

Metric categories and Fraïssé limits

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Continuous Logic and Functional Analysis

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Outline

- 1 Introduction
- 2 Classical framework
- 3 Almost homogeneous objects
- 4 Metric-enriched categories
 - Approximate back-and-forth argument
- 5 Application: The pseudo-arc
- 6 Metric categories
- 7 Application: The Gurariï space
- 8 The end

Fraïssé-Jónsson theory

Given a class \mathfrak{K} of “small” objects, we are asking for a “large” object, reachable from \mathfrak{K} , that is universal for \mathfrak{K} and homogeneous with respect to \mathfrak{K} -substructures.

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- Droste & Göbel 1989: Category-theoretic approach
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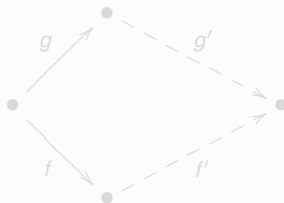
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Typical assumptions:

We work in a category \mathfrak{K} of “small” objects, satisfying the following conditions:

- 1 \mathfrak{K} has the **Joint Embedding Property**, that is, for each \mathfrak{K} -objects a, b there is a \mathfrak{K} -object c satisfying $\mathfrak{K}(a, c) \neq \emptyset \neq \mathfrak{K}(b, c)$.
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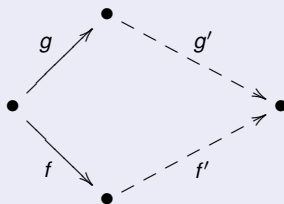


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Almost homogeneous objects

Motivation:

Theorem (Gurariĭ 1966)

There exists a separable Banach space \mathbb{G} satisfying the following condition.

Given finite dimensional spaces $E \subseteq F$, given an isometric embedding $f: E \rightarrow \mathbb{G}$, for every $\varepsilon > 0$ there exists an extension $g: F \rightarrow \mathbb{G}$ of f such that $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$.

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The Gurariĭ space is unique up to a linear isometry.

Solecki & K. 2012: Elementary proof of the uniqueness of \mathbb{G} .

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Another motivation: The pseudo-arc

A **continuum** is a nonempty connected compact metric space.

Theorem

There exists a unique, up to a homeomorphism, snake-like continuum \mathbb{P} with the following property:

- (\star) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1: \mathbb{P} \rightarrow \mathbb{I}$, there exists a homeomorphism $h: \mathbb{P} \rightarrow \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

Furthermore, \mathbb{P} is hereditarily indecomposable and maps onto every snake-like continuum.

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Metric-enriched categories

Categories enriched over metric spaces

Main definition

Let \mathcal{M}_\leq denote the category of metric spaces with non-expansive maps. A category \mathcal{K} is **enriched over** \mathcal{M}_\leq if

- 1 For every \mathcal{K} -objects a, b the hom-set $\mathcal{K}(a, b)$ has a metric ϱ .
- 2 Given compatible \mathcal{K} -arrows f_0, f_1, g, h , the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leq \varrho(f_0, f_1) \quad \text{and} \quad \varrho(f_0 \circ h, f_1 \circ h) \leq \varrho(f_0, f_1).$$

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Basic example:

$\mathfrak{K} := \mathfrak{M}_1$ (metric spaces with 1-Lipschitz mappings).

Joint Embedding and Amalgamation

- 1 \mathfrak{K} has the **Joint Embedding Property (JEP)** if for every \mathfrak{K} -objects a, b there exists a \mathfrak{K} -object d such that

$$\mathfrak{K}(a, d) \neq \emptyset \quad \text{and} \quad \mathfrak{K}(b, d) \neq \emptyset.$$

- 2 \mathfrak{K} has the **Amalgamation Property (AP)** if given $\varepsilon > 0$, given \mathfrak{K} -arrows $f: c \rightarrow a$, $g: c \rightarrow b$ there exist \mathfrak{K} -arrows $f': a \rightarrow w$ and $g': b \rightarrow w$ such that $\varrho(f' \circ f, g' \circ g) < \varepsilon$, that is, the diagram

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Separability

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We say that $\mathcal{F} \subseteq \mathcal{K}$ is **dominating** if

- 1 For every \mathcal{K} -object x there is an \mathcal{F} -object a such that $\mathcal{K}(x, a) \neq \emptyset$.
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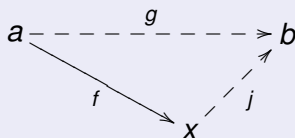
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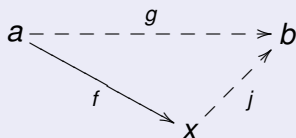
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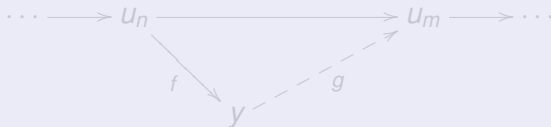
Fraïssé sequences

Definition

Let \mathfrak{K} be a metric-enriched category. A **Fraïssé sequence** in \mathfrak{K} is a sequence $\vec{u}: \omega \rightarrow \mathfrak{K}$ satisfying:

- (U) For every \mathfrak{K} -object x there is n such that $\mathfrak{K}(x, u_n) \neq \emptyset$.
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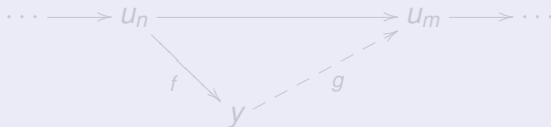
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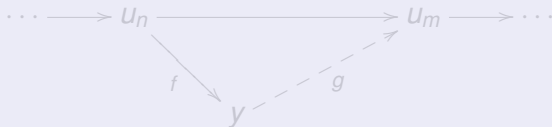
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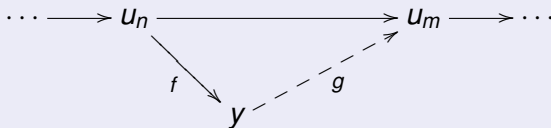
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Theorem

For a metric-enriched category \mathfrak{K} the following conditions are equivalent:

- 1 \mathfrak{K} is separable and has both the Joint Embedding Property and the Amalgamation Property.*
- 2 \mathfrak{K} has a Fraïssé sequence.*

From now on, we fix a separable metric-enriched category \mathfrak{K} with JEP and AP.

We assume that $\sigma\mathfrak{K} \supseteq \mathfrak{K}$ is a category satisfying

- 1 For every $\sigma\mathfrak{K}$ -object X there is a sequence \vec{x} in \mathfrak{K} such that $X = \lim \vec{x}$.
- 2 If $X = \lim \vec{x}$, $Y = \lim \vec{y}$, then $\sigma\mathfrak{K}$ -arrows $F: X \rightarrow Y$ correspond to **approximate sequences** of \mathfrak{K} -arrows:

$$\begin{array}{ccccccc}
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where

$$(\forall \varepsilon > 0)(\exists n_0)(\forall m > n > n_0) \varrho\left(y_{\varphi(n)}^{\varphi(n+1)} \circ f_n, f_m \circ x_n^m\right) < \varepsilon.$$

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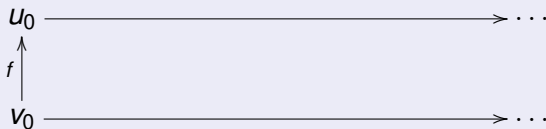
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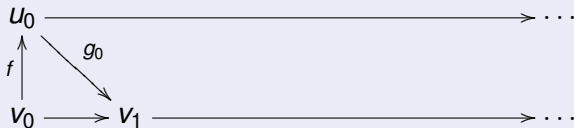
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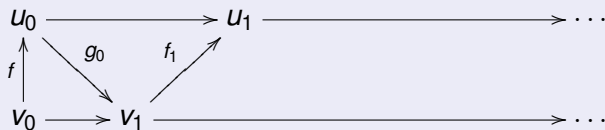
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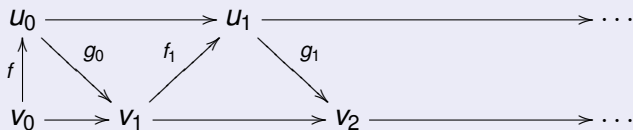
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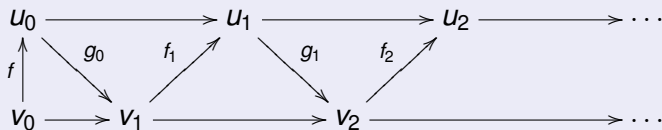
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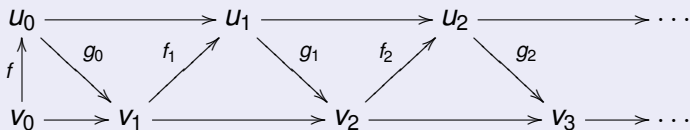
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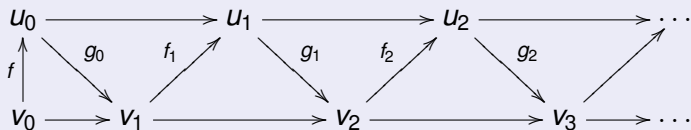
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Call a $\sigma\mathfrak{K}$ -object U the **Fraïssé limit** of \mathfrak{K} if $U = \lim \vec{u}$ for a Fraïssé sequence \vec{u} in \mathfrak{K} .

Theorem

The Fraïssé limit U is unique. Moreover, U is almost \mathfrak{K} -homogeneous, that is, given $\varepsilon > 0$, given \mathfrak{K} -objects a, b , given $i: a \rightarrow U, j: b \rightarrow U$, given a \mathfrak{K} -arrow $h: a \rightarrow b$, there exists an automorphism $H: U \rightarrow U$ for which the square

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Universality

Theorem

Let U be the Fraïssé limit of \mathfrak{K} . Then for every $\sigma\mathfrak{K}$ -object X there is a $\sigma\mathfrak{K}$ -arrow

$$F: X \rightarrow U.$$

Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

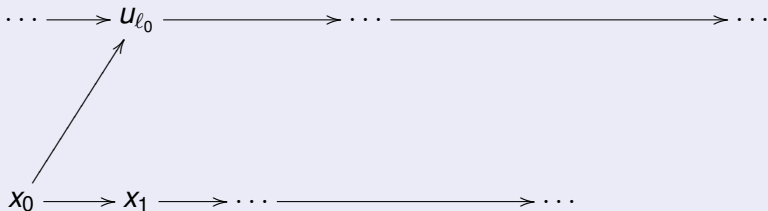
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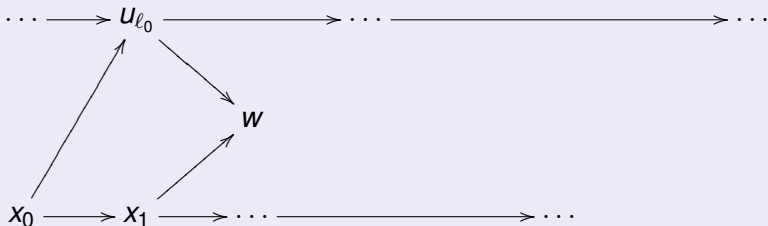
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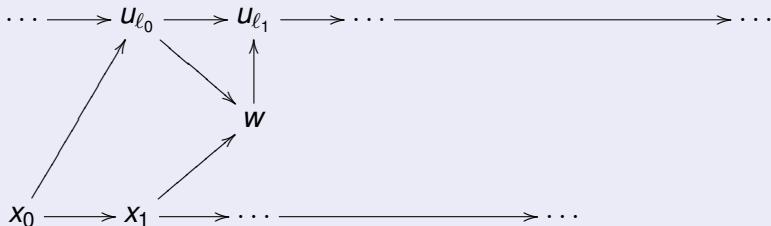
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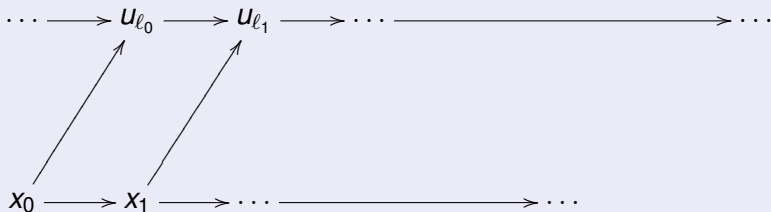
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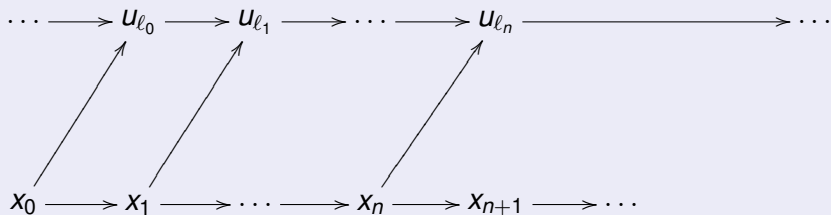
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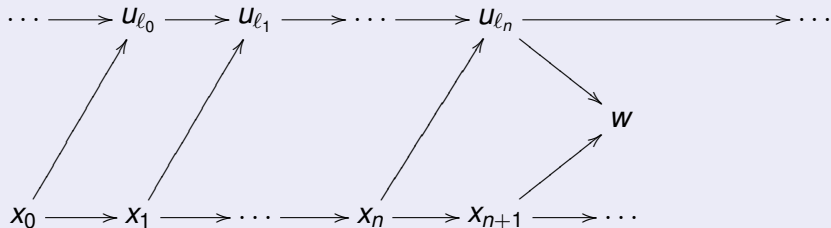
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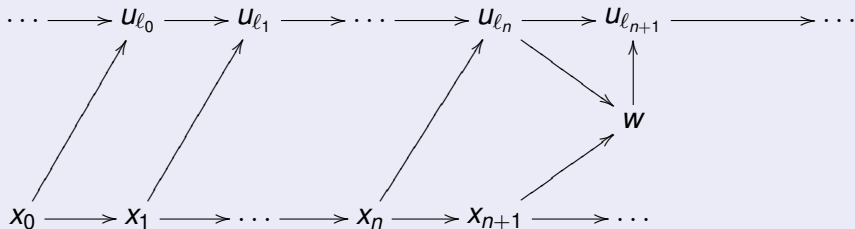
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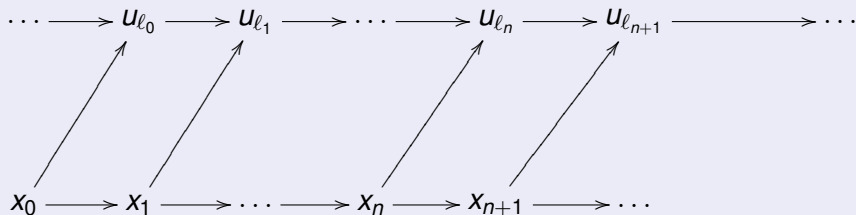
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Application: The pseudo-arc

The category:

Let \mathfrak{K} be the category whose objects are metric spaces $\langle \mathbb{I}, d \rangle$, where d is a metric on $\mathbb{I} := [0, 1]$ satisfying $d \leq 1$. The arrows of \mathfrak{K} are non-expansive quotients.

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Mountain Climbing Theorem

Let $f, g: \mathbb{I} \rightarrow \mathbb{I}$ be two quotient maps that are not locally constant. Then there exist quotient maps $f', g': \mathbb{I} \rightarrow \mathbb{I}$ for which the square

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Claim

Given continuous maps $f_i: \langle X, d \rangle \rightarrow \langle Y_i, d_i \rangle$ ($i < 2$), the formula

$$d'(s, t) = \max \left\{ d(s, t), f_0(s, t), f_1(s, t) \right\}$$

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Snake-like continua

Definition

A continuum K is **snake-like** if $K = \varprojlim \vec{s}$, where \vec{s} is an inverse sequence of quotients of the unit interval onto itself.

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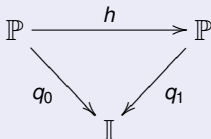
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There exists a unique up to isometry snake-like continuum $\langle \mathbb{P}, e \rangle$ of diameter 1, with the following properties:

- 1 Every snake-like continuum of diameter ≤ 1 is a non-expansive quotient of \mathbb{P} .
- 2 Given $\varepsilon > 0$, given non-expansive quotients $q_0, q_1 : \mathbb{P} \rightarrow \mathbb{I}$, where \mathbb{I} is endowed with some compatible metric ϱ , there exists an isometry $h : \mathbb{P} \rightarrow \mathbb{P}$ such that $\varrho(q_0(t), q_1(h(t))) < \varepsilon$ for every $t \in \mathbb{P}$.



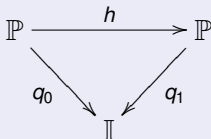
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Metric categories

Let \mathfrak{K} be a metric-enriched category.

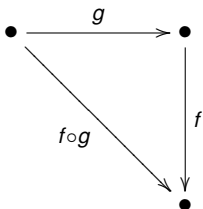
A *metric* on \mathfrak{K} is a function $\mu: \mathfrak{K} \rightarrow [0, +\infty]$ satisfying the following conditions:

(M₁) $\mu(\text{id}_x) = 0$ for every object x .

(M₂) $\mu(f \circ g) \leq \mu(f) + \mu(g)$.

(M₃) $\mu(g) \leq \mu(f \circ g) + \mu(f)$.

(M₄) μ is uniformly continuous with respect to ϱ .



A prototype example:

$\mathfrak{K} := \mathfrak{M}_{\leq}$, the category of metric spaces with non-expansive maps, with

$$\mu(f) = \log \text{Lip}(f^{-1}).$$

Define

$$\ker \mathfrak{K} := \{f \in \mathfrak{K} : \mu(f) = 0\}.$$

Elements of $\ker \mathfrak{K}$ will be called **0-arrows**, **0-isomorphisms**, etc.

Remark

If $h \in \ker \mathfrak{K}$ is an isomorphism then $\mu(h^{-1}) = 0$.

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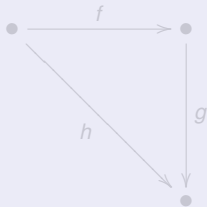
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The Law of Return

Given $\varepsilon > 0$, there is $\eta > 0$, such that whenever f is a \mathfrak{K} -arrow with $\mu(f) < \eta$, then there exist \mathfrak{K} -arrows $g, h \in \ker \mathfrak{K}$ such that

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holds.



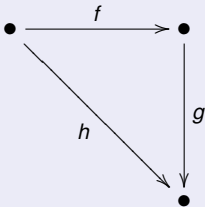
We shall say briefly that \mathfrak{K} has the **LRP**.

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Existence

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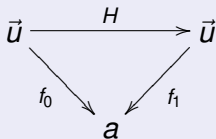
Assume \mathfrak{K} is a metric category with JEP, AP and LRP, such that $\ker \mathfrak{K}$ is separable. Then \mathfrak{K} has a unique, up to a 0-isomorphism, Fraïssé sequence.

Universality and almost homogeneity

Theorem

Let \mathfrak{K} be as above, and let \vec{u} be a Fraïssé sequence in \mathfrak{K} . Then:

- 1 For every sequence \vec{x} in \mathfrak{K} there exists a $\sigma\mathfrak{K}$ -arrow $F: \vec{x} \rightarrow \vec{u}$ with $\mu(F) = 0$.
- 2 Given $\varepsilon > 0$, given a \mathfrak{K} -object a , given $\sigma\mathfrak{K}$ -arrows $f_0, f_1: a \rightarrow \vec{u}$ with $\mu(f_0) = 0 = \mu(f_1)$, there exists a 0-automorphism $H: \vec{u} \rightarrow \vec{u}$ for which the square



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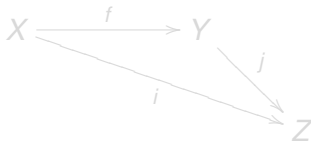
Application: The Gurariĭ space

Lemma (Solecki & K. 2011)

Let $f: X \rightarrow Y$ be an ε -isometric embedding of finite-dimensional Banach spaces. Then there exist a finite-dimensional Banach space Z and isometric embeddings $i: X \rightarrow Z, j: Y \rightarrow Z$ such that

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In fact: $Z = X \oplus Y$ with a suitable norm.



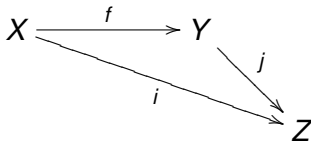
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Proposition

A separable Banach space E is Gurariĭ iff there is a chain $\{E_n\}_{n \in \omega}$ of finite-dimensional subspaces of E with $\bigcup_{n \in \omega} E_n$ dense in E , such that the following condition is satisfied:

- Given $n \in \omega$, given $\varepsilon > 0$, for every isometric embedding $f: E_n \rightarrow Y$ with Y finite-dimensional, there are $m > n$ and an ε -isometric embedding $g: Y \rightarrow E_m$ satisfying

$$\|x - g(f(x))\| < \varepsilon$$

for every $x \in E_n$.

Claim

The metric category of finite-dimensional Banach spaces with linear embeddings of norm ≤ 1 has separable kernel. Furthermore, it has the AP, JEP and the LRP.

Corollary (Gurariĭ 1966, Lusky 1976, Solecki & K. 2012)

The Gurariĭ space \mathbb{G} is unique up to isometry, is isometrically universal for separable Banach spaces and satisfies the following condition:

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Another application

Theorem (Garbulińska & K. 2012)

There exists a linear operator $u_\infty: \mathbb{G} \rightarrow \mathbb{G}$ with $\|u_\infty\| = 1$ and with the following property:

- Given a linear operator $T: X \rightarrow Y$ between separable Banach spaces with $\|T\| \leq 1$, there exist isometric embeddings $i: X \rightarrow \mathbb{G}$ and $j: Y \rightarrow \mathbb{G}$ for which the following diagram commutes.

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