Quantum Field Theory and Gravity:

a window on Mathematical Physics

Igor Khavkine

Department of Mathematics
University of Trento,
University of Rome 2 (Tor Vergata)

May 2016

CV summary

- **2004–2009:** My PhD thesis: *Computer simulation of spin foam models of quantum gravity* (UWO, Canada)
- **2009–2011:** Development of an independent research program (Utrecht). (Canadian grant $$$)
- **2011–   :** Active focus on mathematical aspects of Quantum Field Theory and Gravity (Utrecht, Trento). (Dutch grant $$$)
  - **19** published articles (**11** since 2011).
  - **20** international conference presentations, **7** invited (since 2011).
  - **30+** seminar presentations in Europe, Canada, US (since 2011).
- (Co-)Mentoring:
  - **2** MSc projects completed (Utrecht, Trento), **1** in progress (Pavia).
  - **2** PhD projects in progress (Trento, Pavia).
- Teaching:
  - $\sim$**300** hours of classroom instruction, between 2002 and 2015.
  - Courses assisted: *Numerical Analysis*, *Mathematical Methods for Engineers*, *Mathematical Analysis for Physicists and Engineers*.
  - Instruction in **English** and **Italian**.

QFT and Gravity as motivations

- Mathematical Physics is Mathematics motivated by Physics.
- Quantum Field Theory (QFT) and Gravity, in various combinations, are at the forefront of fundamental physics. E.g.:
  - early universe cosmology
  - black hole dynamics and evaporation
- It is a very fertile ground for interesting and challenging mathematical problems.
- They require tools from and stimulate development in
  - PDEs and analysis on manifolds
  - finite and infinite dimensional geometry, super-geometry
  - operator and topological algebras
  - representation and invariant theory
  - commutative algebra, homological algebra
  - category theory, higher geometry, ...
- Mathematical developments feed back into physicists' calculations.

QFT and Gravity as motivations

- Mathematical Physics is Mathematics motivated by Physics.
- Quantum Field Theory (QFT) and Gravity, in various combinations, are at the forefront of fundamental physics. E.g.:
  - early universe cosmology
  - black hole dynamics and evaporation
- It is a very fertile ground for interesting and challenging mathematical problems.
- They require tools from and stimulate development in
  - PDEs and analysis on manifolds
  - finite and infinite dimensional geometry, super-geometry
  - operator and topological algebras
  - representation and invariant theory
  - commutative algebra, homological algebra
  - category theory, higher geometry, . . .
- Mathematical developments feed back into physicists' calculations.

QFT and Gravity as motivations

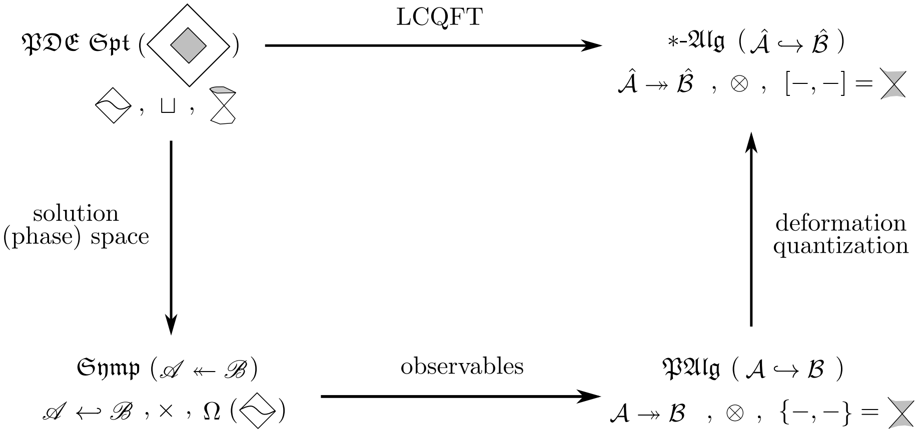
- Mathematical Physics is Mathematics motivated by Physics.
- Quantum Field Theory (QFT) and Gravity, in various combinations, are at the forefront of fundamental physics. E.g.:
  - early universe cosmology
  - black hole dynamics and evaporation
- It is a very fertile ground for interesting and challenging mathematical problems.
- They require tools from and stimulate development in
  - PDEs and analysis on manifolds
  - finite and infinite dimensional geometry, super-geometry
  - operator and topological algebras
  - representation and invariant theory
  - commutative algebra, homological algebra
  - category theory, higher geometry, ...
- Mathematical developments feed back into physicists' calculations.

What is a QFT mathematically?

- Mathematical formalizations of QFTs vary widely. The choice depends on the specific type of field theory under consideration.
- I am mostly interested in (perturbatively) non-linear, gauge theories on curved Lorentzian spacetimes. (Different from 2D CFT, TQFT, instantons, stochastic processes, etc.)
- Such a QFT is mostly determined by a variational PDE system that can be made hyperbolic by suitable gauge-fixing.
- Examples:
  - scalar wave and Klein-Gordon fields, Dirac spinor fields
  - Maxwell theory, Yang-Mills theory
  - General Relativity (most prominent representative)
- The appropriate mathematical formalism is Locally Covariant (Perturbative) Algebraic QFT (LCQFT) on curved spacetimes, axiomatized by Brunetti-Fredenhagen-Verch (2003).

Studying Quantum Field Theory (QFT)

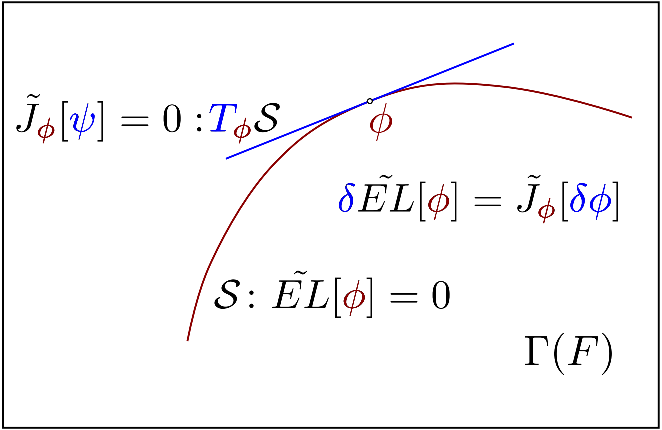
A QFT is constructed roughly as follows:



The Brunetti-Fredenhagen-Verch framework for Locally Covariant QFT; the arrows are functors with specific properties.

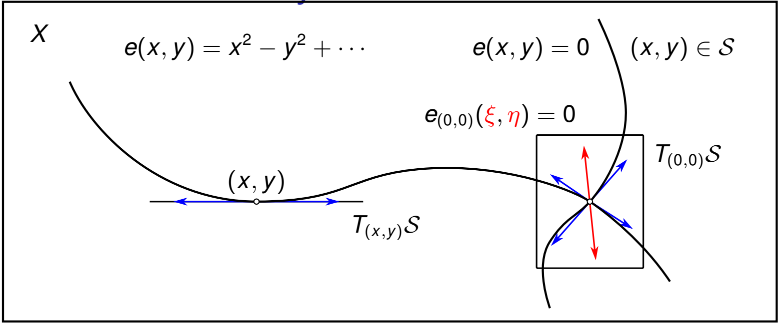
Solution (phase) space

$\tilde{EL}[\phi] = 0$: Euler-Lagrange equations on $F \to M$; $\mathcal{S}$: solution space on M



Under sufficient well-posedness conditions, the solution space $\mathcal{S}$ becomes a phase space, with symplectic and Poisson structures.

Linearization instability



- Solution space of a nonlinear equation:

$$e(x,y) = 0 \Leftrightarrow (x,y) \in \mathcal{S} \subset X.$$

- Formal (Zariski) tangent space $T_{(x,y)}\mathcal{S}$:

$$e(x + t\xi, y + t\eta) = te_{(x,y)}(\xi,\eta) + O(t^2),$$
$$(\xi,\eta) \in T_{(x,y)}\mathcal{S} \Leftrightarrow e_{(x,y)}(\xi,\eta) = 0.$$

- Extendable tangent (cone) vector (ξ,η):

$$e(x_t, y_t) = 0 \text{ with } (x_t, y_t) = (x,y) + t(\xi,\eta) + O(t^2).$$

- Obstruction Q: (ξ,η)-extendable $\Rightarrow Q(\xi,\eta) = 0$.

Linearization instability at (x,y): not all vectors in $T_{(x,y)}\mathcal{S}$ are extendable!

Linearization instability

AHP**16** L.Instab

Def: A Gauge Theory admits a large (gauge) symmetry group, $\mathcal{G} \circlearrowright \Gamma(F)$, locally parametrized by arbitrary functions on M.

Def: In Higher (or Reducible) Gauge Theory, gauge symmetries admit gauge symmetries, etc., $\cdots \circlearrowright \mathcal{G}^2 \circlearrowright \mathcal{G}^1 \circlearrowright \Gamma(F)$; linearize at φ and take cohomologies to get $\mathrm{RSym}^p(\varphi)$, stage-p rigid symmetries.

Theorem

For a sufficiently regular Higher Gauge Theory, at a background solution φ on M, for any higher stage rigid symmetry $\xi \in \mathrm{RSym}^p(\varphi)$ there is a linearization obstruction Q^ξ_p valued in $H^{n-p}_{\mathrm{dR}}(M)$.

Fischer-Marsden (1973): linearization instability in General Relativity

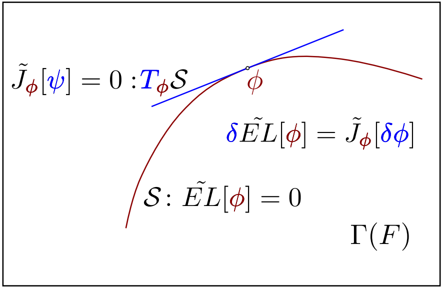
- GR: $\mathrm{RSym}^1(g) \cong$ Killing vectors of g. (Fischer, Marsden, Moncrief)
- YM: $\mathrm{RSym}^1(\nabla) \cong \nabla$-constant $\mathfrak{g}$-valued 0-forms. (Moncrief)
- Freedman-Townsend: $\mathrm{RSym}^2(B = 0) \cong H^0(M, \mathfrak{g} \otimes \mathbb{R})$. (New!)

Poisson structure in Gauge Theories

In a Gauge Theory, we work with equivalence classes $[\phi] \in \bar{\mathcal{S}} = \mathcal{S}/\mathcal{G}$.

Linearizing, we work with equivalence classes $[\psi] \in T_\phi\bar{\mathcal{S}}$.

There is a Poisson bracket $\{-,-\}$ defined on $\bar{\mathcal{S}}$.



Equivalently, we want the Poisson algebra of gauge-invariant functions on $\mathcal{S}$ (observables). It is the starting point for quantization.

The Poisson bracket can be constructed by canonical methods of Hamiltonian mechanics. But it is more convenient to use the Peierls formula (1952), which does not require us to parametrize $\mathcal{S}$ or $\bar{\mathcal{S}}$ by initial data: $\{A,B\}_\phi = \int_{M\times M} \mathrm{d}x \frac{\delta A}{\delta\phi(x)}[G_\phi^+(x,y) - G_\phi^-(x,y)]\frac{\delta B}{\delta\phi(y)}\mathrm{d}y$

The Peierls formula uses gauge-fixing to lift the Poisson bracket to $\mathcal{S}$.

Poisson structure in Gauge Theories

IJMPA**29** Cov.Peierls, AHP*2016* Caus.Cohom

Q: Does $\{-,-\}$ develop degeneracies when restricted to $T^*_\phi \bar{\mathcal{S}} \subset T^*_\phi \mathcal{S}$?

Poisson degeneracies lead to interesting physical effects: central charges, superselection sectors, ...

Theorem

For sufficiently regular Gauge Theories, there exists sheaves $\mathscr{G}$ and $\mathscr{C}'$ on M and degrees $p, q \geq 0$ such that the degeneracy dimension of the Poisson bracket is dominated by $\dim H^p_{sc}(M, \mathscr{G}) \oplus H^q_{sc,sol}(M, \mathscr{C}') < \infty$.

$\mathscr{G}$ is determined by gauge symmetries and $\mathscr{C}'$ by constraints; $p, q > 1$ in Higher Gauge Theories.

- Yang-Mills: $\mathscr{G}, \mathscr{C}' \cong \mathfrak{g} \otimes \underline{\mathbb{R}}$, $p = q = 1$ (twisted de Rham)
- General Relativity: $\mathscr{G}, \mathscr{C}' \cong$ Killing vector sheaf, $p = q = 1$
- Freedman-Townsend, Chern-Simons, Courant σ-model: like YM

Q: Why sheaves? **A:** Sheaf cohomology is a short-cut to counting solutions of complicated PDEs (e.g., de Rham's theorem).

Wick polynomials

- QFT associates a non-commutative $*$-algebra $\mathcal{A}$ (of quantum observables) to a spacetime region M. They are generated by

$$\mathcal{A} \ni \mathbf{1},\ \mathcal{A} \ni \int_M \phi(x) f(x) \mathrm{d}x, \quad f \in C_c^\infty(M).$$

  The quantum field $\phi(x)$ is an $\mathcal{A}$-valued distribution.
- Usually, products $\phi(x) \cdot \phi(x)$ are ill defined, but we can define

$$\text{(Wick powers)} \quad {:}\phi^2(x){:} = \lim_{y \to x} \phi(x)\phi(y) - G(x,y)\,\mathbf{1},$$

  for special scalar distributions $G(x,y)$.
- Wick polynomials are a convenient basis for local physical observables in QFT: energy and momentum density, charge current, etc.
- Replacing $G(x,y)$ by $G'(x,y)$ may give a different prescription ${:}\phi^2(x){:}'$. The difference ${:}\phi^2{:}' - {:}\phi^2{:}$ is a finite renormalization.

Finite renormalizations of Wick polynomials

- Consider a Locally Covariant QFT of a scalar $\phi(x)$, with Wick powers $:\phi^k(x):$, on a Lorentzian spacetime (M, g).
- Different prescriptions $:\cdot:$ and $:\cdot:'$ must differ by

$$:\phi^k(x): - :\phi^k(x):' = \sum_{i=1}^{k-2} C_{k-i}[g](x) :\phi^i(x):,$$

  where $C_i[g](x)$ are local curvature scalars of g.
- Hollands-Wald (2001): The original sufficient conditions required locality, covariance, continuous dependence on metrics and "analytic dependence" on analytic metrics.
- Unnaturality of the analyticity hypothesis has slowed progress beyond the scalar field case.
- Classifying finite renormalizations helps us classify anomalies (symmetries broken by quantum effects).

Finite renormalizations of Wick polynomials

CMP*2016* Analytic Dep, with V.Moretti (Trento)

Q: How is the analyticity hypothesis used?

A: Only to show that $C_i[g](x)$ is a differential operator on $g(x)$. By hypothesis,

$$C_i[g](x) = C_i(g(x), \partial g(x), \partial^2 g(x), \cdots) = \text{convergent series},$$

which is finite by a secondary argument.

Q: Is it possible to remove the analyticity requirement? **A:** Yes.

Proposition (Peetre 1959, Slovak 1988)

A sheaf morphism of smooth functions is given by a differential operator iff it is regular (sends smooth parametrized functions to smooth functions).

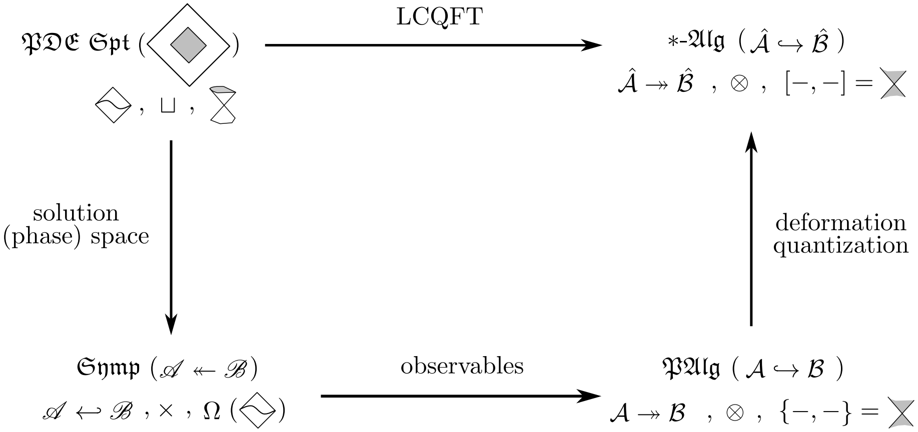
Theorem

By locality, $g \mapsto C_i[g](x)$ is a sheaf morphism. Peetre-Slovak and a regularity hypothesis imply that the $C_i[g]$ are differential operators. By covariance they are curvature scalars.

We have replaced the continuity and analyticity hypotheses by a technically more natural regularity hypothesis.

QFT of Gravity (QG) as motivation

QG is the application of QFT to General Relativity.



The guiding theme of my research program is to make precise and rigorous all of the illustrated steps.

QFT of Gravity (QG) as motivation

QG is difficult because it uniquely combines all of the following challenges:

1. Non-linearity
   - $\lambda\phi^4$, QED, YM, fluids
2. Dynamical Causality
   - gas dynamics, fluids, quasilinear hyperbolic PDE
3. Singularities
   - fluid shocks, breaking waves, wave focusing
4. Gauge Redundancy
   - Maxwell, YM, TQFT, string
5. Non-local Observables
   - Aharonov-Bohm, TQFT, Wilson loops
6. UV Renormalization
   - any interacting QFT
7. IR Renormalization
   - any massless field
8. Non-perturbative Definition
   - any physical QFT

Much is known about each obstacle in isolation. It is an outstanding challenge to understand them better in General Relativity and to combine this understanding together.

Research program panorama

Non-Linearity

Dynamical Causality

Singularities

Gauge Redundancy

Non-local Observables

UV Renormalization

IR Renormalization

Non-perturbative Definition

Past

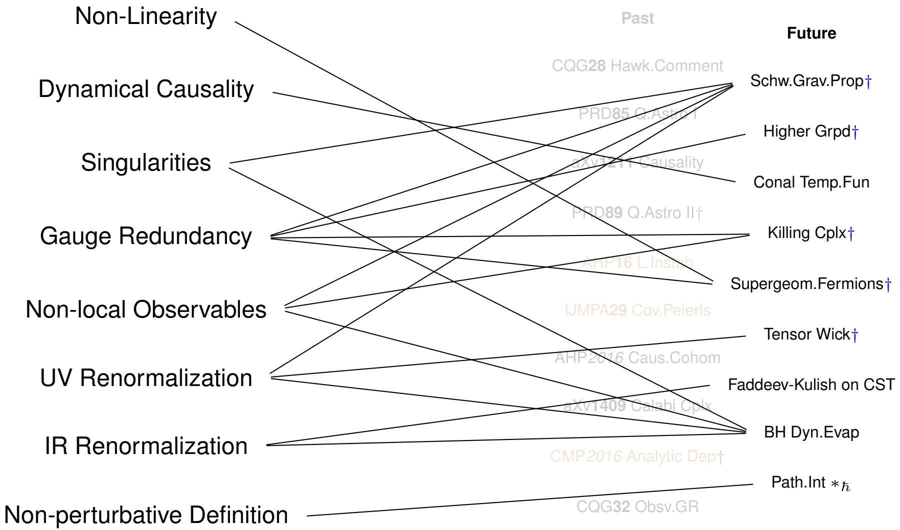
- CQG**28** Hawk.Comment
- PRD**85** Q.Astro I
- aXv**1211** Causality
- PRD**89** Q.Astro II†
- AHP**16** L.Instab
- IJMPA**29** Cov.Peierls
- AHP*2016* Caus.Cohom
- aXv**1409** Calabi Cplx
- CMP*2016* Analytic Dep†
- CQG**32** Obsv.GR

Future

- Schw.Grav.Prop†
- Higher Grpd†
- Conal Temp.Fun
- Killing Cplx†
- Supergeom.Fermions†
- Tensor Wick†
- Faddeev-Kulish on CST
- BH Dyn.Evap
- Path.Int $*_\hbar$

† with collaborators

Research program panorama



† with collaborators

Thank you for your attention!

Current projects

- **Schw.Grav.Prop** Compute the harmonic gauge graviton propagator on Schwarzschild spacetime, via separation of variables and spectral analysis of the radial (highly non-standard) Sturm-Liouville problem. With F. Bussola, C. Dappiaggi (Pavia).
- **Higher Grpd** Reveal the structure of higher groupoids in the gauge symmetries of ordinary and reducible gauge theories. With U. Schreiber (Prague).
- **Conal Temp.Fun** Use de Rham currents and Sullivan's (1976) structure cycles to study temporal functions on conal manifolds.
- **Killing Cplx** Study the cohomology resolution by a complex of differential operators of the sheaf of Killing vectors on curved spacetime, using the Spencer formal theory of PDEs. Initial stages with G. Canepa, C. Dappiaggi (Pavia).
- **Supergeom.Fermions** Use supergeometry and hyperbolic PDEs to construct quasi-linear classical field theories with fermions. With F. Hanisch (Potsdam).
- **Tensor Wick** Extend the new proof, with the use of the Peetre-Slovak theorem and differential invariants, of the characterization of finite renormalizations of Wick polynomials to tensor, spinor and gauge field theories. With A. Melati, V. Moretti (Trento).

Future projects

- **Faddeev-Kulish on CST** Reproduce the success of the UV Renormalizaton program on curved spacetimes by formulating IR Renormalization on curved spacetimes, with the help of the heuristics of Faddeev-Kulish (1970) the modern theory of decay rates of wave-like PDEs on asymptotically flat spacetimes.
- **BH Dyn.Evap** Apply the methods of locally covariant perturbative QFT to study the quantum gravitational back-reaction of Hawking evaporation of black holes.
- **Path.Int $*_\hbar$** Identify a spacetime covariant functional-integral formula for the quantum $*$-product in QFT.