LECTURES: Infinitary combinatorics with applications in mathematical analysis PART 2

Wiesław Kubiś

Mathematical Institute, Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1

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1 Lecture 5

We shall present the classical theorem of Galvin & Prikry on Borel colorings of the space of irrationals. We now look at $[\mathbb{N}]^\omega$ as a natural topological space, namely, the Vietoris hyperspace of the discrete space $\mathbb{N}$. Recall that, given a topological space X, its *hyperspace* $\exp X$ is defined as the family of all nonempty closed subsets of X endowed with the *Vietoris topology* generated by the sets:

$$V_0^- \cap \dots \cap V_{k-1}^- \cap U^+ = \{A \in \exp X : A \subseteq U \text{ and } A \cap V_i \neq \emptyset \text{ for every } i < k\},$$

where each of the sets $V_0, \dots, V_{k-1}, U$ is open and $k \in \omega$. We shall be interested in the smallest non-trivial hyperspace, namely, all nonempty subsets of the discrete space $\mathbb{N}$.

1.1 Ramsey sets

We shall work in the space of all subsets of $\mathbb{N}$, which carries the natural Cantor set topology. Excluding the empty set, it can also be regarded as the Vietoris hyperspace of $\mathbb{N}$. The advantage here is that the Vietoris topology is much richer than the Cantor set topology. This is particularly important when aiming at a partition theorem involving Borel sets.

In what follows, we shall usually denote finite sets by small letters and infinite sets by capital letters. Given $A, B \subseteq \mathbb{N}$, we shall write $A \sqsubset B$ if A is a proper initial segment of B, that is, $A \subseteq B$, $A \neq B$, and $B \cap (-\infty, \sup A) \subseteq A$. Observe that $A \sqsubset B$ implies that A is finite.

The Vietoris topology on $[\mathbb{N}]^\omega$ has a natural basis consisting of the following sets:

$$[s; A]^{\mathscr{V}} := \{B \in [A]^\omega : s \sqsubset B\},$$

where $s \in [\mathbb{N}]^{<\omega}$ and $A \in [\mathbb{N}]^\omega$. Note that $[s; A]^{\mathscr{V}} \neq \emptyset$ if and only if s is a subset of A. Furthermore, $A \in [s; A]^{\mathscr{V}}$ if and only if s is an initial segment of A. Note also that the natural (inherited from the Cantor set) topology on $[\mathbb{N}]^\omega$ has a basis consisting of sets of the form $[s; \mathbb{N}]^{\mathscr{V}}$, where $s \in [\mathbb{N}]^{<\omega}$.

The following fact is an easy exercise:

Proposition 1.1. *The family of all sets of the form $[s; A]^{\mathscr{V}}$, where $s \in [\mathbb{N}]^{<\omega}$, $A \in [\mathbb{N}]^\omega$, forms an open basis for the Vietoris topology on $[\mathbb{N}]^\omega$.*

From now on, we fix $\mathscr{F} \subseteq [\mathbb{N}]^\omega$.

Definition 1.2. Given $A \in [\mathbb{N}]^\omega$, $s \in [\mathbb{N}]^{<\omega}$, we shall say that A *accepts* s (with respect to $\mathscr{F}$) if $s \subseteq A$ and $[s; A]^{\mathscr{V}} \subseteq \mathscr{F}$. We shall say that A *rejects* s (with respect to $\mathscr{F}$) if no $B \in [A]^\omega$ with $A \cap \max(s) \subseteq B$ accepts s.

Finally, we shall say that $A \in [\mathbb{N}]^\omega$ is *decided* if for every $s \subseteq A$ either A accepts s or A rejects s.

The definition of accepting is clear. Rejecting s by A means that it is not possible to "shrink" A by removing some elements on the right-hand side of s so that the smaller set would accept s. Note that every infinite subset of a decided set is decided. The existence of decided sets is crucial.

Lemma 1.3. *Given $N \in [\mathbb{N}]^\omega$, there exists $M \in [N]^\omega$ such that M is decided.*

Proof #1. Given $k \in \mathbb{N}$ and $A \in [\mathbb{N}]^\omega$, we shall say that A *decides* k, provided that for every $s \subseteq k$, A either decides or rejects s. Let $\mathbb{P}$ be the set of all pairs $\langle k, A\rangle$ such that A decides k. Given $\langle k, A\rangle, \langle \ell, B\rangle \in \mathbb{P}$, we define $\langle k, A\rangle \preceq \langle \ell, B\rangle$ iff $k \leqslant \ell$ and $B \subseteq A$ is such that $B \cap k = A \cap k$. Then $\preceq$ is a partial ordering of $\mathbb{P}$. We claim that for every $\langle k, A\rangle \in \mathbb{P}$ there is $\langle \ell, B\rangle \in \mathbb{P}$ such that $\langle k, A\rangle \preceq \langle \ell, B\rangle$ and $k < \ell$.

Fix $\langle k, A\rangle \in \mathbb{P}$ and let $\ell = \min(A \setminus k) + 1$. Suppose $\langle \ell, A\rangle \notin \mathbb{P}$. Then there is $t \subseteq \ell$ such that A neither accepts nor rejects t. Necessarily $\max(t) = \ell - 1$ and, as A does not reject t, there is $A' \subseteq A$ such that $A' \cap \ell = A \cap \ell$ and A' accepts t. Repeating this argument for each possible subset of ℓ, we obtain $B \subseteq A$ such that $B \cap \ell = A \cap \ell$ and $\langle \ell, B\rangle \in \mathbb{P}$. Clearly, $\langle k, A\rangle \preceq \langle \ell, B\rangle$.

Finally, notice that $\langle 0, A\rangle \in \mathbb{P}$ for some $A \in [N]^\omega$. Indeed, either $A = N$ or $A \in [N]^\omega$ accepts $\emptyset$ in case where N does not reject $\emptyset$.

By the arguments above, there is a sequence

$$\langle k_0, A_0\rangle \preceq \langle k_1, A_1\rangle \preceq \langle k_2, A_2\rangle \preceq \dots$$

in $\mathbb{P}$ such that $\langle k_0, A_0\rangle = \langle 0, A\rangle$ and $k_0 < k_1 < k_2 < \dots$. Finally, $A = \bigcap_{n\in\omega} A_n$ is an infinite decided subset of N. $\square$

Proof #2. We construct a strictly increasing sequence of finite sets $\emptyset = s_0 \sqsubset s_1 \sqsubset s_2 \sqsubset \dots$ and a decreasing sequence of infinite sets $N \supseteq M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots$ so that the following conditions are satisfied:

(1) $s_n \sqsubset M_n$,

(2) For every $t \subseteq s_n$, M_n either accepts or rejects t.

If N rejects $\emptyset$ then we set $M_0 = N$, otherwise we find $M_0 \in [N]^\omega$ such that M_0 accepts $\emptyset$. Now suppose s_n and M_n have already been constructed. We set $s_{n+1} = s_n \cup \{\ell_n\}$, where ℓ_n is the minimal element of M_n greater than all elements of s_n.

Fix $t \subseteq s_{n+1}$. If $t \subseteq s_n$ then M_n either accepts or rejects t. Suppose $\ell_n \in t$ and M_n does not reject t. Then there is an infinite set $M_n' \subseteq M_n$ such that $M_n' \cap (\ell_n + 1) = M_n \cap (\ell_n + 1)$ and M_n' accepts t. Repeating this argument finitely many times (for each subset of s_n) we obtain M_{n+1} with the property that $s_{n+1} \sqsubset M_{n+1}$ and condition (2) is satisfied.

Finally, $M = \bigcup_{n\in\omega} s_n$ is as required. $\square$

Lemma 1.4. *Let $M \in [\mathbb{N}]^\omega$ be a decided set and let $s \in [\mathbb{N}]^{<\omega}$ be such that $s \sqsubset M$. If M rejects s then M rejects $s \cup \{n\}$ for all but finitely many $n \in M \setminus s$.*

Proof. Suppose otherwise. Then there is $N \in [s; M]^{\mathscr{V}}$ be such that M accepts $s \cup \{k\}$ whenever $k \in N \setminus s$. Thus, we have

$$[s; N]^{\mathscr{V}} = \bigcup_{k\in N\setminus s} [s \cup \{k\}; N]^{\mathscr{V}} \subseteq \bigcup_{k\in N\setminus s} [s \cup \{k\}; M]^{\mathscr{V}} \subseteq \mathscr{F}.$$

It follows that N accepts s, contradicting the definition of rejecting. $\square$

Lemma 1.5. *Let $M \in [\mathbb{N}]^\omega$ be a decided set. If M rejects $\emptyset$ then there exists $N \in [M]^\omega$ such that N rejects all of its finite subsets.*

Proof. Using Lemma 1.4 inductively, we construct a chain of finite sets

$$\emptyset = s_0 \sqsubset s_1 \sqsubset s_2 \sqsubset \dots \sqsubset M$$

such that M rejects all subsets of s_n for every $n \in \omega$. Finally, $N = \bigcup_{n\in\omega} s_n$ is as required. $\square$

We now come to the main notions.

Definition 1.6. Let $\mathscr{F} \subseteq \mathscr{P}(\mathbb{N})$. We say that $\mathscr{F}$ is *Ramsey* if for every $A \in [\mathbb{N}]^\omega$ there exists $B \in [\mathbb{N}]^\omega$ such that either $[B]^\omega \subseteq \mathscr{F}$ or $[B]^\omega \cap \mathscr{F} = \emptyset$.

Theorem 1.7 (Galvin & Prikry)**.** *Every open set in the Vietoris topology is Ramsey.*

Proof. Let $\mathscr{U} \subseteq \mathscr{P}(\omega)$ be open with respect to the Vietoris topology. Fix $A \in [\mathbb{N}]^\omega$. Shrinking A, we may assume that it is decided with respect to $\mathscr{U}$ (Lemma 1.3). If A accepts $\emptyset$ then $[A]^\omega \subseteq \mathscr{U}$. Otherwise, by Lemma 1.5, we may further shrink A so that it rejects all of its finite subsets. Suppose that $\mathscr{U} \cap [\emptyset; A]^{\mathscr{V}} \neq \emptyset$. As $\mathscr{U}$ is open, there exists a finite set $s \subseteq A$ and $B \in [A]^\omega$ such that $s \sqsubset B$ and $[s; B]^{\mathscr{V}} \subseteq \mathscr{U}$. It follows that B accepts s, contradicting the fact that A rejects s. Thus, $[A]^\omega \cap \mathscr{U} = \emptyset$. $\square$

1.2 Nash-Williams partition theorem

Using Theorem 1.7, we shall now state and prove a partition theorem on finite sets, due to Nash-Williams, which in turn generalizes Ramsey theorem.

Definition 1.8. A family $\mathscr{S}$ of finite subsets of $\mathbb{N}$ will be called *thin* if $s = t$ whenever $s, t \in \mathscr{S}$ and $s \sqsubset t$. In other words, $\mathscr{S}$ is thin if no member of $\mathscr{S}$ is an initial segment of another.

A typical example of a thin family is $[\mathbb{N}]^k$, where $k > 0$ is a natural number.

Theorem 1.9 (Nash-Williams)**.** *Let $\mathscr{S} \subseteq [\mathbb{N}]^{<\omega}$ be a thin family and assume $\mathscr{S} = S_0 \cup \cdots \cup S_{n-1}$. Then there is $M \in [\mathbb{N}]^\omega$ such that $[M]^{<\omega} \cap \mathscr{S} \subseteq S_j$ for some $j < n$.*

Note that setting $\mathscr{S} = [\mathbb{N}]^k$, this gives Ramsey theorem.

Proof. It is sufficient to prove the result for $n = 2$. Define

$$S_0^* = \{X \in [\mathbb{N}]^\omega \colon (\exists\, s \in S_0)\; s \sqsubset X\}.$$

Notice that S_0^* is open in the Vietoris topology (actually, even in the usual topology). By Theorem 1.7, there is $A \in [\mathbb{N}]^\omega$ such that either $[A]^\omega \cap S_0^* = \emptyset$ or $[A]^\omega \subseteq S_0^*$. In the former case we are done, so assume $[A]^\omega \subseteq S_0^*$.

Fix $t \in \mathscr{S} \cap [A]^{<\omega}$ and let

$$A_t = t \cup (A \setminus \{0, 1, \ldots, \max(t)\}).$$

Then $t \sqsubset A_t$ and $A_t \subseteq A$, therefore $A_t \in S_0^*$. Thus, there is $s \in S_0$ such that $s \sqsubset A_t$. Now either $s \sqsubset t$ or $t \sqsubset s$. Recall that $\mathscr{S}$ is thin, therefore $s = t$. This shows that $[A]^{<\omega} \cap \mathscr{S} \subseteq S_0$. $\square$

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