Recursive functions
vs.
classification theory

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Recursive functions and $R$

1. Recursive functions and $R$

2. Model completions

3. Classification theory
Robinson’s theory $R$

**NOT** Robinson’s arithmetic ($Q$), but equally illustrious

Simple presentation: language $\langle 0, S, +, \cdot, < \rangle$, axioms

\[
S^n(0) + S^m(0) = S^{n+m}(0) \\
S^n(0) \cdot S^m(0) = S^{nm}(0) \\
\forall x \ (x < S^n(0) \iff x = 0 \lor \cdots \lor x = S^{n-1}(0))
\]

- Axiomatizes true $\Sigma_1$ sentences
- Essentially undecidable, no r.e. completion
- Locally finitely satisfiable
- Visser ’12: Strongest locally finitely satisfiable r.e. theory up to interpretation
Representability of recursive functions

**Representation** of a (partial) function \( f : \mathbb{N}^k \rightarrow \mathbb{N} \) in \( T \):

Formula \( \varphi(x_1, \ldots, x_k, y) \), constant terms \( n \) for \( n \in \mathbb{N} \) s.t. \( T \) proves

- \( n \neq m \) whenever \( n \neq m \)
- \( \varphi(n_1, \ldots, n_k, z) \leftrightarrow z = m \) whenever \( f(n_1, \ldots, n_k) = m \)

Essential undecidability of \( R \) follows from:

- Theories representing all recursive functions are essentially undecidable
- \( R \) represents all recursive functions (even partial)
_Converse?_

R was designed to represent recursive functions, while being as weak as possible

This suggests the following question:

**Problem**

If a theory represents recursive functions, does it interpret R?
Representability revisited

Representation of \( f \) \( \Longleftrightarrow \) interpretation of a certain theory

The extra requirements are pointless \( \implies \) better definition:

**Definition**

A representation of \( f : \mathbb{N}^k \to \mathbb{N} \) in \( T \) is an interpretation of the following theory \( \text{Rep}_f \) in \( T \):

- **Language**:
  - constants \( n \) for \( n \in \mathbb{N} \)
  - function symbol \( f \)

- **Axioms**:
  - \( n \neq m \) for \( n \neq m \)
  - \( f(n_1, \ldots, n_k) = m \) for \( f(n_1, \ldots, n_k) = m \)
New statement of the problem

**Definition**

\[ \text{PRF} = \bigcup \{ \text{Rep}_f : f \text{ partial recursive function} \} \]

PRF can be equivalently expressed in a finite language:

\[ 0, S(x), \langle x, y \rangle, \phi_x(y) \]

Our question reduces to:

**Problem**

Does PRF interpret R?
1. Recursive functions and $R$

2. Model completions

3. Classification theory
Basic idea

PRF has quantifier-free axioms
\[ \implies \text{shouldn’t interpret much of anything} \]

Trouble: interpretations may use formulas of arbitrary quantifier complexity \[ \implies \text{not easy to analyze directly} \]

Strategy: extend PRF to a theory with quantifier elimination

- get a handle on possible interpretations
- embed the standard model of PRF in a randomly looking structure so that any combinatorial features are dissolved
Definition

Let $T$ be a universal theory. A theory $T^*$ is a

- companion of $T$ if every model of $T$ embeds in a model of $T^*$ and vice versa
  - equivalently: $(T^*)^\forall = T$

- model companion of $T$ if it is a companion, and it is model-complete
  - if $M \subseteq N$ are models of $T^*$, then $M \preceq N$
  - equivalently: over $T^*$, all formulas are existential

- model completion of $T$ if it is a (model) companion, and it has quantifier elimination
The model companion $T^*$ of $T$ is unique if it exists.

Models of $T^*$ are the existentially closed models of $T$:
- $M \models T$
- If an existential formula holds in an extension $M \subseteq N \models T$, it already holds in $M$.

$T$ has a model companion $\iff$ the class of e.c. models of $T$ is elementary.

<table>
<thead>
<tr>
<th>$T$</th>
<th>$T^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>linear orders</td>
<td>dense linear orders</td>
</tr>
<tr>
<td>integral domains</td>
<td>algebraically closed fields</td>
</tr>
<tr>
<td>Boolean algebras</td>
<td>atomless Boolean algebras</td>
</tr>
<tr>
<td>groups</td>
<td>N/A</td>
</tr>
</tbody>
</table>
If the empty $L$-theory $\emptyset_L$ had a model completion $\emptyset^*_L$:

- every $L$-structure extends to a model of $\emptyset^*_L$
- every consistent existential $L$-theory is consistent with $\emptyset^*_L$
- a theory interpretable in a consistent existential $L$-theory is weakly interpretable in $\emptyset^*_L$
- (weak) interpretations in $\emptyset^*_L$ are quantifier-free

$PRF$ is existential, so what we’ll do:

- show that indeed, $\emptyset_L$ has a model completion
- exhibit theories interpretable in $R$ (∼ locally finitely satisfiable) and not weakly interpretable in $\emptyset^*_L$
$\emptyset_L^*$ is well known for relational languages $L$: the theory of random structure(s)

- sentences that hold with asymptotic probability 1 in $n$-element random $L$-structures, $n \to \infty$
- or: the countable random $L$-structure
- Fraïssé limit of the class of all finite $L$-structures
- $\omega$-categorical, quantifier elimination, …
- axiomatized by extension axioms:
  - for any distinct $a_1, \ldots, a_k$, there is another element $b$ that bears any prescribed relations to $a_1, \ldots, a_k$
The general case

If $L$ includes function symbols:

- no 0–1 or limit law; no uniform distribution on $\omega$
- $2^\omega$ quantifier-free types $\implies$ no hope for $\omega$-categoricity
- cannot assign values of terms willy-nilly:
  $f(a) = f(b) \implies g(f(a)) = g(f(b))$

Luckily, it all works out in the end:

**Theorem**

For every language $L$, $\emptyset_L$ has a model completion $\emptyset_L^*$.

**Warning:** $\emptyset_L^*$ may be incomplete (quantifier-free sentences)
Corollary

If \( T \) is interpretable in a consistent existential theory, it is weakly quantifier-free interpretable in \( \emptyset^*_L \) for some \( L \).

A partial converse \( \Rightarrow \) we are on the right track:

Proposition

Let \( T \) be an \( \exists \forall \) theory in a relational language (?). If \( T \) is weakly interpretable in some \( \emptyset^*_L \), it is interpretable in a consistent existential theory.

NB: \( \emptyset^*_L \) is \( \forall \exists \)
1. Recursive functions and $R$

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Classification theory

- Various criteria to separate tame and wild theories ("dividing lines")
- Structure theory for models of tame theories
  - geometry of definable sets and types
  - models with special properties (prime, saturated, ...)
  - interpretable algebraic structures (groups, ...)
- Uncountable categoricity, stability,
o-minimality, simplicity ...
- Shelah

Why is it relevant here?

- Many dividing lines amount to weak interpretability of
  \( \exists \forall \) locally finitely satisfiable theories!
Metaterminology

Let $T$ be a theory:

- A formula $\varphi$ has the xghg xiljxa property (XXP) in $T$ if there are
  - a model $M \models T$
  - tuples $\overline{a}_i \in M$ $(i \in I)$

  such that hflijesai ff jai l jklf ajlifa $\varphi(\overline{a}_i, \overline{x})$ kah f h ahfdj k

- $T$ has XXP if some formula has it in $T$

- $T$ has the no xghg xiljxa property (NXXP) if it doesn’t have XXP

- NXXP is good, XXP is bad
Main classes

picture

missing

see http://forkinganddividing.com
Order and independence properties

- **order property (OP)**
  \[ M \models T, \varphi(\bar{x}, \bar{y}), (\bar{a}_i)_{i \in \mathbb{N}} \text{ s.t.} \]
  \[ M \models \varphi(\bar{a}_i, \bar{a}_j) \iff i < j \]

  NOP = stable = NIP & NSOP

- **strict order property (SOP)**
  - \( \varphi(\bar{x}, \bar{y}) \) defines a strict partial order
  - \( M \models \varphi(\bar{a}_i, \bar{a}_j) \) for \( i < j \)

- **\( k \)-strong order property (SOP\( _k \)), \( k \geq 3 \)**
  - \( \{ \varphi(\bar{x}_1, \bar{x}_2), \varphi(\bar{x}_2, \bar{x}_3), \ldots, \varphi(\bar{x}_k, \bar{x}_1) \} \) is inconsistent
  - \( M \models \varphi(\bar{a}_i, \bar{a}_j) \) for \( i < j \)

- **independence property (IP)**
  \[ \varphi(\bar{x}, \bar{y}), (\bar{a}_i)_{i \in \mathbb{N}}, (\bar{b}_X)_{X \subseteq \mathbb{N}} \text{ s.t.} \]
  \[ M \models \varphi(\bar{a}_i, \bar{b}_X) \iff i \in X \]
\( \mathbb{N}^{<\omega} = \) tree of finite sequences over a countable alphabet

▶ tree property (TP)
\( M, \varphi(\overline{x}, \overline{y}), (\overline{a}_s)_{s \in \mathbb{N}^{<\omega}} \) s.t.

▶ \{\varphi(\overline{x}, \overline{a}_{\sigma|n}) : n \in \omega\} is consistent for each path \( \sigma \in \mathbb{N}^\omega \)

▶ \{\varphi(\overline{x}, \overline{a}_{s \cup i}), \varphi(\overline{x}, \overline{a}_{s \cup j})\} is inconsistent for \( s \in \mathbb{N}^{<\omega}, i < j \)

\[ \text{NTP} = \text{simple} = \text{NTP}_1 \& \text{NTP}_2 \]

▶ TP\(_1\) (\(= \text{“SOP}_2\)”)

▶ \{\varphi(\overline{x}, \overline{a}_{\sigma|n}) : n \in \omega\} is consistent for \( \sigma \in \mathbb{N}^\omega \)

▶ \{\varphi(\overline{x}, \overline{a}_s), \varphi(\overline{x}, \overline{a}_t)\} is inconsistent for \( s, t \) incomparable

▶ TP\(_2\)
\( M, \varphi(\overline{x}, \overline{y}), (\overline{a}_{n,i})_{n,i \in \omega} \)

▶ \{\varphi(\overline{x}, \overline{a}_{n,\sigma(n)}) : n \in \omega\} is consistent for \( \sigma \in \mathbb{N}^\omega \)

▶ \{\varphi(\overline{x}, \overline{a}_{n,i}), \varphi(\overline{x}, \overline{a}_{n,j})\} is inconsistent for \( n \in \omega, i < j \)
\[ \emptyset^*_L \text{ not quite domesticated} \]

NB: random relational structures are supersimple

**Observation**

Any consistent extension of

- \( PRF \), or
- \( \emptyset^*_L \) if \( L \) contains a binary function

is \( TP_2 \) (hence IP and non-simple).

Proof: Take \( \bar{a}_{n,i} = (n, i) \), and

\[
(x)_{y_1} = y_2
\]

for the formula \( \varphi(x, y_1, y_2) \)
**Elimination of infinity**

**Definition**

*T* has elimination of infinity if for every formula \( \varphi(\overline{z}, x) \), there is a bound \( n \) such that

\[
|\varphi(\overline{a}, M)| > n \implies |\varphi(\overline{a}, M)| \geq \aleph_0
\]

for every \( M \models T \) and \( \overline{a} \in M \)

Elimination of infinity \( \iff \) FO formulas are closed under \( \exists^\infty \):

\[
M \models \exists^\infty x \varphi(\overline{a}, x) \iff \varphi(\overline{a}, M) \text{ is infinite}
\]
## Main theorem

For any language $L$:

- $\emptyset^*_L$ has $\text{NSOP}_3$ (hence $\text{NSOP}$)
- $(\emptyset^*_L)^{eq}$ eliminates infinity
Consequences

**Corollary**

The following theories are interpretable in $R$, but not in $PRF$:

- (partial) orders with arbitrarily long chains
- “for each standard $n$, there is a set with $n$ elements”
- directed graphs with arbitrarily long transitive chains, and no directed 3-cycle
Problems

- Does $PRF$ interpret all consistent r.e. existential theories?
- Is the random graph interpretable in a consistent existential theory?
- Does $\emptyset^*_L$ have $\text{NTP}_1$, or even $\text{NSOP}_1$?
- Does $\emptyset^*_L$ have weak elimination of imaginaries?
Thank you for attention!
References

- S. Shelah: Classification theory and the number of nonisomorphic models, 2nd ed., Elsevier, 1990