A remarkable connection between the Jeans' theory of gravitational instabilities and the Newtonian limit of the General Theory of Relativity

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Abstract. We investigate both the Newtonian limit of the General Theory of Relativity and the Jeans' theory of gravitational instabilities in order to obtain some new conclusions on these two classical topics. We use the standard method for calculating the Ricci tensor by means of the Christoffel symbols of the second kind in the Newtonian limit. We assume that both the gravitational potential and the mean mass density can depend on time. The density of the non-relativistic dust can also be time dependent. As an impact on the Jeans' theory, we confirm that the modifications done by Lifshitz in 1946 and by Bonnor in 1957 are perfect and hold. In the Poisson equation the time dependence of both sides is an original new development. The application of this on the Jeans' theory is then straightforward.
Key words: Jean's theory, Newtonian limit, gravitational potential, Poisson equation

Introduction

At the beginning of the last century, James Hopwood Jeans published a controversial study (Jeans 1902) on gravitational instabilities. The static background that he suggested does not fulfill the Poisson equation.

A remarkable solution of this problem was proposed in Lifshitz (1946) and Bonnor (1957). The key idea was to assume an expanding background in accordance with the spatially flat cosmological Friedmannian model. Then all formulae work quite well – except for the fact that in the Poisson equation both the gravitational potential and also the mass density depend on time. The Poisson equation should be static.

The aim of this article is to show by mathematical considerations that the Poisson equation – as the limit in the General Theory of Relativity – may well be depending on time. The application of this result then shows that the solution proposed by Lifshitz and by Bonnor adequately solves the problem of background.

The paper is organized as follows. Section 1 recapitulates some known facts. Section 2 contains the main mathematical calculations. Section 3 discusses the new results. Section 4 summarizes these results.

1. Recapitulation of some known facts

1.1. The Newtonian limit

In this subsection we repeat, how the Newtonian limit is obtained in the frame of the General Theory of Relativity. We follow the approach of Landau & Lifshitz (1975, Chapt. 96).

The covariant metric tensor is assumed to be

$$g_{ij} = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1}$$

In the general case, $g_{00} = g_{00}(x^0, x^1, x^2, x^3)$ is a function of all four coordinates. The spatial part corresponds to the three-dimensional Euclidean space except for the minus sign. Hence, the contravariant metric tensor takes the form

$$g^{00} = \frac{1}{g_{00}}, \qquad g^{\alpha\beta} = g_{\alpha\beta} = -\delta_{\alpha\beta}, \qquad g^{\alpha 0} = g_{\alpha 0} = 0 \tag{2}$$

for all $\alpha, \beta = 1, 2, 3$.

Further, we suppose that the energy-momentum tensor has only one non-zero component

$$T_0^0 = \mu c^2, \tag{3}$$

where μ is the density of the non-relativistic dust and c is the speed of light in a vacuum. In the general case, T_0^0 is also a function of all four coordinates.

Note that Landau & Lifshitz (1975, Chapt. 96) consider only a special form of g_{00}, namely,

$$g_{00} = 1 + \frac{2\phi}{c^2}, \tag{4}$$

where ϕ is also a function of all four coordinates, in general. Its physical dimension is $\mathrm{m}^2/\mathrm{s}^2$.

The Newtonian limit is based on two assumptions:

1) Both ϕ and μ are *not* dependent on time.
2) It holds that $|2\phi/c^2| \ll 1$.

In what follows we will at first not use approximation 2) in order to see, which terms can be neglected. The Christoffel symbols of the first kind are defined as follows:

$$\mathit{\Gamma}_{ijk} = \frac{1}{2}(g_{ij,k} + g_{ki,j} - g_{jk,i}), \quad i, j, k = 0, 1, 2, 3,$$

where the index after the comma denotes the first partial derivative. From the forty Christoffel symbols only six are non-zeros:

$$\mathit{\Gamma}_{\alpha 00} = -\frac{1}{2} g_{00,\alpha} \tag{5}$$

for $\alpha = 1, 2, 3$ and

$$\mathit{\Gamma}_{0\alpha 0} = \mathit{\Gamma}_{00\alpha} = \frac{1}{2} g_{00,\alpha} \tag{6}$$

for $\alpha = 1, 2, 3$.

The Christoffel symbols of the second kind for the diagonal metric tensor (1) are defined as follows:

$$\Gamma^{i}{}_{jk} = g^{ii}\Gamma_{ijk} \tag{7}$$

with no sum over the index i. By (2) we find that the six non-zero symbols are

$$\Gamma^{\alpha}{}_{00} = \frac{1}{2} g_{00,\alpha} \tag{8}$$

for $\alpha = 1, 2, 3$ and

$$\Gamma^{0}{}_{\alpha 0} = \Gamma^{0}{}_{0\alpha} = \frac{g_{00,\alpha}}{2g_{00}} \tag{9}$$

for $\alpha = 1, 2, 3$.

Using the standard definition of the Ricci tensor (see Landau & Lifshitz 1975, Chapt. 92) for the 00 component (the only one that we need), we get

$$R_{00} = \Gamma^{i}{}_{00,i} - \Gamma^{i}{}_{0i,0} + \Gamma^{i}{}_{00}\Gamma^{j}{}_{ij} - \Gamma^{i}{}_{0j}\Gamma^{j}{}_{0i}, \tag{10}$$

where Einstein's summation convention over all the values of repeated indices is used. Therefore,

$$R_{00} = \Gamma^{\alpha}{}_{00,\alpha} - \Gamma^{\alpha}{}_{0\alpha,0} + \Gamma^{\alpha}{}_{00}\Gamma^{0}{}_{\alpha 0} - \Gamma^{\alpha}{}_{00}\Gamma^{0}{}_{0\alpha} - \Gamma^{0}{}_{0\alpha}\Gamma^{\alpha}{}_{00}. \tag{11}$$

Since $\Gamma^{0}{}_{0\alpha,0} = 0$ and $\Gamma^{0}{}_{\alpha 0} = \Gamma^{0}{}_{0\alpha}$, we obtain by (8) and (9) that

$$R_{00} = \sum_{\alpha=1}^{3} \Big[\frac{1}{2} g_{00,\alpha,\alpha} - \Big(\frac{1}{2g_{00}} g_{00,\alpha} \Big) \Big(\frac{1}{2} g_{00,\alpha} \Big) \Big]. \tag{12}$$

The second term can be neglected as a product of three small quantities. The first term gives

$$R_{00} = \frac{\Delta\phi}{c^2}, \tag{13}$$

where Δ is the Laplace operator. Because here the trace R of the Ricci tensor is simply $R^0_0 = R$, Einstein's equations without the cosmological constant give by (3) and (13)

$$R^0_0 - \frac{1}{2} R = \frac{8\pi G}{c^4} T^0_0, \tag{14}$$

$$\Delta\phi = 4\pi G\mu, \tag{15}$$

where G is the gravitational constant. The above Poisson equation (15) is the basis of the Newtonian theory of gravitation.

1.2. Jeans' theory – the original version

The Jeans' theory of the gravitational instability (Jeans 1902) is based on the classical equations of hydrodynamics. They are as follows:

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\boldsymbol{v}) = 0,$$
$$\frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\operatorname{grad})\boldsymbol{v} = -\operatorname{grad}\phi - \frac{\operatorname{grad}p}{\rho},$$
$$\Delta\phi = 4\pi G\rho.$$

The first equation is the continuity equation, in which $\rho > 0$ is the density and $\boldsymbol{v}$ is the velocity. The second one is the vector Navier-Stokes equation, in which ϕ is the gravitational potential and p is the pressure. The third one is the Poisson equation.

We assume that there is a static homogeneous and isotropic background without any motion,

$$\rho_0 = \text{const.} > 0, \quad \boldsymbol{v} = 0, \quad \phi_0 = \text{const.}, \quad p_0 = \text{const.}$$

An essential problem is that such a background does not fulfill the Poisson equation, since the left-hand side is zero, whereas the right-hand side is positive. This problem is ignored in Jeans (1902) and the existence of perturbations $\delta = \delta(x^0, x^1, x^2, x^3)$ is of the following form

$$\rho = \rho_0(1+\delta), \quad \boldsymbol{v}_1 \neq 0, \quad \phi = \phi_0 + \phi_1, \quad p = p_0 + p_1.$$

Then we get

$$\frac{\partial\delta}{\partial t} + \operatorname{div}((1+\delta)\boldsymbol{v}_1) = 0,$$
$$\frac{\partial\boldsymbol{v}_1}{\partial t} + (\boldsymbol{v}_1\cdot\operatorname{grad})\boldsymbol{v}_1 = -\operatorname{grad}\phi_1 - \frac{\operatorname{grad}p_1}{\rho_0(1+\delta)},$$
$$\Delta\phi_1 = 4\pi G\rho_0\delta.$$

Obviously, Jean's theory has a serious defect, because to assume that the Poisson equation does not hold for the background, but simultaneously to assume that the Poisson equation does hold for the perturbations, is an unsatisfactory situation. Sometimes this controversy is called the "Jeans swindle" (Binney & Tremaine 1987).

Note that for the perturbations of the gravitational potential ϕ_1 and the density $\rho_0\delta$, the Poisson equation is already assumed to be fulfilled. Concerning the perturbation of the pressure, it is assumed that there is an adiabatic perturbation and that it can be written as

$$p_1 = b^2\rho_0\delta,$$

where b is the sound speed. It is also assumed that all perturbations are small. This means that all terms containing products of two terms are neglected. After that we get

$$\frac{\partial \delta}{\partial t} + \operatorname{div} \boldsymbol{v}_1 = 0,$$
$$\frac{\partial \boldsymbol{v}_1}{\partial t} = -\operatorname{grad} \phi_1 - b^2 \operatorname{grad} \delta,$$
$$\Delta \phi_1 = 4\pi G \rho_0 \delta.$$

Taking the partial time derivative of the first equation and the divergence of the second equation, we find that

$$\frac{\partial^2 \delta}{\partial t^2} - b^2 \Delta \delta - 4\pi G \rho_0 \delta = 0.$$

Thus, for $b > 0$ we obtain a linear second order partial differential equation of hyperbolic type. Its solution can be searched as a superposition of the plane waves. Hence, we have for one single term that

$$\delta = \tilde{\delta} \exp(\mathrm{i}(\omega t - kx)),$$

where $\tilde{\delta}$ is the constant amplitude, ω is the angular frequency, k is the magnitude of the wavevector (or the wavenumber). Without loss of generality the direction of the wave motion can be in the x direction for $x = x^1$. Then we obtain the dispersion relation

$$\omega^2 - b^2 k^2 + 4\pi G \rho_0 = 0$$

and there is a limiting wavevector

$$k_J = \frac{\sqrt{4\pi G \rho_0}}{b}.$$

If $k > k_J$, then we have ordinary sound waves, because $\omega^2 > 0$, i.e., ω is real. If $k < k_J$, then we have a gravitational instability, because $\omega^2 < 0$, i.e. ω is imaginary. For $k \to 0$ we have an exponential growth of δ as $\propto \exp(\sqrt{4\pi G \rho_0} t)$.

The limiting wavelength

$$\lambda_J = \frac{2\pi}{k_J} = b \left(\frac{\pi}{G \rho_0} \right)^{1/2}$$

defines the order of the Jeans' length. The sizes bigger than this scale should be unstable.

1.3. Jeans' theory – a possible solution

Recall that Lifshitz (1946) presented a possible solution of the Jeans swindle, see also (Bonnor 1957). The key change concerns the background. It is assumed that there is an expanding background in accordance with the spatially flat pressure-less cosmological model.

In cosmology, this model is described by the following metric:

$$ds^2 = c^2 d\tau^2 - a^2(\tau)((dq_1)^2 + (dq_2)^2 + (dq_3)^2),$$

where time is denoted as τ in order to avoid any confusion with time e.g. in (1), $a(\tau)$ is the scaling factor having the physical dimension m and q_1, q_2, q_3 are the dimensionless comoving coordinates. The position $[q_1, q_2, q_3]$ is denoted as $\boldsymbol{q}$. The Hubble parameter is here $H(\tau) = \dot{a}(\tau)/a(\tau) = 2/(3\tau)$. For the density $\rho(\tau)$ we have $\rho(\tau)a^3(\tau) = \text{const.}$ and $6\pi G\rho\tau^2 = 1$. For the deceleration we obtain

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G\rho}{3}.$$

Then the background is taken as follows:

$$\begin{aligned} \boldsymbol{u}_0 = H\boldsymbol{r}, \quad \boldsymbol{r} = a\boldsymbol{q}, \quad \phi_0 = \frac{2}{3}\pi G\rho_0 r^2, \\ r = |\boldsymbol{r}|, \quad \rho_0 r^3 = \text{const.}, \quad p_0 = 0. \end{aligned} \tag{16}$$

This background fulfills the continuity equation, the Navier-Stokes equation, and also the Poisson equation.

The only problem here is still with the Poisson equation – in Subsection 2.1 it was formulated that neither ϕ nor ρ are functions of the time.

For perturbations we obtain

$$\boldsymbol{v} = \boldsymbol{u}_0 + \boldsymbol{v}_1, \quad \phi = \phi_0 + \phi_1, \quad p = p_1 = b^2\rho_0\delta\rho = \rho_0(1+\delta).$$

On the other hand, the sound velocity b can depend on time.

Neglecting all products of small quantities (for details see e.g. Mészáros 1993, Chapt. 2) we get

$$\begin{aligned} \frac{\partial\delta}{\partial\tau} + \boldsymbol{u}_0 \operatorname{grad}\delta + \operatorname{div}\boldsymbol{v}_1 = 0, \\ \frac{\partial\boldsymbol{v}_1}{\partial\tau} + (\boldsymbol{u}_0\cdot\operatorname{grad})\boldsymbol{v}_1 + H\boldsymbol{v}_1 + b^2\operatorname{grad}\delta + \operatorname{grad}\phi_1 = 0, \\ \Delta\phi_1 = 4\pi G\rho_0\delta. \end{aligned}$$

Introducing the comoving coordinates, applying another partial derivative with respect to τ to the first equation and the divergence operator to the second equation, we find that

$$\left.\frac{\partial^2\delta}{\partial\tau^2}\right|_q + 2H\left.\frac{\partial\delta}{\partial\tau}\right|_q - a^{-2}(b^2\Delta_q + 4\pi G\rho_0)\delta = 0. \tag{17}$$

The index q means that the derivatives are taken with respect to comoving coordinates. This is the key equation for the perturbation of the density (see Weinberg 1972, (15.9.23)).

If $b = 0$ and we consider the spatially flat solution, we get

$$\frac{\partial^2 \delta}{\partial \tau^2} + \frac{4}{3\tau}\frac{\partial \delta}{\partial \tau} - \frac{2}{3\tau^2}\delta = 0.$$

Its increasing solution is $\delta \propto t^{2/3} \propto a$ (see Weinberg 1972, (15.9.25), (15.9.29)).

If $b = 0$ and we have the remaining two Friedmannian models, then the exact solutions are also obtainable (Weinberg 1972, (15.9.27), (15.9.31)). For the hyperbolic case the growth is smaller than $\delta \propto t^{2/3}$, for the remaining model – in the best case – the growth can be $\delta \propto 1.45\, t^{2/3}$.

For the spatially flat model also with the pressure term a solution is known (see Weinberg 1972, (15.9.39), (15.9.41)). The Jeans' length is recovered from the behavior of the Bessel function (see Weinberg 1972, (15.9.44)–(15.9.46)). Only for the scales much larger than the Jeans' length the growth can be maximally $\delta \propto t^{2/3}$.

Summing up it can be said that this much slower $\delta \propto t^{2/3}$ growth of the density instability compared with the exponential one, which is the only difference in the behavior of perturbations.

2. The Newtonian limit with time dependence

In this section we again assume that equations (1)–(4) hold. Now the key difference concerns the restriction on time. Here we assume that both ϕ and μ may be dependent on time. In addition, similarly to Section 1, no restrictions on the sizes of ϕ and μ are assumed at beginning. In fact, we had the most general case at the beginning.

Beyond the six Christoffel symbols from (5)–(9) we have the seventh Christoffel symbol

$$\Gamma_{000} = -\frac{1}{2} g_{00,0}$$

and

$$\Gamma^0_{00} = \frac{g_{00,0}}{2g_{00}}.$$

Using (10), we obtain

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{00}\Gamma^{0}_{\alpha 0} - \Gamma^{\alpha}_{00}\Gamma^{0}_{0\alpha} - \Gamma^{0}_{0\alpha}\Gamma^{\alpha}_{00}.$$

The most surprising result is that the $\Gamma^0_{00,0}$ *terms are cancelled*, and thus we have

$$R_{00} = \sum_{\alpha=1}^{3}\Big[\frac{1}{2} g_{00,\alpha,\alpha} - \Big(\frac{1}{2g_{00}} g_{00,\alpha}\Big)\Big(\frac{1}{2} g_{00,\alpha}\Big)\Big].$$

This equation and equation (12) are identical. But here the time dependence was assumed. *This is the main result of the article.*

Neglecting all the products of three small quantities, we again obtain (12)–(15). The Poisson equation also holds. *The new key result is that in it both ϕ and μ may be well dependent on time.*

3. Discussion

The straightforward consequence of the time dependence of the Poisson equation in Lifshitz' and Bonnor's theory, whose background defined by equations (16) in this theory, is without any problems. The time dependence of ϕ is not a defect.

One more important note is needed here. The comparison of Lifshitz' and Bonnor's theory with the general relativistic theory of perturbations (see Weinberg 1972, Chapt. 15.10) shows that these two different procedures are very similar for the subhorizon scales (compare equation (17) of this paper and equation (15.10.57) of Weinberg (1972), Chapt. 15.10)). All this further strengthens the usefulness of Lifshitz' and Bonnor's theory.

4. Conclusions

The main results of the paper can be summarized as follows.

1. We have proven mathematically that in the Poisson equation – as in the Newtonian limit of the General Theory Relativity – both sides may well depend on time.

2. We have shown – as the consequence of this behavior – that Lifshitz' and Bonnor's modification of the Jeans' theory is correct.

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