# Parameter-free induction in bounded arithmetic

Emil Jeřábek

jerabek@math.cas.cz

http://math.cas.cz/~jerabek/

Institute of Mathematics of the Academy of Sciences, Prague

Proof Complexity 2014, Vienna, July 2014

# Parameters in induction axioms

In arithmetic, induction (and other) schemata usually allow formulas with free parameters:

$$\varphi(0, y) \wedge \forall x \left(\varphi(x, y) \to \varphi(x+1, y)\right) \to \forall x\, \varphi(x, y)$$

Examples: $I\Sigma_i$, $S^i_2$, $T^i_2$, $\dots$

For full induction, this makes no difference.

What about fragments?

# Strong fragments

Notation: $I\Gamma^-$ = induction for parameter-free $\Gamma$-formulas

A lot is known about $I\Sigma_n^-$, $I\Pi_n^-$: [KPD'88, B'97, B'99, ...]

- $I\Sigma_n \to I\Sigma_n^- \to I\Sigma_{n-1}$
  $I\Pi_{n+1}^- \to I\Sigma_n^- \to I\Pi_n^-$
  $I\Sigma_{n+1}$ and $I\Pi_n^-$ are incomparable
- $I\Sigma_n$ is $\Sigma_{n+2}$-conservative over $I\Sigma_n^-$
  $I\Pi_{n+1}^-$ is $\mathcal{B}(\Sigma_{n+1})$-conservative over $I\Sigma_n^-$
- Unlike $I\Sigma_n$, neither $I\Sigma_n^-$ nor $I\Pi_n^-$ is finitely axiomatizable
- $I\Sigma_n$ is equivalent to the $\Sigma_{n+1}$ uniform reflection principle
  $I\Gamma^-$ can be characterized using local reflection principles
- $I\Sigma_n^-$ and $I\Pi_n^-$ are intimately related to induction rules

# Theories and rules

We consider theories axiomatized not just by axioms, but by more general rules of the form

$$\frac{\varphi_1, \dots, \varphi_k}{\varphi} \qquad (*)$$

Let $T$ be an ordinary FO theory, and $R$ a set of rules:

- $[T, R]$ denotes the closure of $T$ under unnested $R$-rules (axiomatized by $T$ + those $\varphi$ s.t. $T \vdash \varphi_1 \wedge \dots \wedge \varphi_k$)
- $[T, R]_0 := T$, $[T, R]_{n+1} := [[T, R]_n, R]$
  $T + R := \bigcup_n [T, R]_n$
- $R$ is reducible to $R'$ ($R \le R'$) if $[T, R] \subseteq [T, R']$ for all $T$
- $R$ and $R'$ are equivalent ($R \equiv R'$) if $R \le R' \le R$

# Induction rules

$$\frac{\varphi(0) \qquad \varphi(x) \to \varphi(x+1)}{\varphi(x)}$$

Notation: $I\Gamma^R$, $\Gamma = \Sigma_n, \Pi_n$

- $I\Gamma^R$ is equivalent to its parameter-free variant
- $I\Gamma^-$ is the least theory whose all extensions are closed under $I\Gamma^R$
  - conservation results for $I\Gamma^-$ follow from conservation results for $I\Gamma^R$
- $T + I\Sigma_n$ is $\Pi_{n+1}$-conservative over $T + I\Sigma_n^R$ for $T \subseteq \Pi_{n+2}$
- $[T, I\Sigma_n^R] = [T, I\Pi_{n+1}^R]$ for $T \subseteq \Pi_{n+1} \cup \Sigma_{n+1}$ (essentially)

[B'97]

# Bounded arithmetic

Parameter-free induction and rules in weak fragments:

- [K'90] $IE_i$ is $\exists\forall E_i$-conservative over $IE_i^-$
- [BI'92] studied $\Sigma_i^b$ parameter-free rules
- [CFL'09] proved conservation results for $\hat{\Sigma}_i^b$ rules and parameter-free schemata

This makes a rather patchy knowledge:

- $\hat{\Pi}_i^b$ rules and parameter-free schemata?
- nesting (number of instances)?
- reflection principles?

# This talk

On each level $i > 0$ of Buss's hierarchy, we can consider the following rules and parameter-free schemata (along with standard $T^i_2$, $S^i_2$):

- $\hat\Sigma^b_i$-$PIND^R$, $\hat\Sigma^b_i$-$PIND^-$
- $\hat\Pi^b_i$-$PIND^R$, $\hat\Pi^b_i$-$PIND^-$
- $\hat\Sigma^b_i$-$IND^R$, $\hat\Sigma^b_i$-$IND^-$
- $\hat\Pi^b_i$-$IND^R$, $\hat\Pi^b_i$-$IND^-$

We will try to systematically investigate their properties

Warning: work in progress

# Why these?

- $S^i_2$ and $T^i_2$ can be equivalently axiomatized by various other schemata ($LIND$, $MIN$, $\dots$)
- A single schema can be rulified or deprived of parameters in several different ways
- Fortunately, most variants turn out to be equivalent to one of the 10 mentioned
  - A few pathological exceptions: $LIND^-$
- In particular:

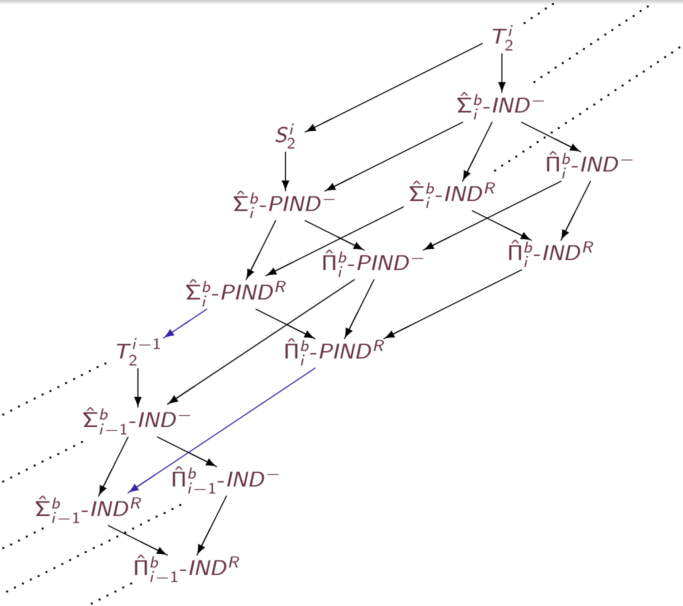
$$\Gamma\text{-}(P)IND^{R-} \equiv \Gamma\text{-}(P)IND^{R}, \quad \Gamma = \hat\Sigma^b_i, \hat\Pi^b_i$$

# Basic reductions

One can check with varying degree of easiness:

- $\Gamma\text{-}(P)IND^R \le \Gamma\text{-}(P)IND^- \le \Gamma\text{-}(P)IND$
- $\hat\Pi^b_i\text{-}(P)IND^{(R/-)} \le \hat\Sigma^b_i\text{-}(P)IND^{(R/-)}$
- $\Gamma\text{-}PIND^{(R/-)} \le \Gamma\text{-}IND^{(R/-)}$
- $\hat\Sigma^b_i\text{-}IND^{(R/-)} \le \hat\Pi^b_{i+1}\text{-}PIND^{(R/-)}$
- $T^i_2 = \hat\Sigma^b_i\text{-}IND \le \hat\Sigma^b_{i+1}\text{-}PIND^R$
  - In fact, $T^i_2 = PV_1 + \hat\Sigma^b_{i+1}\text{-}PIND^R$
  - However, likely $\hat\Sigma^b_{i+1}\text{-}PIND^R \not\le T^i_2$
  - Similar situation: $PV_1 + \hat\Sigma^b_i\text{-}IND^R = PV_1 + \hat\Pi^b_{i+1}\text{-}PIND^R$

# At a glance


$T^i_2$
$\hat\Sigma^b_i$-$IND^-$
$S^i_2$
$\hat\Pi^b_i$-$IND^-$
$\hat\Sigma^b_i$-$PIND^-$
$\hat\Sigma^b_i$-$IND^R$
$\hat\Pi^b_i$-$PIND^-$
$\hat\Pi^b_i$-$IND^R$
$\hat\Sigma^b_i$-$PIND^R$
$T^{i-1}_2$
$\hat\Pi^b_i$-$PIND^R$
$\hat\Sigma^b_{i-1}$-$IND^-$
$\hat\Pi^b_{i-1}$-$IND^-$
$\hat\Sigma^b_{i-1}$-$IND^R$
$\hat\Pi^b_{i-1}$-$IND^R$

# Axiom complexity

- $S^i_2$ and $T^i_2$ are finite $\forall\hat\Sigma^b_{i+1}$ theories
- $\hat\Sigma^b_i$-$(P)IND^R$ is $\forall\hat\Sigma^b_i$ / $\forall\hat\Sigma^b_i$
  $\hat\Pi^b_i$-$(P)IND^R$ is $\forall\hat\Sigma^b_i$ / $\forall\hat\Sigma^b_{i-1}$
- $\hat\Sigma^b_i$-$(P)IND^-$ is $\exists\hat\Pi^b_i \lor \forall\hat\Sigma^b_i$
  $\hat\Pi^b_i$-$(P)IND^-$ is $\exists\hat\Pi^b_i \lor \forall\hat\Sigma^b_{i-1}$
- $\hat\Pi^b_i$-$(P)IND^-$ is also $\forall\hat\Sigma^b_{i+1}$: equivalent to

$$\forall x\,\bigl(\varphi(0) \land \forall y < x\,(\varphi(y) \to \varphi(y+1)) \to \varphi(x)\bigr)$$

  - This doesn't work for $\hat\Sigma^b_i$-$(P)IND^-$
    —presumably not even $\forall\Sigma^b_\infty$?
- $\Gamma$-$(P)IND^-$ appear not to be finitely axiomatizable

# Conservativity for $\hat\Sigma^b_i$ rules

The following was proved by [CFL'09], based on [K'90,BI'92]:

**Theorem**

If $T$ is $\forall\exists\hat\Sigma^b_{i+1}$, then $T + T^i_2$ $(S^i_2)$ is $\forall\hat\Sigma^b_i$-conservative over $T + \hat\Sigma^b_i$-$(P)IND^R$

**Corollary**

- $T^i_2$ $(S^i_2)$ is $\exists\forall\hat\Sigma^b_i$-conservative over $\hat\Sigma^b_i$-$(P)IND^-$
- If $T$ is $\forall\hat\Sigma^b_i$, $T + \hat\Pi^b_{i+1}$-$PIND^R = T + \hat\Sigma^b_i$-$IND^R$
- [Buss]: ... and $T + \hat\Sigma^b_{i+1}$-$PIND^R = T + T^i_2$

# Conservativity for $\hat{\Pi}^b_i$ rules

**Theorem**

If $T$ is $\forall\hat{\Sigma}^b_i$, then $T + S^{i+1}_2$ $(S^i_2)$ is $\forall\exists\hat{\Sigma}^b_{i-1}$-conservative over $T + \hat{\Pi}^b_i$-$(P)IND^R$.

**Corollary**

$S^{i+1}_2$ $(S^i_2)$ is $\exists\hat{\Sigma}^b_{i+1} \vee \forall\exists\hat{\Sigma}^b_{i-1}$ conservative over $\hat{\Pi}^b_i$-$(P)IND^-$.

# Conservative fragments of $S_2^{i+1}$

| theory | axiom. by | cons. under $S_2^{i+1}$ for |
|---|---|---|
| $PV_1 + \hat{\Sigma}^b_{i+1}\text{-}PIND^-$ | $\exists\hat{\Sigma}^b_{i+2} \vee \forall\hat{\Sigma}^b_{i+1}$ | $\exists\forall\hat{\Sigma}^b_{i+1}$<br>$\exists\hat{\Sigma}^b_{i+3} \vee \forall\exists\hat{\Sigma}^b_{i+1}$ |
| $PV_1 + \hat{\Sigma}^b_{i+1}\text{-}PIND^R$<br>$= T^i_2$ | $\forall\hat{\Sigma}^b_{i+1}$ | $\forall\exists\hat{\Sigma}^b_{i+1}$ |
| $PV_1 + \hat{\Pi}^b_{i+1}\text{-}PIND^-$ | $\exists\hat{\Sigma}^b_{i+2} \vee \forall\hat{\Sigma}^b_{i}$<br>$\forall\hat{\Sigma}^b_{i+2}$ | $\exists\hat{\Sigma}^b_{i+2} \vee \forall\exists\hat{\Sigma}^b_{i}$ |
| $PV_1 + \hat{\Sigma}^b_{i}\text{-}IND^-$ | $\exists\hat{\Sigma}^b_{i+1} \vee \forall\hat{\Sigma}^b_{i}$ | $\exists\hat{\Sigma}^b_{i+1} \vee \forall\exists\hat{\Sigma}^b_{i}$ * |
| $PV_1 + \hat{\Pi}^b_{i+1}\text{-}PIND^R$<br>$= PV_1 + \hat{\Sigma}^b_{i}\text{-}IND^R$ | $\forall\hat{\Sigma}^b_{i}$ | $\forall\exists\hat{\Sigma}^b_{i}$ |
| $PV_1 + \hat{\Pi}^b_{i}\text{-}IND^-$ | $\exists\hat{\Sigma}^b_{i+1} \vee \forall\hat{\Sigma}^b_{i-1}$<br>$\forall\hat{\Sigma}^b_{i+1}$ | $\exists\hat{\Sigma}^b_{i+1} \vee \forall\exists\hat{\Sigma}^b_{i-1}$ |

# Nesting of rules

For $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, every $\varphi \in [T, \Gamma\text{-}(P)IND^R]_k$ can be proved using $k$ instances of $\Gamma$-$(P)IND^R$

**Theorem**

- If $T$ is $\forall\Sigma^b_\infty$: $T + \hat{\Pi}^b_i\text{-}(P)IND^R = [T, \hat{\Pi}^b_i\text{-}(P)IND^R]$
- If $T$ is $\forall\hat{\Sigma}^b_i$: $T + \hat{\Sigma}^b_i\text{-}(P)IND^R = [T, \hat{\Sigma}^b_i\text{-}(P)IND^R]$

Moreover, if $T + \hat{\Sigma}^b_i\text{-}IND^R \vdash \varphi(x) \in \hat{\Sigma}^b_i$, there are $t(x)$ and $\psi(y) \in \hat{\Sigma}^b_i$ s.t.

$$T \vdash \psi(0) \wedge \forall y\,(\psi(y) \to \psi(y+1))$$
$$PV_1 \vdash \psi(t(x)) \to \varphi(x)$$

Similarly for $PIND^R$

# Parameter-free conservativity

Conservativity of $T + \Gamma\text{-}(P)IND$ over $T + \Gamma\text{-}(P)IND^R$ implies conservativity of $T + \Gamma\text{-}(P)IND^-$ over $T + \Gamma\text{-}(P)IND^R$

We can do better by a direct argument:

**Theorem**

Let $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, and $T$ be of any complexity:

- $T + \Gamma\text{-}(P)IND^-$ is $\forall\Gamma$-conservative over $T + \Gamma\text{-}(P)IND^R$
- All $\forall\Gamma$ consequences of $T$ + arbitrary $k$ instances of $\Gamma\text{-}(P)IND^-$ are in $[T, \Gamma\text{-}(P)IND^R]_k$

If $\Gamma\text{-}(P)IND^-$ is finitely axiomatizable, there is $k$ s.t. $T + \Gamma\text{-}(P)IND^R = [T, \Gamma\text{-}(P)IND^R]_k$ for every $T$

# Propositional proof systems

$G_i$ = $\Sigma^q_i$-fragment of quantified propositional sequent calculus

$\mathrm{RFN}_j(P)$ = "every $P$-provable $\Sigma^q_j$ sequent is valid"

$\varphi(x) \in \hat{\Sigma}^b_i \implies$ propositional translations $[\![\varphi]\!]_n(p_0, \dots, p_{n-1})$

**Definition**

Let $\xi \in \hat{\Sigma}^b_i$.

- $G_i[\xi]$ denotes $G_i$ with extra initial sequents

$$\implies [\![\xi]\!]_n(A_0, \dots, A_{n-1}),$$

  where $A_0, \dots, A_{n-1}$ are quantifier-free
- $G_i^*[\xi]$ is its tree-like version

# Correspondence

By extension of standard results, one can show easily

**Theorem**

Let $\xi, \varphi \in \hat{\Sigma}^b_i$.

- If $T^i_2$ $(S^i_2) + \forall x\, \xi(x) \vdash \varphi(x)$, then ($PV_1$-provably) there are poly-size $G_i[\xi]$ $(G^*_i[\xi])$ proofs of $[\![\varphi]\!]_n$
- $T^i_2$ $(S^i_2) + \forall x\, \xi(x)$ proves $\mathrm{RFN}_i(G^{(*)}_i[\xi])$

# Induction rules vs. reflection principles

**Theorem**

The rules on the LHS are equivalent to the rules on the RHS for $\xi \in \hat{\Sigma}^b_i$:

$$
\begin{array}{ll}
\hat{\Sigma}^b_i\text{-}(P)IND^R & \forall x\,\xi(x) \;/\; \mathrm{RFN}_i(G_i^{(*)}[\xi]) \\
\hat{\Sigma}^b_i\text{-}(P)IND^- & \forall x\,\xi(x) \to \mathrm{RFN}_i(G_i^{(*)}[\xi]) \\
\hat{\Pi}^b_i\text{-}(P)IND^R & \forall x\,\xi(x) \;/\; \mathrm{RFN}_{i-1}(G_i^{(*)}[\xi]) \\
\hat{\Pi}^b_i\text{-}(P)IND^- & \forall x\,\xi(x) \to \mathrm{RFN}_{i-1}(G_i^{(*)}[\xi])
\end{array}
$$

# Finite closure

Recall: If $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$ and $T$ is $\forall\hat{\Sigma}^b_i$, then
$T + \Gamma\text{-}(P)IND^R = [T, \Gamma\text{-}(P)IND^R]$

The equivalence with reflection rules implies

### Corollary

If $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$ and $T = PV_1 + \forall x\, \xi(x)$ with $\xi \in \hat{\Sigma}^b_i$, then $T + \Gamma\text{-}(P)IND^R$ is finitely axiomatizable:

$$T + \hat{\Sigma}^b_i\text{-}(P)IND^R = PV_1 + \mathrm{RFN}_i(G_i^{(*)}[\xi])$$
$$T + \hat{\Pi}^b_i\text{-}(P)IND^R = T + \mathrm{RFN}_{i-1}(G_i^{(*)}[\xi])$$

# Separations?

Any unexpected reduction or inclusion would subsume one of

- (i) $PV_1 + \hat{\Pi}^b_i\text{-}IND^R \subseteq S^i_2$
- (ii) $S^i_2 \subseteq \hat{\Pi}^b_{i+1}\text{-}IND^-$
- (iii) $\hat{\Pi}^b_i\text{-}PIND^- \subseteq PV_1 + \hat{\Pi}^b_{i+1}\text{-}IND^R$
- (iv) $[\hat{\Pi}^b_i\text{-}PIND^R \le T^{i-1}_2 \implies \hat{\Pi}^b_i\text{-}PIND^- \subseteq T^{i-1}_2 \implies$ (iii) $]$
- $\pm$ some exceptional cases on the lowest level of the hierarchy

We want to make sure that (i)–(iii) are implausible

# Separations? (cont'd)

Most extra reductions/inclusions are false when relativized:

- essentially, one can simulate parameters by the oracle

$$A(\alpha) \vdash B^-(\alpha) \implies A(\alpha) \vdash B(\alpha)$$

- feels like cheating

Unrelativized complexity consequences:

(i) $G_i \le_p G_{i-1}$, $\mathrm{GI}_i \le \mathrm{GI}_{i-1}$

(ii) $\mathrm{P}^{\Sigma^p_i[\log n]} = \mathrm{P}^{\Sigma^p_i[O(1)]}$, $\mathrm{PH} = \mathrm{P}^{\Sigma^p_{i+1}[O(1)]}$

(iii) ? Seems quite subtle

# Thank you for attention!

# References

[B'97] L. D. Beklemishev: *Induction rules, reflection principles, and provably recursive functions*, APAL 85 (1997), 193–242

[B'99] L. D. Beklemishev: *Parameter free induction and provably total computable functions*, TCS 224 (1999), 13–33

[Bl'92] S. A. Bloch: *Divide and conquer in parallel complexity and proof theory*, Ph.D. thesis, UCSD (1992)

[CFL'09] A. Cordón-Franco, A. Fernández-Margarit, F. F. Lara-Martín: *Existentially closed models and conservation results in bounded arithmetic*, JLC 19 (2009), 123–143

[K'90] R. Kaye: *Diophantine induction*, APAL 46 (1990), 1–40

[KPD'88] R. Kaye, J. Paris, C. Dimitracopoulos: *On parameter free induction schemas*, JSL 53 (1988), 1082–1097