Could we orbit the expanding universe in finite time?

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Abstract. According to Einstein's cosmological principle, our universe at any fixed time is homogeneous and isotropic on large scales. It is thus modeled up to scaling by one of the following three three-dimensional maximally symmetric manifolds: the sphere, the Euclidean space, or the hyperbolic pseudosphere, none of which contain any privileged points or directions. We present several arguments which indicate why the last two unbounded manifolds cannot be good models of our physical universe at any fixed time. We shall then concentrate only on the expanding three-dimensional sphere. We prove that if its radius expands linearly, then the total travel time of a shuttle along geodesics around the entire universe is finite. If the shuttle has a constant velocity $v > 0$, its trajectory on such an expanding sphere is a logarithmic spiral.

Keywords: cosmological parameters; cosmic microwave background; maximally symmetric manifold; sphere; logarithmic spiral

1 Introduction

According to the *Copernican principle*, humans on the Earth are not privileged observers of the universe. According to *Einstein's cosmological principle*, our universe on each isochrone is homogeneous and isotropic on large scales. Roughly speaking, at any fixed time its curvature is constant at any point and in any direction. These principles are, in fact, assumptions.

More precisely, *homogeneity* is the assumed property of the universe that at any fixed time instant and on large spatial scales the universe appears the same to all observers, wherever they are. In other words, at any fixed time the *translation symmetry* of the universe is required. In the Hubble test of a local homogeneity of the universe, one has to measure the apparent magnitude (energy flux f from a given galaxy). By Carpenter (1938), the number of observed galaxies in the sky brighter than f should vary as $f^{-3/2}$, see also Baryshev (2012), Peebles (1993), Weinberg (1972). For a modification of this test to γ-ray bursts see Li (2015).

Similarly, *isotropy* is the assumed property of the universe in which the universe at large spatial scales would seem to an observer at any point in space to be the same in all directions, i.e., at any fixed time the *rotational symmetry* of the universe is required. A local isotropy is continually verified by analyzing the Hubble Deep Field, Webb's First Deep Field, Cosmic Microwave Background radiation (CMB), γ-ray bursts, etc. According to the Copernican principle, we are not at some umbilic point, see Křížek (2003), as is the tip of the egg which satisfies the required isotropy only at this point and its antipodal.

Note that isotropy at all points implies homogeneity at all points on the space manifold, see e.g. Weinberg (1993). The converse statement is not true. For instance, a crystal of $CaCO_3$ is homogeneous, but anisotropic. For the Gödel homogeneous and anisotropic universe we refer to Stephani (2004).

Due to Einstein's cosmological principle our universe at any fixed time is usually modeled up to scaling by one of the following three three-dimensional maximally symmetric manifolds: the sphere $\mathbb{S}^3$, the Euclidean space $\mathbb{E}^3$, or the hyperbolic pseudosphere $\mathbb{H}^3$, none of which contain any privileged points or directions. Their curvature indexes are $k = 1$, $k = 0$, and $k = -1$, respectively. Recall that an *n-dimensional manifold* with $n \in \{1, 2, 3, \dots\}$ is a set of points for which there exists an open neighborhood that can be continuously mapped onto an open set in $\mathbb{E}^n$ such that the inverse is continuous, too.

Already in 1900 Karl Schwarzschild conjectured that the universe can be described as a huge three-dimensional sphere, see Schwarzschild (1900). He also speculated that our universe could be even possibly hyperbolic.

Theorem 1. *For any dimension $n > 1$ there exist exactly three maximally symmetric manifolds, namely $\mathbb{S}^n$, $\mathbb{E}^n$, and $\mathbb{H}^n$.*

For the proof see e.g. Weinberg (1972), Penrose (2005). Now let us ask a purely theoretical question:

$$\textit{Could we orbit the universe in finite time?} \tag{1}$$

For $\mathbb{E}^3$ and $\mathbb{H}^3$ the answer is NO, since these manifolds are unbounded. In the next section, we show that these two manifolds are not likely models of our universe. In Section 3, we concentrate on the three-dimensional sphere. In Section 4, we prove that if it expands linearly, then the travel time of a shuttle along geodesics around the entire universe is finite. Finally, in Section 5, we present some generalizations and give a few essential notes.

2 Arguments against unbounded physical universes

The global curvature of our universe is not definitely established yet. Anyway, below we present several arguments against Euclidean and hyperbolic geometries. The first nonvacuum solution of Einstein's equations was found by Schwarzschild, see Schwarzschild (1916). He assumed that a ball with coordinate radius $R > 0$ is formed by an ideal incompressible nonrotating fluid with constant density to avoid a possible internal mechanical stress in the solid that may have a nonnegligible influence on the resulting gravitational field. For the line element $\mathrm{d}l$ of the interior of the homogeneous nonrotating mass ball he derived that, see also Stephani (2004), Ellis (2012) and Florides (1974),

$$\mathrm{d}l^2 = \frac{1}{1 - s^2r^2}\,\mathrm{d}r^2 + r^2\mathrm{d}a^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2, \tag{2}$$

where $sr \in [0, 1)$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$, $s = \sqrt{S/R^3}$, and S is the Schwarzschild radius. This relation is very similar to

$$\mathrm{d}l^2 = \frac{1}{1 - kr^2}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2$$

for the unit sphere

$$\mathbb{S}^3 = \{(x, y, z, w) \in \mathbb{E}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$$

with curvature parameter $k = 1$, see Křížek (2016). Here $r \in [0, 1)$ is a dimensionless parameter. We observe that the cases $k = -1$ and $k = 0$ do not match (2) when $s > 0$. Thus, the presence of a homogeneous mass distribution causes a positive curvature globally and the most probable model of our universe at any fixed time is the sphere $\mathbb{S}^3$ up to scaling.

Another argument against the case $k \leq 0$ is as follows. It is very unlikely that an unbounded universe would have at every point and every time instant almost the same temperature, density, pressure, curvature, etc., on large scales. This would require an infinite speed of information transfer.

Moreover, the hyperbolic manifold $\mathbb{H}^3$ has a somewhat counter-intuitive geometry which is difficult to imagine. David Hilbert proved that the hyperbolic plane $\mathbb{H}^2$ cannot be isometrically imbedded into $\mathbb{E}^3$, see Hilbert (1901). His statement is usually formulated as follows: *There is no complete regular two-dimensional manifold of negative constant Gaussian curvature imbedded into* $\mathbb{E}^3$. Thus the hyperbolic plane should be deformed somehow to get some idea of how it looks like. There are at least six basic ways to perform such a deformation, see Cannon (1997). One way is, for example, the well-known two-dimensional Poincaré disc in which all angles between geodesics in $\mathbb{H}^2$ are preserved, but distances are not preserved.

Danilo Blanuša proved that the hyperbolic plane $\mathbb{H}^2$ can be isometrically imbedded into the space $\mathbb{E}^6$, see Blanuša (1955). However, for the time being it is not known whether the dimension six can be reduced. Blanuša's assertion was generalized in Brander (2003):

Theorem 2 *For* $n > 1$ *the pseudosphere* $\mathbb{H}^n$ *can be isometrically imbedded into* $\mathbb{E}^{6n-6}$.

It is again not known whether the exponent $6n - 6$ can be reduced. The manifolds $\mathbb{S}^3$ and $\mathbb{H}^3$, which possibly model our universe at any fixed time, can be isometrically imbedded to the Euclidean space $\mathbb{E}^4$ and $\mathbb{E}^{12}$, respectively, i.e.

$$\mathbb{S}^3 \hookrightarrow \mathbb{E}^4, \quad \mathbb{H}^3 \hookrightarrow \mathbb{E}^{12}.$$

Here the symbol $\hookrightarrow$ denotes the isometric imbedding. Consequently, in a 12-dimensional Euclidean space the distances in $\mathbb{H}^3$ are undeformed. The pseudosphere $\mathbb{H}^3$ is thus a rather exotic object. Visualization and construction of $\mathbb{H}^3$ without any deformations of distances is therefore extremely difficult, see Thurston (1984). Note that by Brander (2007) there exists a local isometric imbedding from $\mathbb{H}^n$ to $\mathbb{E}^{2n-1}$. Hence, $\mathbb{H}^3 \hookrightarrow \mathbb{E}^5$ only locally.

3 Nonuniqueness of the notion universe

From now on we shall consider only the case $k = 1$ and assume that the cosmological constant $\Lambda \approx 10^{-52}$ m^{-2}. For Einstein's static universe with radius $a = 1/\sqrt{\Lambda}$, i.e.

$$\mathbb{S}^3_a = \{(x, y, z, w) \in \mathbb{E}^4 \,|\, x^2 + y^2 + z^2 + w^2 = a^2\}, \tag{3}$$

the answer to the question (1) is obviously YES. Traveling along a given great circle (geodesic) with constant velocity $v = 0.999\,c$, the total travel time T can be estimated as follows

$$T \approx \frac{2\pi a}{v} \approx \frac{2\pi 10^{26}}{3 \cdot 10^8}\ \mathrm{s} \approx \frac{2\pi 10^{26}}{3 \cdot 10^8 \cdot \pi \cdot 10^7}\ \mathrm{yr} = 6.666 \cdot 10^{10}\ \mathrm{yr}.$$

This is five times the present age of the universe $t_0 = 13.82$ Gyr according to the standard ΛCDM cosmological model. However, the proper time of the traveler would be much smaller than T due to time dilation.

Now we shall consider an expanding universe, where the radius a in (3) is a continuous increasing expansion function $a = a(t) \geq 0$ depending on cosmic time t. However, the term "universe" is used in cosmology with various meanings: true spacetime, true space (i.e. a part of spacetime at any fixed time instant), and the observable universe, which is seen as a projection on the celestial sphere. These are three different objects. Their mathematical models are also three completely different manifolds (see Figure 1). Thus altogether we have $6 = 3+3$ meanings of the problematic notion "universe" for which the terminology is not fixed yet. The first three contain real matter, whereas the other three are only abstract mathematical idealizations of physical reality.

In accordance with the Einstein cosmological principle, we shall understand by the *universe* a cross-section of spacetime at any fixed time instant, i.e., the universe will be an isochrone in spacetime for constant t. For $k = 1$ the corresponding model of the universe is the sphere $\mathbb{S}_a^3$ for some fixed radius $a = a(t) > 0$, which is a three-dimensional manifold in the four-dimensional Euclidean space $\mathbb{E}^4$. The model of the observable universe has dimension three, too. The model of spacetime in $\mathbb{E}^5$ has dimension four (cf. Figure 1),

$$\mathbb{M} = \{(t, x, y, z, w) \in \mathbb{E}^5 \mid x^2 + y^2 + z^2 + w^2 = a^2(t),\ t \in [0, t_0]\}.$$

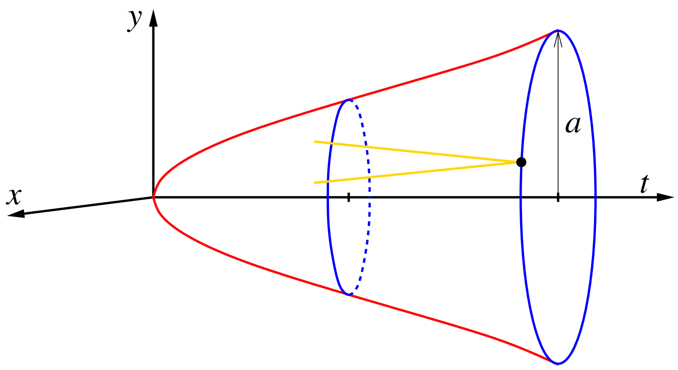

Fig 1. Schematic illustration of three different manifolds that are used in the Big Bang model for our universe with positive curvature index. For simplicity, the space dimensions are reduced by two. Hence, the sphere $\mathbb{S}_a^3$ with radius $a = a(t) > 0$ at any fixed time instant t is replaced only by its great (blue) circle $\mathbb{S}_a^1$ for $z = w = 0$. This is the model of the space (universe) with positive constant curvature $1/a$. The model of spacetime can be obtained

by rotation of the (red) graph of the expansion function about the time axis t. The observable universe is marked by the (yellow) past light cone whose vertex corresponds to an observer. This light cone is deformed near the origin, but we do not know how. Each of these three models has a different center.

All six above-mentioned objects have to be carefully distinguished, otherwise we may come to various confusions. For instance, the observable universe is not homogeneous, since for different cosmological redshifts z, it has a different mass density. Thus, it is an entirely different object than the universe as a space. From a similar reason the spacetime is also not homogeneous. Therefore, the expansion of the universe is a completely different notion than the expansion of the observable universe.

We often hear that the universe has no center. This is similar to the statement that a circle has no center. The circle, of course, has its center even though it does not belong to it. In Figure 1, the centers of the blue circles lie on the axis t. Therefore, also the model of the universe $\mathbb{S}^3_a$ has its center at the origin $(0,0,0,0)$ of coordinates of the space $\mathbb{E}^4$ although $(0,0,0,0) \notin \mathbb{S}^3_a$. On the other hand, the Earth is at the center (see the yellow vertex in Figure 1) of the observable universe which is finite for any $k \in \{-1,0,1\}$. Its horizon can be modeled by a two-dimensional sphere corresponding to the CMB. The center of the red model of the expanding universe corresponds to the Big Bang at an initial time (see also Figure 1). The Big Bang thus happened everywhere.

Finally note that Albert Einstein is a part of the entire spacetime which is modeled by the red manifold $\mathbb{M}$ in Figure 1. However, he is not a part of the current physical universe not even of the observable universe modeled by the blue and yellow manifolds, respectively.

4 Flight around the expanding universe

Further, we shall consider a variable radius $a = a(t)$ of the sphere $\mathbb{S}^3_a$. For simplicity, we assume that it increases linearly with constant velocity $V > 0$, cf. Remark 1 below. The expansion velocity V can be even faster than light and there is no contradiction with Special Theory of Relativity outside the observer's inertial system, see Davis (2004). Furthermore, assume that a space shuttle has also a constant velocity $v > 0$ with respect to its local neighborhood (i.e. intergalactic dust). In this particular case, its trajectory will be one turn of the logarithmic spiral instead of the great circle, see Figure 2. Recall that the *logarithmic spiral* is a self-similar curve making a constant *slope angle* $\vartheta \in (0, \pi/2]$ with the polar radii at all its points.

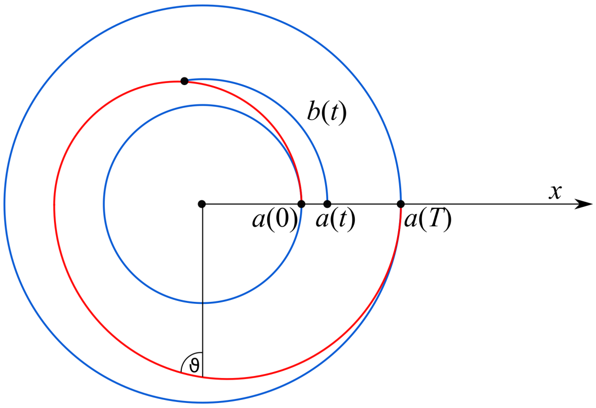

Fig 2. Trajectory of a space shuttle in an expanding universe modeled by the sphere $\mathbb{S}^3_{a(t)}$ for $a(T) = 2a(0)$. It is described by the logarithmic spiral of the length $L = (a(T) - a(0))/\cos\vartheta$, where the slope angle ϑ fulfills $\tan\vartheta = v/V$.

The following surprising theorem inspired by Rokyta (2012) states that if the space shuttle has an arbitrarily small constant velocity $v > 0$ with respect to V, then it always needs finite time to travel around a linearly expanding universe, see also Remark 4. Therefore, we will deal with the nonrelativistic addition of velocities in Theorem 3 below. Note that the spiraling trajectory can be easily unfolded into a straight line. Thus, in fact, we shall investigate only a one-dimensional problem. For brevity, we also make a linear shift of cosmic time so that $t = 0$ corresponds to the start time.

Theorem 3. *Let $\mathbb{S}^3_{a(t)}$ expand with constant velocity $V > 0$, i.e. $\dot{a}(t) = V$ for all $t \geq 0$, and let $v > 0$ be a constant velocity of the space shuttle. Then its total cosmic travel time around the universe is*

$$T = a(0)\frac{\exp(2\pi V/v) - 1}{V}. \tag{4}$$

Proof. Assume that the space shuttle will launch from the axis x at time $t = 0$. The circumference of a great circle (geodesic) of $\mathbb{S}^3_a$ at time $t \geq 0$ is clearly given by

$$2\pi a(t) = 2\pi(a_0 + Vt), \tag{5}$$

where $a_0 = a(0)$ is the initial radius. Let $b(t)$ be the instant distance of the space shuttle at time $t \geq 0$ along the great circle with radius $a(t)$ from the axis x, i.e. from the launch point in the expanding universe (see Figure 2). It satisfies the following initial problem

$$\dot{b}(t) = v + w(t), \quad b(0) = 0, \tag{6}$$

where $w = w(t)$ is the drifting velocity of a linearly expanding space. We see that the ratio between the velocity $w(t)$ and the expansion velocity $2\pi V$ of the great circle is the same as the ratio between the distance $b(t)$ and the circumference $2\pi a(t)$, namely,

$$\frac{w(t)}{2\pi V} = \frac{b(t)}{2\pi a(t)}.$$

From this, (5), and (6) we get the initial problem

$$\dot{b}(t) = v + V\frac{b(t)}{a_0 + Vt} \quad \text{and} \quad b(0) = 0. \tag{7}$$

The right-hand side of this first order differential equation contains the velocity v which is increased by the drifting velocity $w(t)$ of space expansion. By the substitution $t = 0$ and differentiation, we can easily verify that

$$b(t) = (a_0 + Vt)\frac{v}{V}\ln\frac{a_0 + Vt}{a_0} \tag{8}$$

is the solution to problem (7). The space shuttle will return to the launch point from the opposite direction at time T when

$$2\pi a(T) = b(T).$$

From this, (5), and (8) we obtain the equation

$$\frac{v}{V}\ln\frac{a_0 + VT}{a_0} = 2\pi. \tag{9}$$

Consequently,

$$a_0 + VT = a_0 \exp\frac{2\pi V}{v} \tag{10}$$

and thus (4) follows. Q.E.D.

5 Final remarks

Remark 1. The assumption that $V > 0$ is constant in Theorem 3 is not too restrictive, since the expansion function $a = a(t)$ in Figure 3 is almost linear during the last 10 Gyr, see Křížek (2015). By l'Hospital's rule, (4) converges for $V \to 0$ to $T = 2\pi a(0)/v$ corresponding to Einstein's static universe (3). Moreover, a finite travel time $\tilde{T}$ can be preserved if there exists a linear upper bound $a = a(t)$ of a nonlinear expansion function $\tilde{a} = \tilde{a}(t)$ of a particular cosmological model. Hence, if $\tilde{a}(t) \le a(t)$ for all t, then

$$\tilde{T} \le T,$$

where T is finite by Theorem 3.

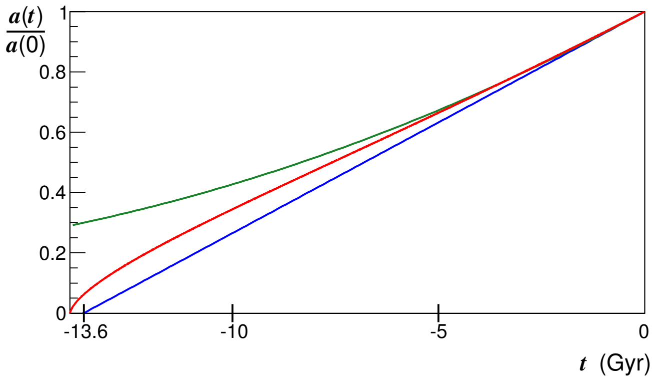

Fig. 3. The assumed behavior of the normalized expansion function $a(t)/a(0)$. The time variable is shifted for simplicity such that $t_0 = 0$ corresponds to the present time. The lower blue graph corresponds to the linear function $1 + H_0 t$ on the interval $[-1/H_0, 0]$, where H_0 is the Hubble constant and $1/H_0 = 13.6$ Gyr is the Hubble time. The upper green graph shows the quadratic function $1 + H_0 t - \frac{1}{2} q_0 H_0^2 t^2$, where $q_0 = -0.6$ is the current value of the deceleration parameter. The middle red graph illustrates the behavior of the normalized expansion function calculated numerically from the Friedmann equation for $k = 1$ and measured cosmological parameters, see Planck (2014). We observe that the accelerated expansion differs only very little from the linear expansion during the last few Gyr.

Remark 2. The assumption that $v > 0$ is constant in Theorem 3 is also not restrictive, since for each variable velocity of the space shuttle greater than v we clearly get a smaller (i.e. finite) travel time.

Remark 3. For simplicity, we have chosen $a(T) = 2a(0)$ in Figure 2. From this and (9) it follows that $v/V = 2\pi/\ln 2$. Hence, the constant slope angle ϑ between the polar radii and the tangent line to the spiral is only slightly less than $\pi/2$, where $\vartheta = \arctan(v/V)$. Nevertheless, Theorem 3 covers also the case $0 < v \ll c < V$ which is more realistic.

Remark 4. Now we present an independent proof of formula (4). Recall the equation of the logarithmic spiral with slope angle ϑ in polar coordinates (r, φ),

$$r(\varphi) = r_0 \exp(\varphi \operatorname{cotan} \vartheta), \tag{11}$$

where $r_0 > 0$ is a given constant. If $\vartheta = \pi/2$, then the logarithmic spiral reduces to a circle with radius r_0. If $\vartheta \in (0, \pi/2)$ and $\varphi_2 \geq \varphi_1$, then the length of the logarithmic spiral is given by

$$L(\varphi_1, \varphi_2) = \frac{r(\varphi_2) - r(\varphi_1)}{\cos \vartheta}, \tag{12}$$

see Figure 2 for $\varphi_2 = 2\pi$ and $\varphi_1 = 0$. Since the space expands in the radial direction at the constant velocity V and since the shuttle (a nonrelativistic massive particle) moves in the perpendicular direction at the constant velocity

v, the total velocity along the logarithmic spiral $\sqrt{v^2+V^2}$ is also constant and the slope angle is $\vartheta = \arctan(v/V)$. Hence, $V \sin\vartheta = v\cos\vartheta$ and thus $V^2 = (v^2+V^2)\cos^2\vartheta$, i.e.

$$V = \sqrt{v^2+V^2}\cos\vartheta.$$

Consequently, from (11) and (12) we find that the length of the entire trajectory is $L = (a(T)-a(0))/\cos(\arctan(v/V))$ and thus, the total travel time is

$$T = \frac{L}{\sqrt{v^2+V^2}} = \frac{a(T)-a(0)}{\sqrt{v^2+V^2}\cos(\arctan(v/V))} = a(0)\frac{\exp(2\pi V/v)-1}{V}.$$

This result independently confirms the formula (4).

There are several theories about the origin of the CMB, see e.g. Planck (2014), Vavryčuk (2018). Now we show that the CMB radiation (the cosmological horizon) might be just the image of the antipodal point of our neighborhood ≈ 13.8 Gyr ago.

Example. To illustrate the main idea of the above hypothesis, consider the trajectory of a photon which is moving at the velocity $v = c$ with respect to its local space. Then for the trajectory from the antipodal point of $\mathbb{S}^3_{a(0)}$ to our present location we find by (5) and similarly to (10) that

$$a(t) = a(0) + Vt = a(0)\exp\frac{\pi V}{c},$$

where t is the travel time of CMB photons along trajectories of length (12) for $\varphi_2 = \pi$ and $\varphi_1 = 0$. From this the cosmological redshift of the photon is given by

$$z = \frac{a(t)}{a(0)} - 1 = \exp(2.22\,\pi) - 1 \approx 1075 \quad \text{for } V = 2.22c.$$

This value is quite close to the observed redshift $z = 1089$ of the CMB, see Planck (2014), even though the actual expansion velocity of the universe is variable and not a fixed constant V.

Remark 5. It would be a mistake to believe that the well-known map of the CMB radiation shows the entire universe, how it looked like 380 000 years after the Big Bang. This map shows only a two-dimensional slice of a three-dimensional manifold corresponding to the universe for $z \approx 1089$ when its radius was $z+1 = 1090$ times smaller than at present. This radius is the same as the radius of any great circle (e.g. equator) at the time when relict photons were emitted. Moreover, we observe everything only on the projection on the celestial sphere. For example, the relict radiation produced at that time in our neighborhood is not on the map of the cosmic microwave background radiation. We also do not see any relict radiation from the places where all 10^{12} galaxies in the observable universe are to date. At each of these galaxies we would observe at present completely different maps of the cosmic microwave background fluctuations. So on the Earth, an observer may have an idea about how only a tiny part of the early universe looked like.

Remark 6. The question of whether photons can orbit a closed universe near the initial singularity was investigated by Misner in Misner (1969), see also Doroshkevich (1971). The answer is yes. Our main result is different, since we accept also subluminal velocities at the present era. Moreover, we can cover many cosmological models as explained in Remark 1.

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