{PSPACE}$-hard with 2 parameters

Theorem:
$L$ bd-dp-wd logic $\Rightarrow$
$L$-inadmissibility with $O(1)$ parameters is NP-complete
More exotic classes

Exponential version of \( \Theta_2^p \):

\[
\Theta_2^{\exp} = \text{EXP}^{\text{NP}[\text{poly}]} = \text{EXP} \parallel \text{NP} = \text{P}^{\text{NEXP}} = \text{PSPACE}^{\text{NEXP}}
\]

Exponential version of the Boolean hierarchy:

- \( \text{BH}^{\exp} \) = closure of NEXP under Boolean operations
- Stratified into levels:
  - \( \text{BH}_1^{\exp} = \text{NEXP} \)
  - \( \text{BH}_k^{\exp} = \{ A \setminus B : A \in \text{NEXP}, B \in \text{BH}_k^{\exp} \} \)
- Special case:
  - \( \text{DEXP} = \text{BH}_2^{\exp} = \{ A \cap B : A \in \text{NEXP}, B \in \text{coNEXP} \} \)

\[
\text{NEXP}, \text{coNEXP} \subseteq \text{BH}^{\exp} \subseteq \Theta_2^{\exp} \subseteq \Delta_2^{\exp}
\]
\(\Theta_2^{\exp}\) bounds

**Lower bound:**
\[\forall n \exists \text{graph-connected } L\text{-frame of cluster size } \geq n \text{ and width } \geq 2 \implies L\text{-unifiability is } \Theta_2^{\exp}\text{-hard}\]

**Upper bound:**

\(L\)-admissibility is in \(\Theta_2^{\exp}\) if

- \(L\) is a tclx logic of bounded inner cluster size, or
- \(L\) is a bd-dp-wd logic, doesn’t satisfy the \(\Sigma_2^{\exp}\) LB condition

**Example:** \(S4.2 \oplus S4.1.4\)
The NEXP and coNEXP lower bounds imply:

**Lower bound:**
$L$ nonlinear logic of unbounded cluster size
$\implies L$-unifiability is DEXP-hard

Rare case where unifiability and inadmissibility (with parameters) have different complexity:

**Upper bound:**
$L$ bd-dp-wd logic, doesn't satisfy the $\Theta_2^{\exp}$ LB condition
$\implies$

- $L$-unifiability is in DEXP
- single-conclusion $L$-inadmissibility is in $BH_4^{\exp}$
- multiple-conclusion $L$-inadmissibility is in $EXP^{NP[\log n]}$
Full classification?

**Known:** complexity of unifiability for (t)clx, bd-dp-wd logics

Could it be determined for all logics $L \supseteq K4$?

▶ **Hopeless:**
 already $\vdash_L$ can be undecidable, arbitrary Turing degree

▶ **The form of results that we’ve seen:**
  - **Upper bounds:** tame, nicely-behaved logics
  - **Lower bounds:** logics allowing certain frame patterns
    $\implies$ downward-closed classes of logics

▶ **Determine minimal** complexity of unifiability among sublogics of $L$?

**Definition:** Unifiability has **hereditary hardness** $C$ below $L$ if

▶ $L'$-unifiability is $C$-hard for all $L' \subseteq L$

▶ $L'$-unifiability is in $C$ for some $L' \subseteq L$
**Hereditary hardness**

**Theorem:** Below any $L \supseteq K4$, one of the following applies:

<table>
<thead>
<tr>
<th>Width</th>
<th>Logic $L$</th>
<th>Hereditary hardness of unifiability</th>
<th>Witness $L' \subseteq L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; $\infty$</td>
<td>depth $d$</td>
<td>$\Pi_2^d$</td>
<td>$= L$</td>
</tr>
<tr>
<td>$\infty$</td>
<td>depth $\infty$</td>
<td>PSPACE</td>
<td>$K4.3BC_k$</td>
</tr>
<tr>
<td>$\leq 2$</td>
<td>certain conditions</td>
<td>DEXP</td>
<td>$K4BC_k$</td>
</tr>
<tr>
<td>$\geq 2$</td>
<td>(as before)</td>
<td>$\Theta_2^{\exp}$</td>
<td>$K4$</td>
</tr>
</tbody>
</table>

($D4 = K4 \oplus \Box \perp$, $BIC_k = \text{bounded inner cluster size}$)

Except for DEXP, also applies to inadmissibility
Thank you for attention!
References


E. J.: Rules with parameters in modal logic II, coming soon
E. J.: Rules with parameters in modal logic III, planned
R. E. Ladner: The computational complexity of provability in systems of modal propositional logic, SIAM J. Comput. 6 (1977), 467–480
V. V. Rybakov: Admissibility of logical inference rules, Elsevier, 1997