Perfect cliques and $G_δ$ colorings of Polish spaces

Wiesław Kubis

Ben-Gurion University of the Negev, Beer-Sheva
Israel

and

University of Silesia, Katowice
Poland

E-mail: kubis@math.bgu.ac.il

July, 2001

Abstract

A coloring of a set $X$ is any subset $C$ of $[X]^N$, where $N > 1$ is a natural number. We give some sufficient conditions for the existence of a perfect $C$-homogeneous set, in case where $C$ is $G_δ$ and $X$ is a Polish space. In particular, we show that it is sufficient that there exist $C$-homogeneous sets of arbitrarily large countable Cantor-Bendixson rank. We apply our methods to show that an analytic subset of the plane contains a perfect 3-clique if it contains any uncountable $k$-clique, where $k$ is a natural number or $ℵ_0$ (a set $K$ is a $k$-clique in $X$ if the convex hull of any of its $k$-element subsets is not contained in $X$).

2000 AMS Subject Classification: Primary: 52A37, 54H05, Secondary: 03E02, 52A10.

Keywords: Open ($G_δ$) coloring, perfect homogeneous set, clique.

1 Introduction

For a set $X$ and natural number $N$, $[X]^N$ denotes the collection of all $N$-element subsets of $X$. A (two-color) coloring of $X$ is (represented by) a set $C ⊆ [X]^N$. We identify $[X]^N$ with a suitable subspace of the product $X^N$. We are interested in the following problem: find sufficient conditions for the existence of a perfect $C$-homogeneous set $P ⊆ X$, where $X$ is a Polish space and $C ⊆ [X]^N$ is open (or more generally $G_δ$). A natural example for this problem is the following: let $X ⊆ ℝ^N$ be closed and $C = \{ s ∈ [X]^k : \text{conv } s ⊆ X \}$. Then $C$ is open and a $C$-homogeneous set is called a $k$-clique in $X$. It is known (see [3]) that there exists a closed set $X ⊆ ℝ^2$ such that $X$ is not a countable union of convex sets but every $k$-clique in $X$ is countable for every $k < ω$. On the other hand, it is proved in [3] that if a closed set $X ⊆ ℝ^2$ contains an uncountable $k$-clique for some $k$ then it contains a perfect 3-clique. We prove that if $C$ is a $G_δ$ coloring of a Polish space and there are no perfect $C$-homogeneous sets, then there is a countable ordinal $γ$ such that the Cantor-Bendixson rank of every $C$-homogeneous set is $< γ$. In the context of cliques, this strengthens the result of Kojman [2].
(see Theorem 3.1(a) below). From our result it follows that if $C$ is a $G_δ$ coloring of an analytic space then either there exists a perfect $C$-homogeneous set or all $C$-homogeneous sets are countable. This is not true for $F_σ$ colorings: a result of Shelah [4] states that consistently there exist $F_σ$ 2-colorings with uncountable but not perfect homogeneous sets. Concerning cliques, we investigate analytic subsets of the plane. We prove that if an analytic set $X \subseteq \mathbb{R}^2$ contains an uncountable $\aleph_0$-clique then $X$ contains also a perfect 3-clique.

1.1 Notation

Any subset of $|X|^N$ is called a coloring (or an $N$-coloring) of $X$. We write $\neg C$ instead of $|X|^N \setminus C$. A set $S \subseteq X$ is $C$-homogeneous if $|A|^N \subseteq C$. We identify $|X|^N$ with the subspace of $X^N$ consisting of all $N$-tuples $(x_0, \ldots , x_{N-1})$ with $x_i \neq x_j$ for $i \neq j$. Thus we may consider topological properties of colorings. If $f : X \to Y$ is a function then we write $f[S]$ for the image of a set $S \subseteq X$ and $f(s)$ for the value at a point $s \in X$. By a perfect set we mean a compact, nonempty, topological space with no isolated points.

2 On colorings

First we recall a simple result on open 2-colorings of analytic spaces (see Todorčević-Farah’s book [5, p. 81]). We present a proof for completeness.

Proposition 2.1. Let $X$ be an analytic space and let $C \subseteq |X|^2$ be open. Then either there exists a perfect $C$-homogeneous set or else $X$ is a countable union of $\neg C$-homogeneous sets, i.e. $X = \bigcup_{n\in\omega} A_n$ where $|A_n|^2 \cap C = \emptyset$ for every $n \in \omega$.

Proof. Let $f : \omega^\omega \to X$ be continuous and onto $X$. Define

$$C' = \{ s \in [\omega^\omega]^2 : f[s] \in C \}.$$

Note that if $\{x, y\} \in C'$ then $f(x) \neq f(y)$. Now observe that if $\omega^\omega$ is a union of countably many $\neg C'$-homogeneous sets, then the same holds for $X$. Also, if $P$ is a compact, perfect, $C'$-homogeneous subset of $\omega^\omega$ then $f \upharpoonright P$ is one-to-one and hence $f[P]$ is a perfect $C$-homogeneous set. Thus we may assume that $X = \omega^\omega$ and that $X$ cannot be covered by countably many $\neg C$-homogeneous sets.

Let $V$ consist of all $x \in \omega^\omega$ such that some neighborhood of $x$ is a countable union of $\neg C$-homogeneous sets. By assumption, it follows that $V \neq \omega^\omega$. Let $B = \omega^\omega \setminus V$. Now we are working in $B$: construct a tree $T = \{ u_s : s \in 2^{<\omega} \}$ of open subsets of $B$ such that $T$ defines a Cantor set and $\{ x, y \} \in C$ whenever $x \in u_s$, $y \in u_t$ and $s, t \in 2^k$ are distinct, $k < \omega$. Coming to split $u_s$, where $s \in 2^k$, we first find a pair $\{ x, y \} \in [u_s]^2 \cap C$ (this is possible since $u_s$ is not $\neg C$-homogeneous). Next, using the fact that $C$ is open, enlarge $x, y$ to open sets $u_s^{-0}, u_s^{-1}$, preserving $C$-homogeneity. The perfect set obtained from $T$ is evidently $C$-homogeneous.

The above result is no longer valid when we replace the word "open" with "closed", see [5, p. 83]. Also, the above proposition cannot be strengthened for colorings of triples: there exists a clopen 3-coloring of $2^\omega$ such that there are no uncountable homogeneous sets neither of this
color nor of its complement, see Blass’ example [1]. In this example, the Cantor-Bendixson rank of any homogeneous set is at most 1. Below we show that in this situation, there always exists a countable ordinal which bounds the Cantor-Bendixson ranks of all homogeneous sets. In fact this is true for \( G_\delta \) colorings.

For a topological space \( Y \) and an ordinal \( \alpha \) we denote by \( Y^{(\alpha)} \) the \( \alpha \)-derivative of \( Y \); the Cantor-Bendixson rank of \( Y \) is the minimal ordinal \( \gamma \) such that \( Y^{(\gamma+1)} \) is empty.

**Theorem 2.2.** Let \( C \) be a \( G_\delta \) \( N \)-coloring of a Polish space \( X \). If for every countable ordinal \( \gamma \) there exists a \( C \)-homogeneous set of the Cantor-Bendixson rank \( \geq \gamma \) then \( X \) contains a perfect \( C \)-homogeneous set.

**Proof.** Fix a countable base \( B \) in \( X \) and fix a complete metric on \( X \). Let \( C = \bigcap_{n \in \omega} C_n \), where each \( C_n \) is open and \( C_{n+1} \subseteq C_n \). We will construct a tree of open sets \( T = \{ u_s : s \in 2^{<\omega} \} \) with the following properties:

1. \( \text{cl } u_{s^{-1}} \subseteq u_s \), \( \text{cl } u_s \cap \text{cl } u_t = \emptyset \) if \( s, t \) are incompatible and \( \text{diam}(u_s) < 2^{-\text{length}(s)} \);

2. if \( k < \omega \) and \( s_0, \ldots, s_{N-1} \in 2^k \) are pairwise distinct then
   \[
   \{ x_0, \ldots, x_{N-1} \} \in C_k
   \]
   whenever \( x_i \in u_{s_i}, i < N \);

3. if \( k < \omega \) then for each \( \gamma < \omega_1 \) there exists a \( C \)-homogeneous set \( P = P_{k,\gamma} \) such that \( P^{(\gamma)} \cap u_s \neq \emptyset \) for each \( s \in 2^k \).

We start with \( u_0 = X \). Suppose that \( u_s \) has been defined for all \( s \in 2^{\leq k} \). Fix \( \gamma < \omega_1 \) and consider \( P = P_{k,\gamma+1} \), as in (iii). Then for each \( s \in 2^k \) the set \( P^{(\gamma)} \cap u_s \) is infinite. Fix \( S \subseteq P^{(\gamma)} \) such that \( |S \cap u_s| = 2 \) for each \( s \in 2^k \). Next, enlarge each \( x \in S \cap u_s \) to a small open set \( v_x \in B \), contained in \( u_s \), such that \( \{ y_0, \ldots, y_{N-1} \} \in C_{k+1} \) whenever \( y_t \) are taken from pairwise distinct \( v_x \)'s. This is possible, because \( C_{k+1} \) is open. Let \( \varphi(\gamma) = \{ v_x : x \in S \} \). This defines a mapping \( \varphi : \omega_1 \to [B]^{<\omega} \). As \( B \) is countable, there is unbounded \( F \subseteq \omega_1 \) such that \( \varphi \restriction F \) is constant, say \( \{ v_{s^{-1}} : s \in 2^k, i < 2 \} \), where \( v_{s^{-1}} \subseteq u_s \). Set \( u_{s^{-1}} = v_{s^{-1}} \). Observe that (i) holds if we let \( v_x \)'s to be small enough. Also (ii) holds, by the definition of \( v_x \)'s. Finally, (iii) holds, because \( P_{k,\gamma+1}^{(\gamma)} \cap u_t \neq \emptyset \) for \( t \in 2^{k+1} \) whenever \( \gamma \in F \). By (ii) the perfect set obtained from this construction is \( C \)-homogeneous.

Using the above theorem and arguments from the proof of Proposition 2.1 we obtain the following (see Shelah [4, Remark 1.14]):

**Corollary 2.3.** Let \( 1 \leq N < \omega \) and let \( C \) be a \( G_\delta \) \( N \)-coloring of an analytic space \( X \). If there exists an uncountable \( C \)-homogeneous set then there exists also a perfect one.
3 Applications to convexity

Let $X \subseteq E$, where $E$ is a real vector space. A subset $K$ of $X$ is a $k$-clique ($k$ can be a cardinal or just a natural number, we will use this notion for $k < \omega$ and $k = \aleph_0$) if $\text{conv } S \nsubseteq X$ whenever $S \in [K]^k$. If $E$ is finite-dimensional and $k > \dim E$ then we can define the notion of a strong $k$-clique replacing $\text{conv } S$ by $\text{int } \text{conv } S$ in the definition. A finite set $S \subseteq X$ is (strongly) defected in $X$ if $\text{conv } S \nsubseteq X$ (int $\text{conv } S \nsubseteq X$). It is clear that the relation of strong defectedness is open and defectedness is open provided that $X$ is closed.

Applying the results of the previous section we get the following:

**Theorem 3.1.** (a) Let $X$ be a closed set in a Polish linear space and let $N < \omega$. If $X$ does not contain a perfect $N$-clique then all $N$-cliques in $X$ are countable. Moreover, there exists an ordinal $\gamma < \omega_1$ which bounds the Cantor-Bendixson ranks of all $N$-cliques in $X$.

(b) Let $X$ be an analytic subset of $\mathbb{R}^m$. If $m < N < \omega$ and $X$ contains an uncountable strong $N$-clique then $X$ contains also a perfect one.

Theorem 3.1(a) was proved, under the stronger assumption that $X$ is a countable union of convex sets, by Kojman in [2].

In [3] we proved, in particular, that in a closed planar set either all cliques are countable or there exists a perfect 3-clique. Here we prove the same for analytic sets, namely:

**Theorem 3.2.** Let $X \subseteq \mathbb{R}^2$ be analytic. If $X$ contains an uncountable $\aleph_0$-clique then $X$ contains a perfect 3-clique.

**Proof.** Fix a continuous function $f: \omega^\omega \to X$ onto $X$ and fix an uncountable $\aleph_0$-clique $K \subseteq X$. We may assume that every line contains only countably many points of $L$: otherwise, for some line $L$, $X \cap L$ contains an uncountable $\aleph_0$-clique, so it contains a perfect 2-clique (Proposition 2.1), which is also a 3-clique in $X$. Fix uncountable $K' \subseteq \omega^\omega$ such that $f \upharpoonright K'$ is a bijection onto $K$.

A finite collection $\{u_0, \ldots, u_{k-1}\}$ of open subsets of $\omega^\omega$ will be called relevant if each $u_i$ contains uncountably many points of $K'$, $\text{cl } u_i \cap \text{cl } u_j = \emptyset$ whenever $i < j < k$ and

$$\text{int } \text{conv } \{f(x_0), f(x_1), f(x_2)\} \nsubseteq X$$

whenever $x_0, x_1, x_2$ are taken from pairwise distinct $u_i$’s. To find a perfect 3-clique in $X$, it suffices to construct a perfect tree of open sets in $\omega^\omega$ with relevant levels. If $P$ is a perfect set obtained from such a tree then $f \upharpoonright P$ is one-to-one and $f[P]$ is a perfect strong 3-clique.

Suppose that we have a relevant collection $\{u_0, \ldots, u_k\}$. We have to show that it is possible to split each $u_i$ to obtain again a relevant collection. We will split $u_k$. Let $L = K' \cap u_k$ and pick $y_i \in u_i$ for $i < k$. Define $c_i: [L]^2 \to 2$ by letting $c_i(x_0, x_1) = 1$ if $\text{conv } \{f(x_0), f(x_1), f(y_i)\} \nsubseteq X$. Observe that there are no infinite $c_i$-homogeneous sets of color 0: if $S \subseteq L$ is infinite then, by Carathéodory’s theorem, there is $s \in [S]^3$ such that $f[s]$ is defected in $X$ (because $f[S]$ is defected) and hence for some $x_0, x_1 \in s$ we have $\text{conv } \{f(x_0), f(x_1), f(y_i)\} \nsubseteq X$, because $\text{conv } T \subseteq \bigcup_{x,y \in T} \text{conv } \{x, y, p\}$ for $T \subseteq \mathbb{R}^2$, $p \in \mathbb{R}^2$. Using $k$ times the theorem of Dushnik-Miller we obtain uncountable $L' \subseteq L$ which is $c_i$-homogeneous of color 1 for $i < k$. Shrinking $L'$ we may assume that each nonempty open subset of $L'$ is uncountable. Now choose disjoint
open sets \( v_0, v_1 \) with \( \text{cl} v_j \subseteq u_k \) and \( v_j \cap L' \neq \emptyset \) for \( j < 2 \). To finish the proof we need the following geometric property of the plane:

**Claim 3.3.** Let \( A, B \subseteq X \subseteq \mathbb{R}^2 \) and \( c \in \mathbb{R}^2 \) be such that \( A, B \) are uncountable, each line contains countably many points of \( A \cup B \) and \( \text{conv}\{a, b, c\} \nsubseteq X \) whenever \( a \in A, b \in B \). Then there are \( a_0 \in A, b_0 \in B \) such that \( \text{int} \ \text{conv}\{a_0, b_0, c\} \nsubseteq X \).

**Proof.** Suppose this is not true. Observe that, replacing \( a \) with \( b \) the right side of \( a \) and not in \( [b, c] \). Now, if some vertical line contains two elements of \( A \) then we are done: we take \( a_0 \in A \) such that some \( a_1 \in A \) is below \( a_0 \), then the relative interiors of segments \( [b_0, a_1], [c, a_1] \) are contained in the interior of \( \text{conv}\{a_0, b_0, c\} \).

Assume that each vertical line contains at most one element of \( A \). As \( A \) is uncountable, there is \( a_1 \in A \) such that arbitrarily close to \( a_1 \) there are uncountably many points both on the left and the right side of \( a_1 \). Suppose now that e.g. \( \{b_0, a_1\} \) is defected in \( X \). As \( [b_0, a_1] \) contains only countably many points of \( A \), we can find \( a_2 \in A \) which is close enough to \( a_1 \), on the left side of \( a_1 \) and not in \( [b_0, a_1] \). If \( a_2 \) is below \( [b_0, a_1] \) then we can set \( a_0 = a_1 \), otherwise we can set \( a_0 = a_2 \).

Let \( i = 0 \). Using Claim 3.3 for \( A = f[v_0 \cap L'], B = f[v_1 \cap L'] \) and \( c = f(y_i) \) we get \( x_j \in v_j \) such that \( \text{int} \ \text{conv}\{f(x_0), f(x_1), f(y_i)\} \nsubseteq X \). By continuity, shrink \( v_0, v_1 \) and enlarge \( y_i \) to an open set \( u'_i \subseteq u_i \) such that each triple selected from \( f[v_0] \times f[v_1] \times f[u'_i] \) is (strongly) defected in \( X \). Repeat the same argument for each \( i < k \), obtaining a relevant collection \( \{u'_0, \ldots, u'_{k-1}, v'_0, v'_1\} \) which realizes the splitting of \( u_k \). This completes the proof.

Actually, we have proved that if an analytic planar set \( X \) contains any uncountable \( \aleph_0 \)-clique then either \( X \) contains a perfect strong 3-clique or else, \( X \cap L \) contains a perfect 2-clique for some line \( L \).

**References**


