# Metric categories and Fraïssé limits

Wiesław Kubiś

Czech Academy of Sciences (CZECH REPUBLIC)
and
Jan Kochanowski University, Kielce (POLAND)
http://www.ujk.edu.pl/~wkubis/

Continuous Logic and Functional Analysis
Lyon, 16 – 20 July 2012

# Outline

# Fraïssé-Jónsson theory

Given a class $\mathfrak{K}$ of "small" objects, we are asking for a "large" object, reachable from $\mathfrak{K}$, that is universal for $\mathfrak{K}$ and homogeneous with respect to $\mathfrak{K}$-substructures.

- Fraïssé 1954; Jónsson 1960
- Droste & Göbel 1989: Category-theoretic approach
- Irwin & Solecki 2006: Reversed Fraïssé limits

# Fraïssé-Jónsson theory

Given a class $\mathfrak{K}$ of "small" objects, we are asking for a "large" object, reachable from $\mathfrak{K}$, that is universal for $\mathfrak{K}$ and homogeneous with respect to $\mathfrak{K}$-substructures.

- Fraïssé 1954; Jónsson 1960
- Droste & Göbel 1989: Category-theoretic approach
- Irwin & Solecki 2006: Reversed Fraïssé limits

# Fraïssé-Jónsson theory

Given a class $\mathfrak{K}$ of "small" objects, we are asking for a "large" object, reachable from $\mathfrak{K}$, that is universal for $\mathfrak{K}$ and homogeneous with respect to $\mathfrak{K}$-substructures.

- Fraïssé 1954; Jónsson 1960
- Droste & Göbel 1989: Category-theoretic approach
- Irwin & Solecki 2006: Reversed Fraïssé limits

# Fraïssé-Jónsson theory

Given a class $\mathfrak{K}$ of “small” objects, we are asking for a “large” object, reachable from $\mathfrak{K}$, that is universal for $\mathfrak{K}$ and homogeneous with respect to $\mathfrak{K}$-substructures.

- Fraïssé 1954; Jónsson 1960
- Droste & Göbel 1989: Category-theoretic approach
- Irwin & Solecki 2006: Reversed Fraïssé limits

# Typical assumptions:

We work in a category $\mathfrak{K}$ of “small” objects, satisfying the following conditions:

1. $\mathfrak{K}$ has the Joint Embedding Property, that is, for each $\mathfrak{K}$-objects $a, b$ there is a $\mathfrak{K}$-object $c$ satisfying $\mathfrak{K}(a, c) \neq \emptyset \neq \mathfrak{K}(b, c)$.
2. $\mathfrak{K}$ has the Amalgamation Property.

## The Amalgamation Property:

$$\begin{tikzcd} & \bullet \arrow[dr, dashed, "g'"] & \\ \bullet \arrow[ur, "g"] \arrow[dr, "f"'] & & \bullet \\ & \bullet \arrow[ur, dashed, "f'"'] & \end{tikzcd}$$

# Typical assumptions:

We work in a category $\mathfrak{K}$ of "small" objects, satisfying the following conditions:

1. $\mathfrak{K}$ has the Joint Embedding Property, that is, for each $\mathfrak{K}$-objects $a, b$ there is a $\mathfrak{K}$-object $c$ satisfying $\mathfrak{K}(a, c) \neq \emptyset \neq \mathfrak{K}(b, c)$.
2. $\mathfrak{K}$ has the Amalgamation Property.

## The Amalgamation Property:

$$\begin{tikzcd}
& \bullet \arrow[dr, dashed, "g'"] & \\
\bullet \arrow[ur, "g"] \arrow[dr, "f"'] & & \bullet \\
& \bullet \arrow[ur, dashed, "f'"'] &
\end{tikzcd}$$

Almost homogeneous objects

# Motivation:

**Theorem (Gurariĭ 1966)**

*There exists a separable Banach space $\mathbb{G}$ satisfying the following condition.*

- *Given finite dimensional spaces $E \subseteq F$, given an isometric embedding $f\colon E \to \mathbb{G}$, for every $\varepsilon > 0$ there exists an extension $g\colon F \to \mathbb{G}$ of $f$ such that $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$.*

**Theorem (Lusky 1976)**

*The Gurariĭ space is unique up to a linear isometry.*

Solecki & K. 2012: Elementary proof of the uniqueness of $\mathbb{G}$.

# Motivation:

**Theorem (Gurariĭ 1966)**

*There exists a separable Banach space $\mathbb{G}$ satisfying the following condition.*

- *Given finite dimensional spaces $E \subseteq F$, given an isometric embedding $f\colon E \to \mathbb{G}$, for every $\varepsilon > 0$ there exists an extension $g\colon F \to \mathbb{G}$ of $f$ such that $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$.*

**Theorem (Lusky 1976)**

*The Gurariĭ space is unique up to a linear isometry.*

Solecki & K. 2012: Elementary proof of the uniqueness of $\mathbb{G}$.

# Motivation:

## Theorem (Gurariĭ 1966)

*There exists a separable Banach space $\mathbb{G}$ satisfying the following condition.*

- *Given finite dimensional spaces $E \subseteq F$, given an isometric embedding $f\colon E \to \mathbb{G}$, for every $\varepsilon > 0$ there exists an extension $g\colon F \to \mathbb{G}$ of $f$ such that $\|g\| \cdot \|g^{-1}\| < 1 + \varepsilon$.*

## Theorem (Lusky 1976)

*The Gurariĭ space is unique up to a linear isometry.*

Solecki & K. 2012: Elementary proof of the uniqueness of $\mathbb{G}$.

# Another motivation: The pseudo-arc

A continuum is a nonempty connected compact metric space.

**Theorem**

*There exists a unique, up to a homeomorphism, snake-like continuum $\mathbb{P}$ with the following property:*

($\star$) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1 \colon \mathbb{P} \to \mathbb{I}$, there exists a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

*Furthermore, $\mathbb{P}$ is hereditarily indecomposable and maps onto every snake-like continuum.*

**History:**

- Bing 1948-1959, Moise 1949, Mioduszewski 1962
- Irwin & Solecki 2006

# Another motivation: The pseudo-arc

A continuum is a nonempty connected compact metric space.

**Theorem**

*There exists a unique, up to a homeomorphism, snake-like continuum $\mathbb{P}$ with the following property:*

($\star$) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1 : \mathbb{P} \to \mathbb{I}$, there exists a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

*Furthermore, $\mathbb{P}$ is hereditarily indecomposable and maps onto every snake-like continuum.*

**History:**

- Bing 1948-1959, Moise 1949, Mioduszewski 1962
- Irwin & Solecki 2006

# Another motivation: The pseudo-arc

A continuum is a nonempty connected compact metric space.

**Theorem**

*There exists a unique, up to a homeomorphism, snake-like continuum $\mathbb{P}$ with the following property:*

($\star$) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1 : \mathbb{P} \to \mathbb{I}$, there exists a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

*Furthermore, $\mathbb{P}$ is hereditarily indecomposable and maps onto every snake-like continuum.*

**History:**

- Bing 1948-1959, Moise 1949, Mioduszewski 1962
- Irwin & Solecki 2006

# Another motivation: The pseudo-arc

A continuum is a nonempty connected compact metric space.

**Theorem**

*There exists a unique, up to a homeomorphism, snake-like continuum $\mathbb{P}$ with the following property:*

($\star$) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1 : \mathbb{P} \to \mathbb{I}$, there exists a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

*Furthermore, $\mathbb{P}$ is hereditarily indecomposable and maps onto every snake-like continuum.*

**History:**

- Bing 1948-1959, Moise 1949, Mioduszewski 1962
- Irwin & Solecki 2006

# Another motivation: The pseudo-arc

A continuum is a nonempty connected compact metric space.

**Theorem**

*There exists a unique, up to a homeomorphism, snake-like continuum $\mathbb{P}$ with the following property:*

($\star$) *Given $\varepsilon > 0$, given quotient maps $f_0, f_1 \colon \mathbb{P} \to \mathbb{I}$, there exists a homeomorphism $h \colon \mathbb{P} \to \mathbb{P}$ such that $|f_0 - f_1 \circ h| < \varepsilon$.*

*Furthermore, $\mathbb{P}$ is hereditarily indecomposable and maps onto every snake-like continuum.*

**History:**

- Bing 1948-1959, Moise 1949, Mioduszewski 1962
- Irwin & Solecki 2006

Metric-enriched categories

# Categories enriched over metric spaces

## Main definition

Let $\mathfrak{Ms}$ denote the category of metric spaces with non-expansive maps. A category $\mathfrak{K}$ is enriched over $\mathfrak{Ms}$ if

1. For every $\mathfrak{K}$-objects $a, b$ the hom-set $\mathfrak{K}(a, b)$ has a metric $\varrho$.
2. Given compatible $\mathfrak{K}$-arrows $f_0, f_1, g, h$, the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leqslant \varrho(f_0, f_1) \quad \text{and} \quad \varrho(f_0 \circ h, f_1 \circ h) \leqslant \varrho(f_0, f_1).$$

We say briefly that $\mathfrak{K}$ is a metric-enriched category.

**History:**

- Eilenberg & Steenrod 1952
- Mioduszewski 1963 (in the context of compact metric spaces)

# Categories enriched over metric spaces

## Main definition

Let $\mathfrak{Ms}$ denote the category of metric spaces with non-expansive maps. A category $\mathfrak{K}$ is enriched over $\mathfrak{Ms}$ if

1. For every $\mathfrak{K}$-objects $a, b$ the hom-set $\mathfrak{K}(a, b)$ has a metric $\varrho$.
2. Given compatible $\mathfrak{K}$-arrows $f_0, f_1, g, h$, the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leqslant \varrho(f_0, f_1) \quad \text{and} \quad \varrho(f_0 \circ h, f_1 \circ h) \leqslant \varrho(f_0, f_1).$$

We say briefly that $\mathfrak{K}$ is a metric-enriched category.

**History:**

- Eilenberg & Steenrod 1952
- Mioduszewski 1963 (in the context of compact metric spaces)

# Categories enriched over metric spaces

## Main definition

Let $\mathfrak{Ms}$ denote the category of metric spaces with non-expansive maps. A category $\mathfrak{K}$ is enriched over $\mathfrak{Ms}$ if

1. For every $\mathfrak{K}$-objects $a, b$ the hom-set $\mathfrak{K}(a, b)$ has a metric $\varrho$.
2. Given compatible $\mathfrak{K}$-arrows $f_0, f_1, g, h$, the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leqslant \varrho(f_0, f_1) \quad \text{and} \quad \varrho(f_0 \circ h, f_1 \circ h) \leqslant \varrho(f_0, f_1).$$

We say briefly that $\mathfrak{K}$ is a metric-enriched category.

**History:**

- Eilenberg & Steenrod 1952
- Mioduszewski 1963 (in the context of compact metric spaces)

# Categories enriched over metric spaces

## Main definition

Let $\mathfrak{Ms}$ denote the category of metric spaces with non-expansive maps. A category $\mathfrak{K}$ is enriched over $\mathfrak{Ms}$ if

1. For every $\mathfrak{K}$-objects $a, b$ the hom-set $\mathfrak{K}(a, b)$ has a metric $\varrho$.
2. Given compatible $\mathfrak{K}$-arrows $f_0, f_1, g, h$, the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leqslant \varrho(f_0, f_1) \qquad \text{and} \qquad \varrho(f_0 \circ h, f_1 \circ h) \leqslant \varrho(f_0, f_1).$$

We say briefly that $\mathfrak{K}$ is a metric-enriched category.

**History:**

- Eilenberg & Steenrod 1952
- Mioduszewski 1963 (in the context of compact metric spaces)

# Categories enriched over metric spaces

## Main definition

Let $\mathfrak{Ms}$ denote the category of metric spaces with non-expansive maps. A category $\mathfrak{K}$ is enriched over $\mathfrak{Ms}$ if

1. For every $\mathfrak{K}$-objects $a, b$ the hom-set $\mathfrak{K}(a, b)$ has a metric $\varrho$.
2. Given compatible $\mathfrak{K}$-arrows $f_0, f_1, g, h$, the following inequalities hold:

$$\varrho(g \circ f_0, g \circ f_1) \leqslant \varrho(f_0, f_1) \qquad \text{and} \qquad \varrho(f_0 \circ h, f_1 \circ h) \leqslant \varrho(f_0, f_1).$$

We say briefly that $\mathfrak{K}$ is a metric-enriched category.

**History:**

- Eilenberg & Steenrod 1952
- Mioduszewski 1963 (in the context of compact metric spaces)

**Basic example:**

$\mathfrak{K} := \mathfrak{Ms}$ (metric spaces with 1-Lipschitz mappings).

# Joint Embedding and Amalgamation

1. $\mathfrak{K}$ has the **Joint Embedding Property** (JEP) if for every $\mathfrak{K}$-objects $a, b$ there exists a $\mathfrak{K}$-object $d$ such that

$$\mathfrak{K}(a,d) \neq \emptyset \qquad \text{and} \qquad \mathfrak{K}(b,d) \neq \emptyset.$$

2. $\mathfrak{K}$ has the **Amalgamation Property** (AP) if given $\varepsilon > 0$, given $\mathfrak{K}$-arrows $f\colon c \to a$, $g\colon c \to b$ there exist $\mathfrak{K}$-arrows $f'\colon a \to w$ and $g'\colon b \to w$ such that $\varrho(f' \circ f, g' \circ g) < \varepsilon$, that is, the diagram

$$\begin{tikzcd} b \arrow[r, dashed, "g'"] & w \\ c \arrow[u, "g"] \arrow[r, "f"'] & a \arrow[u, dashed, "f'"'] \end{tikzcd}$$

is $\varepsilon$-commutative.

# Joint Embedding and Amalgamation

1. $\mathfrak{K}$ has the Joint Embedding Property (JEP) if for every $\mathfrak{K}$-objects $a, b$ there exists a $\mathfrak{K}$-object $d$ such that

$$\mathfrak{K}(a, d) \neq \emptyset \qquad \text{and} \qquad \mathfrak{K}(b, d) \neq \emptyset.$$

2. $\mathfrak{K}$ has the Amalgamation Property (AP) if given $\varepsilon > 0$, given $\mathfrak{K}$-arrows $f \colon c \to a$, $g \colon c \to b$ there exist $\mathfrak{K}$-arrows $f' \colon a \to w$ and $g' \colon b \to w$ such that $\varrho(f' \circ f, g' \circ g) < \varepsilon$, that is, the diagram

$$\begin{tikzcd} b \arrow[r, dashed, "g'"] & w \\ c \arrow[u, "g"] \arrow[r, "f"'] & a \arrow[u, dashed, "f'"'] \end{tikzcd}$$

is $\varepsilon$-commutative.

# Separability

## Definition

We say that $\mathscr{F} \subseteq \mathfrak{K}$ is dominating if

1. For every $\mathfrak{K}$-object $x$ there is an $\mathscr{F}$-object $a$ such that $\mathfrak{K}(x, a) \neq \emptyset$.
2. Given $\varepsilon > 0$, given an $\mathscr{F}$-object $a$, a $\mathfrak{K}$-arrow $f\colon a \to x$, there exist an $\mathscr{F}$-arrow $g\colon a \to b$ and a $\mathfrak{K}$-arrow $j\colon x \to b$ such that $\varrho(g, j \circ f) < \varepsilon$, that is, the triangle

$$\begin{tikzcd} a \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & b \\ & x \arrow[ur, dashed, "j"'] & \end{tikzcd}$$

is $\varepsilon$-commutative.

$\mathfrak{K}$ is separable if it has a countable dominating subcategory.

# Separability

## Definition

We say that $\mathscr{F} \subseteq \mathfrak{K}$ is dominating if

1. For every $\mathfrak{K}$-object $x$ there is an $\mathscr{F}$-object $a$ such that $\mathfrak{K}(x, a) \neq \emptyset$.
2. Given $\varepsilon > 0$, given an $\mathscr{F}$-object $a$, a $\mathfrak{K}$-arrow $f \colon a \to x$, there exist an $\mathscr{F}$-arrow $g \colon a \to b$ and a $\mathfrak{K}$-arrow $j \colon x \to b$ such that $\varrho(g, j \circ f) < \varepsilon$, that is, the triangle

$$\begin{tikzcd} a \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & b \\ & x \arrow[ur, dashed, "j"'] & \end{tikzcd}$$

is $\varepsilon$-commutative.

$\mathfrak{K}$ is separable if it has a countable dominating subcategory.

# Separability

## Definition

We say that $\mathscr{F} \subseteq \mathfrak{K}$ is dominating if

1. For every $\mathfrak{K}$-object $x$ there is an $\mathscr{F}$-object $a$ such that $\mathfrak{K}(x, a) \neq \emptyset$.
2. Given $\varepsilon > 0$, given an $\mathscr{F}$-object $a$, a $\mathfrak{K}$-arrow $f\colon a \to x$, there exist an $\mathscr{F}$-arrow $g\colon a \to b$ and a $\mathfrak{K}$-arrow $j\colon x \to b$ such that $\varrho(g, j \circ f) < \varepsilon$, that is, the triangle

$$\begin{tikzcd} a \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & b \\ & x \arrow[ur, dashed, "j"'] & \end{tikzcd}$$

is $\varepsilon$-commutative.

$\mathfrak{K}$ is separable if it has a countable dominating subcategory.

# Separability

## Definition

We say that $\mathscr{F} \subseteq \mathfrak{K}$ is dominating if

1. For every $\mathfrak{K}$-object $x$ there is an $\mathscr{F}$-object $a$ such that $\mathfrak{K}(x, a) \neq \emptyset$.
2. Given $\varepsilon > 0$, given an $\mathscr{F}$-object $a$, a $\mathfrak{K}$-arrow $f \colon a \to x$, there exist an $\mathscr{F}$-arrow $g \colon a \to b$ and a $\mathfrak{K}$-arrow $j \colon x \to b$ such that $\varrho(g, j \circ f) < \varepsilon$, that is, the triangle

$$\begin{tikzcd} a \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & b \\ & x \arrow[ur, dashed, "j"'] & \end{tikzcd}$$

is $\varepsilon$-commutative.

$\mathfrak{K}$ is separable if it has a countable dominating subcategory.

# Fraïssé sequences

## Definition

Let $\mathfrak{K}$ be a metric-enriched category. A Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u}\colon \omega \to \mathfrak{K}$ satisfying:

(U) For every $\mathfrak{K}$-object $x$ there is $n$ such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

(A) Given $\varepsilon > 0$, given $n \in \omega$, given $\mathfrak{K}$-arrow $f\colon u_n \to y$, there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that

$$\varrho(g \circ f, u_n^m) < \varepsilon.$$

$$\begin{tikzcd} \cdots \arrow[r] & u_n \arrow[rr] \arrow[dr, "f"'] & & u_m \arrow[r] & \cdots \\ & & y \arrow[ur, dashed, "g"'] & & \end{tikzcd}$$

# Fraïssé sequences

## Definition

Let $\mathfrak{K}$ be a metric-enriched category. A Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u} \colon \omega \to \mathfrak{K}$ satisfying:

(U) For every $\mathfrak{K}$-object $x$ there is $n$ such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

(A) Given $\varepsilon > 0$, given $n \in \omega$, given $\mathfrak{K}$-arrow $f \colon u_n \to y$, there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g \colon y \to u_m$ such that

$$\varrho(g \circ f, u_n^m) < \varepsilon.$$

$$\begin{tikzcd} \cdots \arrow[r] & u_n \arrow[rr] \arrow[dr, "f"'] & & u_m \arrow[r] & \cdots \\ & & y \arrow[ur, dashed, "g"'] & & \end{tikzcd}$$

# Fraïssé sequences

## Definition

Let $\mathfrak{K}$ be a metric-enriched category. A Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u}\colon \omega \to \mathfrak{K}$ satisfying:

(U) For every $\mathfrak{K}$-object $x$ there is $n$ such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

(A) Given $\varepsilon > 0$, given $n \in \omega$, given $\mathfrak{K}$-arrow $f\colon u_n \to y$, there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that

$$\varrho(g \circ f, u_n^m) < \varepsilon.$$

$$\begin{tikzcd}
\cdots \arrow[r] & u_n \arrow[rr] \arrow[dr, "f"'] & & u_m \arrow[r] & \cdots \\
& & y \arrow[ur, dashed, "g"'] & &
\end{tikzcd}$$

# Fraïssé sequences

## Definition

Let $\mathfrak{K}$ be a metric-enriched category. A Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u}\colon \omega \to \mathfrak{K}$ satisfying:

(U) For every $\mathfrak{K}$-object $x$ there is $n$ such that $\mathfrak{K}(x, u_n) \neq \emptyset$.

(A) Given $\varepsilon > 0$, given $n \in \omega$, given $\mathfrak{K}$-arrow $f\colon u_n \to y$, there exist $m \geqslant n$ and a $\mathfrak{K}$-arrow $g\colon y \to u_m$ such that

$$\varrho(g \circ f, u_n^m) < \varepsilon.$$

$$\begin{tikzcd}
\cdots \arrow[r] & u_n \arrow[rr] \arrow[dr, "f"'] & & u_m \arrow[r] & \cdots \\
& & y \arrow[ur, dashed, "g"'] & &
\end{tikzcd}$$

# Existence

**Theorem**

*For a metric-enriched category $\mathfrak{K}$ the following conditions are equivalent:*

1. *$\mathfrak{K}$ is separable and has both the Joint Embedding Property and the Amalgamation Property.*
2. *$\mathfrak{K}$ has a Fraïssé sequence.*

From now on, we fix a separable metric-enriched category $\mathfrak{K}$ with JEP and AP.

We assume that $\sigma\mathfrak{K} \supseteq \mathfrak{K}$ is a category satisfying

1. For every $\sigma\mathfrak{K}$-object $X$ there is a sequence $\vec{x}$ in $\mathfrak{K}$ such that $X = \lim \vec{x}$.
2. If $X = \lim \vec{x}$, $Y = \lim \vec{y}$, then $\sigma\mathfrak{K}$-arrows $F \colon X \to Y$ correspond to approximate sequences of $\mathfrak{K}$-arrows:

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d, "f_0"'] & x_1 \arrow[r] \arrow[d, "f_1"'] & x_2 \arrow[r] \arrow[d, "f_2"'] & x_3 \arrow[r] \arrow[d, "f_3"'] & \cdots \\
y_{\varphi(0)} \arrow[r] & y_{\varphi(1)} \arrow[r] & y_{\varphi(2)} \arrow[r] & y_{\varphi(3)} \arrow[r] & \cdots
\end{tikzcd}$$

where

$$(\forall\, \varepsilon > 0)(\exists\, n_0)(\forall\, m > n > n_0)\ \ \varrho\left(y_{\varphi(n)}^{\varphi(n+1)} \circ f_n, f_m \circ x_n^m\right) < \varepsilon.$$

From now on, we fix a separable metric-enriched category $\mathfrak{K}$ with JEP and AP.

We assume that $\sigma\mathfrak{K} \supseteq \mathfrak{K}$ is a category satisfying

1. For every $\sigma\mathfrak{K}$-object $X$ there is a sequence $\vec{x}$ in $\mathfrak{K}$ such that $X = \lim \vec{x}$.
2. If $X = \lim \vec{x}$, $Y = \lim \vec{y}$, then $\sigma\mathfrak{K}$-arrows $F \colon X \to Y$ correspond to approximate sequences of $\mathfrak{K}$-arrows:

$$\begin{tikzcd}
x_0 \arrow[r] \arrow[d, "f_0"'] & x_1 \arrow[r] \arrow[d, "f_1"'] & x_2 \arrow[r] \arrow[d, "f_2"'] & x_3 \arrow[r] \arrow[d, "f_3"'] & \cdots \\
y_{\varphi(0)} \arrow[r] & y_{\varphi(1)} \arrow[r] & y_{\varphi(2)} \arrow[r] & y_{\varphi(3)} \arrow[r] & \cdots
\end{tikzcd}$$

where

$$(\forall\, \varepsilon > 0)(\exists\, n_0)(\forall\, m > n > n_0)\ \ \varrho\left(y_{\varphi(n)}^{\varphi(n+1)} \circ f_n, f_m \circ x_n^m\right) < \varepsilon.$$

From now on, we fix a separable metric-enriched category $\mathfrak{K}$ with JEP and AP.

We assume that $\sigma\mathfrak{K} \supseteq \mathfrak{K}$ is a category satisfying

1. For every $\sigma\mathfrak{K}$-object $X$ there is a sequence $\vec{x}$ in $\mathfrak{K}$ such that $X = \lim \vec{x}$.
2. If $X = \lim \vec{x}$, $Y = \lim \vec{y}$, then $\sigma\mathfrak{K}$-arrows $F\colon X \to Y$ correspond to approximate sequences of $\mathfrak{K}$-arrows:

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d, "f_0"'] & x_1 \arrow[r] \arrow[d, "f_1"'] & x_2 \arrow[r] \arrow[d, "f_2"'] & x_3 \arrow[r] \arrow[d, "f_3"'] & \cdots \\ y_{\varphi(0)} \arrow[r] & y_{\varphi(1)} \arrow[r] & y_{\varphi(2)} \arrow[r] & y_{\varphi(3)} \arrow[r] & \cdots \end{tikzcd}$$

where

$$(\forall\, \varepsilon > 0)(\exists\, n_0)(\forall\, m > n > n_0)\ \ \varrho\Big(y_{\varphi(n)}^{\varphi(n+1)} \circ f_n, f_m \circ x_n^m\Big) < \varepsilon.$$

# Approximate back-and-forth argument

**Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.**

$$
\begin{tikzcd}
u_0 \arrow[r] & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & \cdots
\end{tikzcd}
$$

# Approximate back-and-forth argument

## Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[r] & \cdots
\end{tikzcd}$$

# Approximate back-and-forth argument

Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & u_1 \arrow[r] & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[ur, "f_1"] \arrow[rr] & & \cdots
\end{tikzcd}
$$

# Approximate back-and-forth argument

**Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.**

$$
\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & u_1 \arrow[rr] \arrow[dr, "g_1"] & & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[ur, "f_1"] \arrow[rr] & & v_2 \arrow[r] & \cdots
\end{tikzcd}
$$

# Approximate back-and-forth argument

**Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.**

$$
\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & u_1 \arrow[rr] \arrow[dr, "g_1"] & & u_2 \arrow[rr] & & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[ur, "f_1"] \arrow[rr] & & v_2 \arrow[ur, "f_2"] \arrow[rrr] & & & \cdots
\end{tikzcd}
$$

# Approximate back-and-forth argument

Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & u_1 \arrow[rr] \arrow[dr, "g_1"] & & u_2 \arrow[rr] \arrow[dr, "g_2"] & & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[rr] \arrow[ur, "f_1"] & & v_2 \arrow[rr] \arrow[ur, "f_2"] & & v_3 \arrow[r] & \cdots
\end{tikzcd}$$

# Approximate back-and-forth argument

**Assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.**

$$\begin{tikzcd}
u_0 \arrow[rr] \arrow[dr, "g_0"] & & u_1 \arrow[rr] \arrow[dr, "g_1"] & & u_2 \arrow[rr] \arrow[dr, "g_2"] & & \cdots \\
v_0 \arrow[u, "f"] \arrow[r] & v_1 \arrow[ur, "f_1"] \arrow[rr] & & v_2 \arrow[ur, "f_2"] \arrow[rr] & & v_3 \arrow[ur] \arrow[r] & \cdots
\end{tikzcd}$$

Call a $\sigma\mathfrak{K}$-object $U$ the Fraïssé limit of $\mathfrak{K}$ if $U = \lim \vec{u}$ for a Fraïssé sequence $\vec{u}$ in $\mathfrak{K}$.

**Theorem**

*The Fraïssé limit $U$ is unique. Moreover, $U$ is almost $\mathfrak{K}$-homogeneous, that is, given $\varepsilon > 0$, given $\mathfrak{K}$-objects $a, b$, given $i \colon a \to U$, $j \colon b \to U$, given a $\mathfrak{K}$-arrow $h \colon a \to b$, there exists an automorphism $H \colon U \to U$ for which the square*

$$\begin{tikzcd} U \arrow[r, "H"] & U \\ a \arrow[u, "i"] \arrow[r, "h"'] & b \arrow[u, "j"'] \end{tikzcd}$$

*is $\varepsilon$-commutative.*

Call a $\sigma\mathfrak{K}$-object $U$ the Fraïssé limit of $\mathfrak{K}$ if $U = \lim \vec{u}$ for a Fraïssé sequence $\vec{u}$ in $\mathfrak{K}$.

## Theorem

*The Fraïssé limit $U$ is unique. Moreover, $U$ is almost $\mathfrak{K}$-homogeneous, that is, given $\varepsilon > 0$, given $\mathfrak{K}$-objects $a, b$, given $i \colon a \to U$, $j \colon b \to U$, given a $\mathfrak{K}$-arrow $h \colon a \to b$, there exists an automorphism $H \colon U \to U$ for which the square*

$$\begin{tikzcd} U \arrow[r, "H"] & U \\ a \arrow[u, "i"] \arrow[r, "h"'] & b \arrow[u, "j"'] \end{tikzcd}$$

*is $\varepsilon$-commutative.*

# Universality

**Theorem**

*Let $U$ be the Fraïssé limit of $\mathfrak{K}$. Then for every $\sigma\mathfrak{K}$-object $X$ there is a $\sigma\mathfrak{K}$-arrow*

$$F \colon X \to U.$$

**Proof.**

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[rr] & & \cdots \arrow[rrr] & & & \cdots \\
\\
x_0 \arrow[r] & x_1 \arrow[r] & \cdots \arrow[rr] & & \cdots
\end{tikzcd}
$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & \cdots \arrow[rr] & & \cdots \\
& & & & \\
x_0 \arrow[uur] \arrow[r] & x_1 \arrow[r] & \cdots \arrow[r] & \cdots &
\end{tikzcd}$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] \arrow[dr] & \cdots \arrow[r] & \cdots \\
& & w & \\
x_0 \arrow[uur] \arrow[r] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & \cdots
\end{tikzcd}
$$

□

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] \arrow[dr] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & \cdots \\
 & & w \arrow[u] & & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & \cdots &
\end{tikzcd}$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[rr] & & \cdots \\
x_0 \arrow[r] \arrow[ur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[rr] & & \cdots &
\end{tikzcd}
$$

$\square$

**Proof.**

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[rr] & & \cdots \\
x_0 \arrow[r] \arrow[ur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[ur] & x_{n+1} \arrow[r] & \cdots &
\end{tikzcd}$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[rr] \arrow[dr] & & \cdots \\
 & & & & & w & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[uur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[uur] & x_{n+1} \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[r] \arrow[dr] & u_{\ell_{n+1}} \arrow[r] & \cdots \\
 & & & & & w \arrow[u] & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[uur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[uur] & x_{n+1} \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\square$

## Proof.

Let $U = \lim \vec{u}$, $X = \lim \vec{x}$, and assume $\sum_{n=0}^{\infty} \varepsilon_n < +\infty$.

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[r] & u_{\ell_{n+1}} \arrow[r] & \cdots \\
x_0 \arrow[r] \arrow[ur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[ur] & x_{n+1} \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}
$$

$\square$

# Application: The pseudo-arc

## The category:

Let $\mathfrak{K}$ be the category whose objects are metric spaces $\langle \mathbb{I}, d \rangle$, where $d$ is a metric on $\mathbb{I} := [0,1]$ satisfying $d \leqslant 1$. The arrows of $\mathfrak{K}$ are non-expansive quotients.

**Note:** The arrows are reversed!

## Lemma

*$\mathfrak{K}$ is a separable metric-enriched category with the JEP and AP.*

# Application: The pseudo-arc

## The category:

Let $\mathfrak{K}$ be the category whose objects are metric spaces $\langle \mathbb{I}, d \rangle$, where $d$ is a metric on $\mathbb{I} := [0,1]$ satisfying $d \leqslant 1$. The arrows of $\mathfrak{K}$ are non-expansive quotients.

**Note:** The arrows are reversed!

## Lemma

*$\mathfrak{K}$ is a separable metric-enriched category with the JEP and AP.*

## Mountain Climbing Theorem

Let $f, g \colon \mathbb{I} \to \mathbb{I}$ be two quotient maps that are not locally constant. Then there exist quotient maps $f', g' \colon \mathbb{I} \to \mathbb{I}$ for which the square

$$\begin{tikzcd} \mathbb{I} \arrow[d, "g"'] & \mathbb{I} \arrow[l, "g'"'] \arrow[d, "f'"] \\ \mathbb{I} & \mathbb{I} \arrow[l, "f"'] \end{tikzcd}$$

is commutative.

## Claim

*Given continuous maps $f_i \colon \langle X, d\rangle \to \langle Y_i, d_i\rangle$ $(i < 2)$, the formula*

$$d'(s,t) = \max\Big\{d(s,t), f_0(s,t), f_1(s,t)\Big\}$$

*defines a metric on $X$ for which both $f_0, f_1$ are non-expansive.*

## Mountain Climbing Theorem

Let $f, g\colon \mathbb{I} \to \mathbb{I}$ be two quotient maps that are not locally constant. Then there exist quotient maps $f', g'\colon \mathbb{I} \to \mathbb{I}$ for which the square

$$\begin{tikzcd} \mathbb{I} \arrow[d, "g"'] & \mathbb{I} \arrow[l, "g'"'] \arrow[d, "f'"] \\ \mathbb{I} & \mathbb{I} \arrow[l, "f"'] \end{tikzcd}$$

is commutative.

## Claim

*Given continuous maps $f_i\colon \langle X, d\rangle \to \langle Y_i, d_i\rangle$ ($i < 2$), the formula*

$$d'(s,t) = \max\Big\{d(s,t), f_0(s,t), f_1(s,t)\Big\}$$

*defines a metric on $X$ for which both $f_0, f_1$ are non-expansive.*

# Snake-like continua

## Definition

A continuum $K$ is snake-like if $K = \varprojlim \vec{s}$, where $\vec{s}$ is an inverse sequence of quotients of the unit interval onto itself.

## Observation

A sequence $\vec{s}$ as above "comes" from a sequence in $\mathfrak{K}$.

# Snake-like continua

**Definition**

A continuum $K$ is snake-like if $K = \varprojlim \vec{s}$, where $\vec{s}$ is an inverse sequence of quotients of the unit interval onto itself.

**Observation**

A sequence $\vec{s}$ as above "comes" from a sequence in $\mathfrak{K}$.

## Theorem

*There exists a unique up to isometry snake-like continuum $\langle \mathbb{P}, e \rangle$ of diameter 1, with the following properties:*

1. *Every snake-like continuum of diameter $\leqslant 1$ is a non-expansive quotient of $\mathbb{P}$.*
2. *Given $\varepsilon > 0$, given non-expansive quotients $q_0, q_1 : \mathbb{P} \to \mathbb{I}$, where $\mathbb{I}$ is endowed with some compatible metric $\varrho$, there exists an isometry $h \colon \mathbb{P} \to \mathbb{P}$ such that $\varrho(q_0(t), q_1(h(t))) < \varepsilon$ for every $t \in \mathbb{P}$.*

$$\begin{tikzcd} \mathbb{P} \arrow[rr, "h"] \arrow[dr, "q_0"'] & & \mathbb{P} \arrow[dl, "q_1"] \\ & \mathbb{I} & \end{tikzcd}$$

## Corollary

*$\mathbb{P}$ is the pseudo-arc.*

## Theorem

*There exists a unique up to isometry snake-like continuum* $\langle \mathbb{P}, e \rangle$ *of diameter* $1$, *with the following properties:*

1. *Every snake-like continuum of diameter* $\leqslant 1$ *is a non-expansive quotient of* $\mathbb{P}$.
2. *Given* $\varepsilon > 0$, *given non-expansive quotients* $q_0, q_1 : \mathbb{P} \to \mathbb{I}$, *where* $\mathbb{I}$ *is endowed with some compatible metric* $\varrho$, *there exists an isometry* $h \colon \mathbb{P} \to \mathbb{P}$ *such that* $\varrho(q_0(t), q_1(h(t))) < \varepsilon$ *for every* $t \in \mathbb{P}$.

$$\begin{tikzcd} \mathbb{P} \arrow[rr, "h"] \arrow[dr, "q_0"'] & & \mathbb{P} \arrow[dl, "q_1"] \\ & \mathbb{I} & \end{tikzcd}$$

## Corollary

$\mathbb{P}$ *is the pseudo-arc.*

Metric categories

Let $\mathfrak{K}$ be a metric-enriched category.
A *metric* on $\mathfrak{K}$ is a function $\mu\colon \mathfrak{K} \to [0, +\infty]$ satisfying the following conditions:

(M$_1$) $\mu(\mathrm{id}_x) = 0$ for every object $x$.

(M$_2$) $\mu(f \circ g) \leqslant \mu(f) + \mu(g)$.

(M$_3$) $\mu(g) \leqslant \mu(f \circ g) + \mu(f)$.

(M$_4$) $\mu$ is uniformly continuous with respect to $\varrho$.

$$\begin{tikzcd}
\bullet \arrow[r, "g"] \arrow[dr, "f\circ g"'] & \bullet \arrow[d, "f"] \\
& \bullet
\end{tikzcd}$$

## A prototype example:

$\mathfrak{K} := \mathfrak{Ms}$, the category of metric spaces with non-expansive maps, with

$$\mu(f) = \log \operatorname{Lip}(f^{-1}).$$

Define

$$\ker \mathfrak{K} := \{f \in \mathfrak{K} \colon \mu(f) = 0\}.$$

Elements of $\ker \mathfrak{K}$ will be called 0-arrows, 0-isomorphisms, etc.

**Remark**

If $h \in \ker \mathfrak{K}$ is an isomorphism then $\mu(h^{-1}) = 0$.

Define

$$\ker \mathfrak{K} := \{f \in \mathfrak{K} \colon \mu(f) = 0\}.$$

Elements of $\ker \mathfrak{K}$ will be called 0-arrows, 0-isomorphisms, etc.

**Remark**

If $h \in \ker \mathfrak{K}$ is an isomorphism then $\mu(h^{-1}) = 0$.

## The Law of Return

Given $\varepsilon > 0$, there is $\eta > 0$, such that whenever $f$ is a $\mathfrak{K}$-arrow with $\mu(f) < \eta$, then there exist $\mathfrak{K}$-arrows $g, h \in \ker \mathfrak{K}$ such that

$$\varrho(g \circ f, h) < \varepsilon$$

holds.

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "h"'] & \bullet \arrow[d, "g"] \\ & \bullet \end{tikzcd}$$

We shall say briefly that $\mathfrak{K}$ has the LRP.

## The Law of Return

Given $\varepsilon > 0$, there is $\eta > 0$, such that whenever $f$ is a $\mathfrak{K}$-arrow with $\mu(f) < \eta$, then there exist $\mathfrak{K}$-arrows $g, h \in \ker \mathfrak{K}$ such that

$$\varrho(g \circ f, h) < \varepsilon$$

holds.

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dr, "h"'] & \bullet \arrow[d, "g"] \\
& \bullet
\end{tikzcd}$$

We shall say briefly that $\mathfrak{K}$ has the LRP.

# Existence

**Theorem**

*Assume $\mathfrak{K}$ is a metric category with JEP, AP and LRP, such that $\ker \mathfrak{K}$ is separable. Then $\mathfrak{K}$ has a unique, up to a $0$-isomorphism, Fraïssé sequence.*

# Universality and almost homogeneity

## Theorem

*Let $\mathfrak{K}$ be as above, and let $\vec{u}$ be a Fraïssé sequence in $\mathfrak{K}$. Then:*

1. *For every sequence $\vec{x}$ in $\mathfrak{K}$ there exists a $\sigma\mathfrak{K}$-arrow $F\colon \vec{x} \to \vec{u}$ with $\mu(F) = 0$.*
2. *Given $\varepsilon > 0$, given a $\mathfrak{K}$-object $a$, given $\sigma\mathfrak{K}$-arrows $f_0, f_1 \colon a \to \vec{u}$ with $\mu(f_0) = 0 = \mu(f_1)$, there exists a $0$-automorphism $H\colon \vec{u} \to \vec{u}$ for which the square*

$$\begin{tikzcd} \vec{u} \arrow[rr, "H"] \arrow[dr, "f_0"'] & & \vec{u} \arrow[dl, "f_1"] \\ & a & \end{tikzcd}$$

*is $\varepsilon$-commutative.*

# Application: The Gurariĭ space

**Lemma (Solecki & K. 2011)**

*Let $f\colon X \to Y$ be an $\varepsilon$-isometric embedding of finite-dimensional Banach spaces. Then there exist a finite-dimensional Banach space $Z$ and isometric embeddings $i\colon X \to Z$, $j\colon Y \to Z$ such that*

$$\|j \circ f - i\| < \varepsilon.$$

*In fact: $Z = X \oplus Y$ with a suitable norm.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[drr, "i"'] & Y \arrow[dr, "j"] & \\ & & Z \end{tikzcd}$$

# Application: The Gurariĭ space

## Lemma (Solecki & K. 2011)

*Let $f\colon X \to Y$ be an $\varepsilon$-isometric embedding of finite-dimensional Banach spaces. Then there exist a finite-dimensional Banach space $Z$ and isometric embeddings $i\colon X \to Z$, $j\colon Y \to Z$ such that*

$$\|j \circ f - i\| < \varepsilon.$$

*In fact: $Z = X \oplus Y$ with a suitable norm.*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[drrr, "i"'] & & Y \arrow[dr, "j"] & \\ & & & Z \end{tikzcd}$$

## Proposition

*A separable Banach space $E$ is Gurariĭ iff there is a chain $\{E_n\}_{n\in\omega}$ of finite-dimensional subspaces of $E$ with $\bigcup_{n\in\omega} E_n$ dense in $E$, such that the following condition is satisfied:*

- *Given $n \in \omega$, given $\varepsilon > 0$, for every isometric embedding $f\colon E_n \to Y$ with $Y$ finite-dimensional, there are $m > n$ and an $\varepsilon$-isometric embedding $g\colon Y \to E_m$ satisfying*

  $$\|x - g(f(x))\| < \varepsilon$$

  *for every $x \in E_n$.*

### Claim

*The metric category of finite-dimensional Banach spaces with linear embeddings of norm $\leqslant 1$ has separable kernel. Furthermore, it has the AP, JEP and the LRP.*

### Corollary (Gurariĭ 1966, Lusky 1976, Solecki & K. 2012)

*The Gurariĭ space $\mathbb{G}$ is unique up to isometry, is isometrically universal for separable Banach spaces and satisfies the following condition:*

- *Given $\varepsilon > 0$, given an isometry $h \colon X \to Y$ between finite-dimensional subspaces of $\mathbb{G}$, there is a bijective isometry $H \colon \mathbb{G} \to \mathbb{G}$ such that*

$$\|H \restriction X - h\| < \varepsilon.$$

## Claim

*The metric category of finite-dimensional Banach spaces with linear embeddings of norm $\leqslant 1$ has separable kernel. Furthermore, it has the AP, JEP and the LRP.*

## Corollary (Gurariĭ 1966, Lusky 1976, Solecki & K. 2012)

*The Gurariĭ space $\mathbb{G}$ is unique up to isometry, is isometrically universal for separable Banach spaces and satisfies the following condition:*

- *Given $\varepsilon > 0$, given an isometry $h \colon X \to Y$ between finite-dimensional subspaces of $\mathbb{G}$, there is a bijective isometry $H \colon \mathbb{G} \to \mathbb{G}$ such that*

$$\|H \restriction X - h\| < \varepsilon.$$

# Another application

**Theorem (Garbulińska & K. 2012)**

*There exists a linear operator $u_\infty \colon \mathbb{G} \to \mathbb{G}$ with $\|u_\infty\| = 1$ and with the following property:*

- *Given a linear operator $T \colon X \to Y$ between separable Banach spaces with $\|T\| \leqslant 1$, there exist isometric embeddings $i \colon X \to \mathbb{G}$ and $j \colon Y \to \mathbb{G}$ for which the following diagram commutes.*

$$\begin{tikzcd} \mathbb{G} \arrow[r, "u_\infty"] & \mathbb{G} \\ X \arrow[u, "i"] \arrow[r, "T"'] & Y \arrow[u, "j"'] \end{tikzcd}$$

# Another application

**Theorem (Garbulińska & K. 2012)**

*There exists a linear operator $u_\infty \colon \mathbb{G} \to \mathbb{G}$ with $\|u_\infty\| = 1$ and with the following property:*

- *Given a linear operator $T \colon X \to Y$ between separable Banach spaces with $\|T\| \leqslant 1$, there exist isometric embeddings $i \colon X \to \mathbb{G}$ and $j \colon Y \to \mathbb{G}$ for which the following diagram commutes.*

$$\begin{tikzcd} \mathbb{G} \arrow[r, "u_\infty"] & \mathbb{G} \\ X \arrow[u, "i"] \arrow[r, "T"'] & Y \arrow[u, "j"'] \end{tikzcd}$$

THE END