Bounded induction without parameters

Emil Jeřábek

jerabek@math.cas.cz

http://math.cas.cz/~jerabek/

Institute of Mathematics of the Czech Academy of Sciences, Prague

FEALORA Farewell Workshop
Špindlerův Mlýn, November 2018

Parameters in induction axioms

In arithmetic, induction (and other) schemata usually allow formulas with free parameters:

$$\varphi(0, y) \land \forall x\, (\varphi(x, y) \to \varphi(x + 1, y)) \to \forall x\, \varphi(x, y)$$

Examples: $I\Sigma_i$, S^i_2, T^i_2, ...

- for full induction, parameters make no difference
- fragments become genuinely weaker without parameters
- strong theories: many intriguing results about $I\Sigma^-_n$, $I\Pi^-_n$
- closely related to induction rules
- characterizations using reflection principles
- this talk: parameter-free versions of Buss's theories

Parameter-free bounded arithmetic

- $T^i_2 = \hat{\Sigma}^b_i\text{-}IND = \hat{\Pi}^b_i\text{-}IND$, $S^i_2 = \hat{\Sigma}^b_i\text{-}PIND = \hat{\Pi}^b_i\text{-}PIND$
- $\hat{\Sigma}^b_i\text{-}(P)IND^-$, $\hat{\Pi}^b_i\text{-}(P)IND^-$:

$$\varphi(0) \land \forall x\, \big(\varphi(x) \to \varphi(x+1)\big) \to \forall x\, \varphi(x)$$
$$\varphi(0) \land \forall x\, \big(\varphi(\lfloor x/2 \rfloor) \to \varphi(x)\big) \to \forall x\, \varphi(x)$$

- $\hat{\Sigma}^b_i\text{-}(P)IND^R$, $\hat{\Pi}^b_i\text{-}(P)IND^R$:

$$\frac{\varphi(0) \quad \varphi(x) \to \varphi(x+1)}{\varphi(x)} \qquad \frac{\varphi(0) \quad \varphi(\lfloor x/2 \rfloor) \to \varphi(x)}{\varphi(x)}$$

  - parameters do not matter for $(P)IND^R$
  - this is not a sequent calculus (no "side formulas")

Theories and rules

Theories axiomatized not just by axioms, but by more general rules

$$\frac{\varphi_1, \dots, \varphi_k}{\varphi}$$

T an ordinary FO theory, R a set of rules:

- $[T, R]$ = closure of T under unnested R-rules (axiomatized by T + those φ s.t. $T \vdash \varphi_1 \wedge \dots \wedge \varphi_k$)
- $[T, R]_0 := T$, $[T, R]_{n+1} := [[T, R]_n, R]$
  $T + R := \bigcup_n [T, R]_n$
- R is reducible to R' ($R \leq R'$) if $[T, R] \subseteq [T, R']$ for all T
- R and R' are equivalent ($R \equiv R'$) if $R \leq R' \leq R$

Parameter-free axioms vs. rules

$\Gamma = \hat{\Sigma}^b_i$ or $\hat{\Pi}^b_i$:

- variants of Γ-$(P)IND^R$ with and without parameters equivalent
- Γ-$(P)IND^-$ is the least theory whose all extensions are closed under Γ-$(P)IND^R$
  - conservation results over Γ-$(P)IND^-$ follow from conservation results over Γ-$(P)IND^R$
  - a converse also holds

Previous work

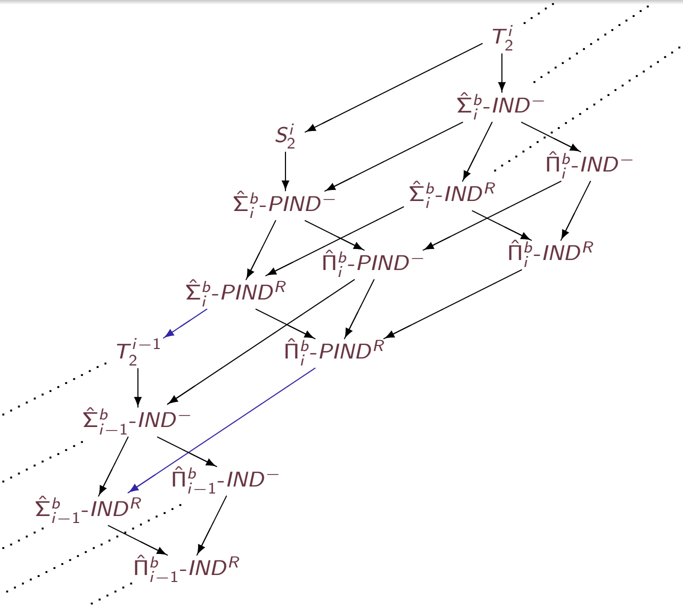
Parameter-free induction in bounded arithmetic:

- [K'90] IE_i is $\exists\forall E_i$-conservative over IE_i^-
- [BI'92] studied Σ_i^b parameter-free rules
- [CFL'09] proved conservation results for $\hat{\Sigma}_i^b$ rules and parameter-free schemata

Many questions left unanswered:

- $\hat{\Pi}_i^b$ rules and parameter-free schemata?
- nesting (number of instances)?
- reflection principles?

At a glance


T^i_2
$\hat\Sigma^b_i$-IND^-
S^i_2
$\hat\Pi^b_i$-IND^-
$\hat\Sigma^b_i$-$PIND^-$
$\hat\Sigma^b_i$-IND^R
$\hat\Pi^b_i$-$PIND^-$
$\hat\Pi^b_i$-IND^R
$\hat\Sigma^b_i$-$PIND^R$
T^{i-1}_2
$\hat\Pi^b_i$-$PIND^R$
$\hat\Sigma^b_{i-1}$-IND^-
$\hat\Pi^b_{i-1}$-IND^-
$\hat\Sigma^b_{i-1}$-IND^R
$\hat\Pi^b_{i-1}$-IND^R

Conservation over $\hat\Sigma^b_i$ rules

The following was proved by [CFL'09], based on [K'90,BI'92]:

Theorem

If T is $\forall\exists\hat\Sigma^b_{i+1}$, then $T + T^i_2$ ($T + S^i_2$) is $\forall\hat\Sigma^b_i$-conservative over $T + \hat\Sigma^b_i$-$(P)IND^R$

Corollary

- T^i_2 (S^i_2) is $\exists\forall\hat\Pi^b_{i+1}$-conservative over $\hat\Sigma^b_i$-$(P)IND^-$
- If T is $\forall\hat\Sigma^b_i$, $T + \hat\Pi^b_{i+1}$-$PIND^R = T + \hat\Sigma^b_i$-$IND^R$
- [Buss]: … and $T + \hat\Sigma^b_{i+1}$-$PIND^R = T + T^i_2$

Conservativity over $\hat{\Pi}^b_i$ rules

Theorem

If T is $\forall\hat{\Sigma}^b_i$, then $T + T^i_2$ ($T + S^i_2$) is $\forall\exists\hat{\Sigma}^b_{i-1}$-conservative over $T + \hat{\Pi}^b_i$-$(P)IND^R$

Corollary

T^i_2 (S^i_2) is $\exists\hat{\Sigma}^b_{i+1} \vee \forall\exists\hat{\Sigma}^b_{i-1}$ conservative over $\hat{\Pi}^b_i$-$(P)IND^-$

Nesting of rules

For $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, every $\varphi \in [T, \Gamma\text{-}(P)IND^R]_k$ can be proved using k instances of $\Gamma\text{-}(P)IND^R$

Theorem

- If T is $\forall\hat{\Sigma}^b_\infty$: $T + \hat{\Pi}^b_i\text{-}(P)IND^R = [T, \hat{\Pi}^b_i\text{-}(P)IND^R]$
- If T is $\forall\hat{\Sigma}^b_i$: $T + \hat{\Sigma}^b_i\text{-}(P)IND^R = [T, \hat{\Sigma}^b_i\text{-}(P)IND^R]$

The instance only depends on axioms of T used:

- If $T = BTC^0 + \forall x\, \xi(x)$ with $\xi \in \hat{\Sigma}^b_i$:

$$T + \hat{\Sigma}^b_i\text{-}(P)IND^R = BTC^0 + \mathrm{RFN}_i(G_i^{(*)} + \xi)$$

$$T + \hat{\Pi}^b_i\text{-}(P)IND^R = T + \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi)$$

  - closure finitely axiomatizable

Parameter-free conservation

Generalizing a result of [K'88]:

Theorem

Let $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, and T be of any complexity:

- $T + \Gamma\text{-}(P)IND^-$ is $\forall\Gamma$-conservative over $T + \Gamma\text{-}(P)IND^R$
- All $\forall\Gamma$ consequences of T + arbitrary k instances of $\Gamma\text{-}(P)IND^-$ are in $[T, \Gamma\text{-}(P)IND^R]_k$

If $\Gamma\text{-}(P)IND^-$ is finitely axiomatizable, there is k s.t. $T + \Gamma\text{-}(P)IND^R = [T, \Gamma\text{-}(P)IND^R]_k$ for every T

Propositional proof systems

G_i = Σ^q_i-fragment of quantified propositional sequent calculus

$\mathrm{RFN}_j(P)$ = "every P-provable Σ^q_j sequent is valid"

$\varphi(x) \in \hat{\Sigma}^b_\infty \implies$ propositional translations $[\![\varphi]\!]_n(p_0, \dots, p_{n-1})$

Definition

Let $\xi \in \hat{\Sigma}^b_i$:

- $G_i + \xi$ = G_i with extra initial sequents

$$\implies [\![\xi]\!]_n(A_0, \dots, A_{n-1})$$

  with $A_0, \dots, A_{n-1}$ quantifier-free
- $G^*_i + \xi$ is its tree-like version

Correspondence

By extension of standard results, one can show easily

Theorem

Let $\xi, \varphi \in \hat{\Sigma}^b_i$:

- If $T^i_2 + \forall x\, \xi(x) \vdash \forall x\, \varphi(x)$, then ($BTC^0$-provably) there are poly-size $G_i + \xi$ proofs of $[\![\varphi]\!]_n$
- $T^i_2 + \forall x\, \xi(x)$ proves $\mathrm{RFN}_i(G_i + \xi)$
- $T^i_2 \rightsquigarrow S^i_2$: $G_i + \xi \rightsquigarrow G^*_i + \xi$

Induction rules vs. reflection principles

Theorem

The rules on the LHS are equivalent to the rules on the RHS for $\xi \in \hat{\Sigma}^b_i$:

$$
\begin{array}{ll}
\hat{\Sigma}^b_i\text{-}(P)IND^R & \forall x\, \xi(x) \;/\; \mathrm{RFN}_i(G_i^{(*)} + \xi) \\
\hat{\Sigma}^b_i\text{-}(P)IND^- & \forall x\, \xi(x) \to \mathrm{RFN}_i(G_i^{(*)} + \xi) \\
\hat{\Pi}^b_i\text{-}(P)IND^R & \forall x\, \xi(x) \;/\; \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi) \\
\hat{\Pi}^b_i\text{-}(P)IND^- & \forall x\, \xi(x) \to \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi)
\end{array}
$$

A witnessing theorem

Theorem

If T^i_2 (S^i_2) proves $\forall x\, \varphi(x)$, $\varphi \in \exists\forall\hat{\Pi}^b_i$, there are $\hat{\Pi}^b_{i-1}$ formulas $\theta_1(x_0, x_1), \dots, \theta_k(x_0, \dots, x_k)$ s.t.

$$\vdash \varphi(x_0) \lor \exists y\, \theta_j(x_0, \dots, x_{j-1}, y) \qquad (j = 1, \dots, k)$$

$$\vdash \bigwedge_{j=1}^{k} \Big[\theta_j(x_0, \dots, x_j) \land \bigwedge_{l=1}^{k} \neg(x_l \prec x_j \land \theta_j(x_0, \dots, x_{j-1}, x_l))\Big] \to \varphi(x_0)$$

where $y \prec x$ denotes $y < x$ ($|y| < |x|$)

A witnessing theorem (contd.)

- k-times iterated $\hat{\Pi}^b_{i-1}$-$(L)MIN$ $(\equiv \hat{\Pi}^b_i$-$(P)IND)$
  - restricted parameters
- Also works in the presence of a $\forall\exists\hat{\Sigma}^b_i$ ground theory
- Has an analogue one level higher (see next slide)
- If $\varphi \in \exists\hat{\Pi}^b_i$, can bound the quantifiers and take $k = 1$:

$$\vdash y \geq t(x) \to \theta(x, y)$$
$$\vdash \big(\theta(x, y) \wedge \neg\exists z \prec y\, \theta(x, z)\big) \to \varphi(x)$$

  - conservation over $[T, \hat{\Pi}^b_i$-$(P)IND^R]$, $[T, \hat{\Sigma}^b_i$-$(P)IND^R]$
- Can we reduce to $k = 1$ in other cases?
  - Is T^i_2 (S^i_2) $\exists\forall\hat{\Pi}^b_i$-conservative over $\hat{\Pi}^b_i$-$(P)IND^-$?
  - $\iff$ Is $T + T^i_2$ $(T + S^i_2)$ $\forall\hat{\Pi}^b_i$-conservative over $T + \hat{\Pi}^b_i$-$(P)IND^R$ for any $T \subseteq \forall\exists\hat{\Sigma}^b_i$?

A predecessor variant

Theorem

If T^i_2 (S^i_2) proves $\forall x\, \varphi(x)$, $\varphi \in \exists\forall\hat{\Pi}^b_{i+1}$, there are $\hat{\Pi}^b_i$ formulas $\theta_1(x_0, x_1), \dots, \theta_k(x_0, \dots, x_k)$ s.t.

$$\vdash \varphi(x_0) \lor \exists y\, \theta_j(x_0, \dots, x_{j-1}, y) \qquad (j = 1, \dots, k)$$

$$\vdash \bigwedge_{j=1}^{k} \Big[\theta_j(x_0, \dots, x_j) \land \neg\big(x_j \neq 0 \land \theta_j(x_0, \dots, x_{j-1}, P(x_j))\big)\Big] \to \varphi(x_0)$$

where $P(x)$ denotes $x - 1$ ($\lfloor x/2 \rfloor$)

Thank you for attention!

References

- L. D. Beklemishev: Induction rules, reflection principles, and provably recursive functions, APAL 85 (1997), 193–242
- ________: Parameter free induction and provably total computable functions, TCS 224 (1999), 13–33
- S. A. Bloch: Divide and conquer in parallel complexity and proof theory, Ph.D. thesis, UCSD, 1992
- A. Cordón-Franco, A. Fernández-Margarit, F. F. Lara-Martín: Existentially closed models and conservation results in bounded arithmetic, JLC 19 (2009), 123–143
- E. Jeřábek: Induction rules in bounded arithmetic, arXiv:1809.10718 [math.LO]
- R. Kaye: Axiomatizations and quantifier complexity, Proc. 6th Easter Conference on Model Theory, Sektion Mathematik der Humboldt-Universität zu Berlin, 1988, 65–84
- ________: Diophantine induction, APAL 46 (1990), 1–40
- R. Kaye, J. Paris, C. Dimitracopoulos: On parameter free induction schemas, JSL 53 (1988), 1082–1097