

Fraïssé sequences

Category-theoretic approach

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Motivations

- Fraïssé-Jónsson theory of universal homogeneous structures (1953)
 - ▶ Cantor's back-and-forth method
 - ▶ Urysohn's universal metric space
- Work of Droste & Göbel (1989)
- Reversed Fraïssé limits: Irwin & Solecki (2005)
- Universal compact spaces “generated” by retractions (Valdivia compacta)



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Valdivia compacta

A **Valdivia compact** is a closed set $K \subseteq [0, 1]^\kappa$ such that

$$K = \text{cl}(K \cap \Sigma(\kappa)),$$

where $\Sigma(\kappa) = \{x \in [0, 1]^\kappa : |\{\alpha : x(\alpha) \neq 0\}| \leq \aleph_0\}$.

Theorem (H.Michalewski & W.K.)

A compact space K of weight \aleph_1 is Valdivia if and only if

$$K = \varprojlim \vec{s},$$

*where $\vec{s} = \langle K_\xi, r_\xi^\eta, \omega_1 \rangle$ is a continuous inverse sequence of metric compacta in which all bonding maps r_ξ^η are retractions.
(**retraction** = right-invertible map)*



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Linearly ordered Valdivia compacta

Example

The linearly ordered space $\omega_1 + 1$ is Valdivia compact.

Claim

Linearly ordered Valdivia compacta can have weight at most \aleph_1 .

Theorem (W.K.)

There exists a zero-dimensional linearly ordered Valdivia compact C_{ω_1} which maps increasingly onto any other nonempty linearly ordered Valdivia compact.



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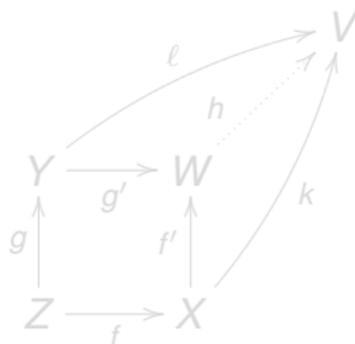
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Categories

Let \mathfrak{K} be a category. We say that \mathfrak{K} has the **amalgamation property** if for every arrows $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ there are arrows $f': X \rightarrow W$ and $g': Y \rightarrow W$ such that $f'f = g'g$. If moreover for every other pair of arrows $k: X \rightarrow V$ and $\ell: Y \rightarrow V$ with $kf = \ell g$ there exists a unique arrow $h: W \rightarrow V$ such that the diagram

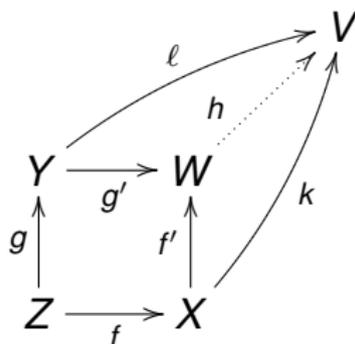


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Cofinality and homogeneity

- A family \mathcal{F} of objects of \mathfrak{K} is said to be **cofinal** in \mathfrak{K} if for every $x \in \mathfrak{K}$ there is $y \in \mathcal{F}$ such that $\mathfrak{K}(x, y) \neq \emptyset$.
- An object $u \in \mathfrak{K}$ is **cofinal** in \mathfrak{K} if for every $x \in \mathfrak{K}$ there is an arrow $f: x \rightarrow u$ in \mathfrak{K} .
- Let \mathcal{L} be a subcategory of \mathfrak{K} . An object $u \in \mathfrak{K}$ is **\mathcal{L} -homogeneous** if for every arrow $f: a \rightarrow b$ in \mathcal{L} and for every arrows $i: a \rightarrow u$, $j: b \rightarrow u$ in \mathfrak{K} there exists an isomorphism $h: u \rightarrow u$ such that the diagram

$$\begin{array}{ccc} u & \xrightarrow{h} & u \\ i \uparrow & & \uparrow j \\ a & \xrightarrow{f} & b \end{array}$$

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Sequences

By a **sequence** in a category \mathfrak{K} we mean a functor \vec{x} from an ordinal λ into \mathfrak{K} . A sequence \vec{x} of length λ can be described as a sequence $\{x_\alpha\}_{\alpha < \lambda}$ together with arrows $i_\alpha^\beta: x_\alpha \rightarrow x_\beta$ for $\alpha \leq \beta < \lambda$, such that

- 1 $i_\alpha^\alpha = \text{id}_{x_\alpha}$,
- 2 $\alpha < \beta < \gamma \implies i_\alpha^\gamma = i_\beta^\gamma i_\alpha^\beta$.

We shall write $\vec{x} = \langle x_\alpha, i_\alpha^\beta, \lambda \rangle$.

Let $\vec{x} = \langle x_\alpha, i_\alpha^\beta, \lambda \rangle$ and $\vec{y} = \langle y_\alpha, j_\alpha^\beta, \delta \rangle$ be sequences in \mathfrak{K} .

A **transformation** of \vec{x} into \vec{y} is a pair $\langle \varphi, \vec{f} \rangle$ such that

- 1 $\varphi: \lambda \rightarrow \delta$ is increasing;
- 2 $\vec{f} = \{f_\alpha\}_{\alpha < \lambda}$, where $f_\alpha: x_\alpha \rightarrow y_{\varphi(\alpha)}$;
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Arrows between sequences

- Let \vec{x}, \vec{y} be sequences in \mathfrak{K} and let $\langle \varphi, \vec{f} \rangle, \langle \psi, \vec{g} \rangle$ be transformations between them. We say that they are **equivalent** if all diagrams like

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are commutative.

- An **arrow of sequences** $\vec{x} \rightarrow \vec{y}$ is an equivalence class of this relation. We write $\vec{f}: \vec{x} \rightarrow \vec{y}$, having in mind the equivalence class of a transformation $\vec{f} = \{f_{\alpha}\}_{\alpha < \lambda}$.



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The κ -completion of a category

Let κ be a regular cardinal and let \mathcal{K} be a category.

We denote by $\mathfrak{S}_\kappa(\mathcal{K})$ the category of all sequences in \mathcal{K} of length $< \kappa$, with arrows of sequences defined above.

A category \mathcal{L} is κ -closed if sequences of length $< \kappa$ have colimits in \mathcal{L} .

Theorem

- 1 $\mathfrak{S}_\kappa(\mathcal{K})$ is a κ -closed category containing \mathcal{K} as a full subcategory.
- 2 For every κ -closed category \mathcal{L} , every covariant functor $F: \mathcal{K} \rightarrow \mathcal{L}$ has a unique extension $F': \mathfrak{S}_\kappa(\mathcal{K}) \rightarrow \mathcal{L}$ to a κ -continuous functor.

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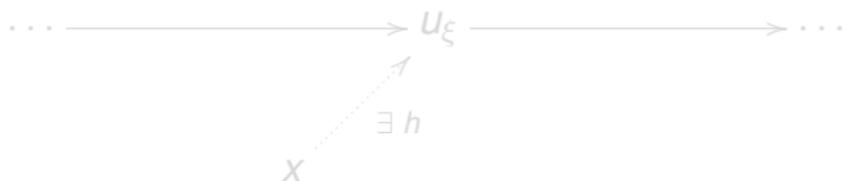
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Definition:

Let \mathfrak{K} be a fixed category. A κ -Fraïssé sequence in \mathfrak{K} is an inductive sequence $\vec{u} = \langle u_\xi, i_\xi^\eta, \kappa \rangle$ satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.



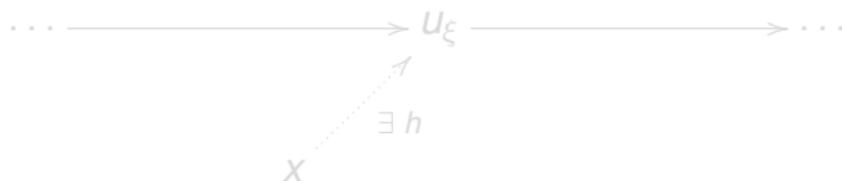
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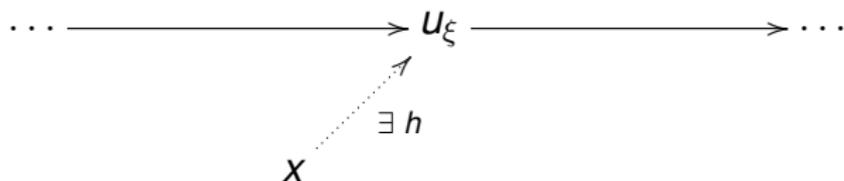
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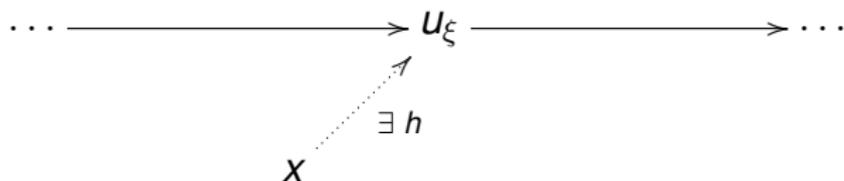
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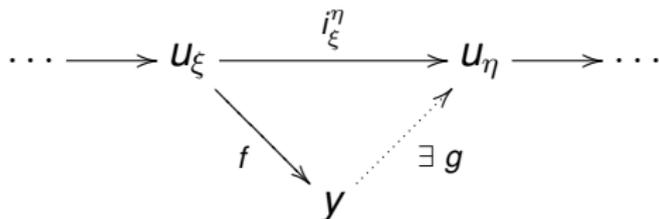
Definition:

Let \mathfrak{K} be a fixed category. A κ -**Fraïssé sequence** in \mathfrak{K} is an inductive sequence $\vec{u} = \langle u_\xi, i_\xi^\eta, \kappa \rangle$ satisfying the following conditions:

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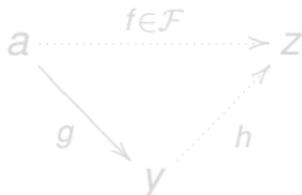
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Let \mathcal{F} be a set of arrows in \mathfrak{K} . Let $\text{Dom}(\mathcal{F}) = \{\text{dom}(f) : f \in \mathcal{F}\}$. We say that \mathcal{F} is **dominating** in \mathfrak{K} if the following conditions are satisfied:

(D1) For every $x \in \mathfrak{K}$ there exists $a \in \text{Dom}(\mathcal{F})$ such that $\mathfrak{K}(x, a) \neq \emptyset$.

$$x \cdots \cdots \cdots \rightarrow a$$

(D2) For every arrow $g: a \rightarrow y$ in \mathfrak{K} with $a \in \text{Dom}(\mathcal{F})$ there exist arrows f, h in \mathfrak{K} such that $f \in \mathcal{F}$ and $f = hg$.



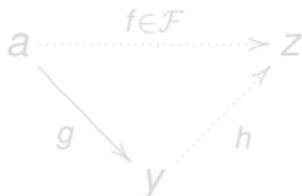
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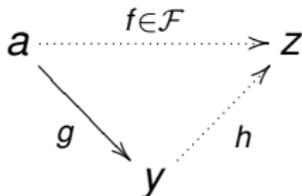
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The existence

A category \mathfrak{K} is κ -bounded if for every sequence $\vec{u} \in \mathfrak{S}_\kappa(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $F: \vec{u} \rightarrow a$.

Theorem

Let $\kappa > 1$ be a regular cardinal and let \mathfrak{K} be a κ -bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathcal{F} \subseteq \text{Arr}(\mathfrak{K})$ is dominating in \mathfrak{K} and $|\mathcal{F}| \leq \kappa$.

Then there exists a Fraïssé sequence $\vec{u} = \langle u_\xi, i_\xi^\eta, \kappa \rangle$ in \mathfrak{K} such that $\{u_\alpha: \alpha < \kappa\} \subseteq \text{Dom}(\mathcal{F})$.



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Linearly ordered Valdivia compacta revisited

Denote by $\mathfrak{L}\mathfrak{V}_0$ the category whose objects are nonempty 0-dimensional metric linearly ordered compacta and arrows are increasing retractions.

Claim

The dual category $\mathfrak{L}\mathfrak{V}_0^{\leftarrow}$ has the amalgamation property.

Let C be the Cantor set with the standard linear order. Let $\pi: C \rightarrow C$ be the (unique) increasing surjection such that

- $\pi^{-1}(p)$ is order isomorphic to the Cantor set whenever $p \in C$ is rational,
- $|\pi^{-1}(p)| = 1$ whenever $p \in C$ is not rational.

Claim

$\{\pi\}$ is dominating in $\mathfrak{L}\mathfrak{V}_0^{\leftarrow}$.

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In fact:

Theorem

Let $\vec{c} = \langle C_\xi, \pi_\xi^\eta, \omega_1 \rangle$ be the inverse sequence in $\mathfrak{L}\mathfrak{B}_0$ described by the following conditions:

- $C_\xi \approx C$ and $\pi_\xi^{\xi+1} \approx \pi$ for every $\xi < \omega_1$.
- \vec{c} is continuous with respect to the category of all compact spaces.

Let

$$C_{\omega_1} = \varprojlim \vec{c}$$

in the category of all linearly ordered compact spaces. Then C_{ω_1} is a linearly ordered Valdivia compact which maps increasingly onto any other nonempty linearly ordered Valdivia compact.

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Countable Fraïssé sequences

Theorem (Countable Cofinality)

Assume $\vec{u} = \langle u_\alpha, i_\alpha^\beta, \kappa \rangle$ is a Fraïssé sequence in a category with amalgamation \mathfrak{K} . Then for every countable sequence \vec{x} in \mathfrak{K} there exists an arrow $\vec{f}: \vec{x} \rightarrow \vec{u}$.

Corollary

Let \vec{u} be a countable Fraïssé sequence in a category \mathfrak{K} . If \mathfrak{K} satisfies amalgamation then \vec{u} is cofinal in $\mathfrak{S}_{\aleph_1}(\mathfrak{K})$.



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Homogeneity & Uniqueness

Theorem

Assume that $\vec{u} = \langle u_m, i_m^n, \omega \rangle$, $\vec{v} = \langle v_m, j_m^n, \omega \rangle$ are Fraïssé sequences in a fixed category \mathfrak{K} .

- (a) Let $f: u_k \rightarrow v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ such that $Fi_k = j_\ell f$. In particular $\vec{u} \approx \vec{v}$.
- (b) Assume \mathfrak{K} has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f: a \rightarrow b$, $i: a \rightarrow \vec{u}$, $j: b \rightarrow \vec{v}$ there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ such that $Fi = jf$.

$$\begin{array}{ccc} \vec{u} & \xrightarrow{F} & \vec{v} \\ i_k \uparrow & & \uparrow j_\ell \\ u_k & \xrightarrow{f} & v_\ell \end{array}$$

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Uncountable Fraïssé sequences

Theorem

Let $\kappa > \aleph_0$ be regular and assume that \mathfrak{K} is a full and cofinal subcategory of a κ -closed category \mathfrak{L} . If \mathfrak{L} has the amalgamation property, then:

- 1 There exists, up to equivalence of sequences, at most one κ -Fraïssé sequence in \mathfrak{K} .
- 2 A κ -Fraïssé sequence in \mathfrak{K} is also a Fraïssé sequence in \mathfrak{L} and it is both \mathfrak{L} -homogeneous and $\mathfrak{S}_{\kappa^+}(\mathfrak{L})$ -cofinal.



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Corollary

Let $\kappa > \aleph_0$ be regular and let \mathfrak{K} be a category. Assume at least one of the following conditions is satisfied:

- 1 \mathfrak{K} is κ -closed.
- 2 \mathfrak{K} has pushouts.
- 3 $\mathfrak{S}_\kappa(\mathfrak{K})$ has the amalgamation property.

Then a possible κ -Fraïssé sequence in \mathfrak{K} is

- unique,
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Trees

A tree $\langle T, \leq \rangle$ is **bounded** if for every $x \in T$ there is $y \in \max(T)$ such that $x \leq y$.

Let \mathfrak{T}_2 be the following category:

- Objects are bounded countable binary trees.
- Arrows are tree embeddings $f: T \rightarrow S$ such that $f[T]$ is a closed initial segment of S .

Claim

\mathfrak{T}_2 has the amalgamation property.

A tree T is **healthy** if

- every $x \in T \setminus \max(T)$ has two immediate successors,
- all maximal elements of T are on the top level of T .



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Theorem

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Let \vec{u}, \vec{v} be \aleph_1 -Fraïssé sequences in \mathfrak{T}_2 , induced by healthy trees U, V respectively. Let $\vec{f}: \vec{u} \rightarrow \vec{v}$ be an arrow of sequences. Then the induced tree embedding $f_{\omega_1}: U \rightarrow V$ is an isomorphism.

Corollary

\mathfrak{T}_2 has at least two incomparable \aleph_1 -Fraïssé sequences.



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Valdivia compacta

Let \mathfrak{K} be the following category:

- The objects of \mathfrak{K} are nonempty metric compacta.
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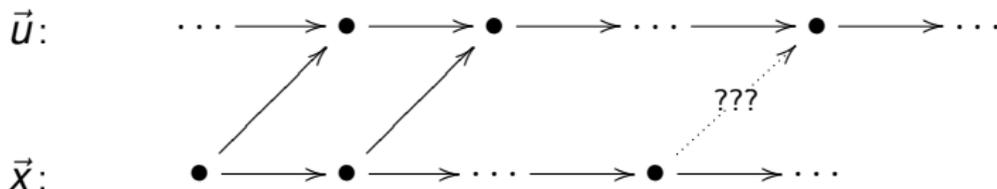
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Solution:

Let $F: \mathfrak{A}\mathfrak{I}\mathfrak{d} \rightarrow \mathfrak{C}\mathfrak{o}\mathfrak{m}\mathfrak{p}^{\leftarrow}$ be the natural functor. Then:

- Under CH, there exists an \aleph_1 -Fraïssé sequence \vec{u} in $\mathfrak{A}\mathfrak{I}\mathfrak{d}$ such that $F[\vec{u}]$ is continuous.
- Let K be a Valdivia compact of weight \aleph_1 . Then there is a sequence \vec{x} in $\mathfrak{A}\mathfrak{I}\mathfrak{d}$ such that $F[\vec{x}]$ is continuous and $K = \lim F[\vec{x}]$.

Claim

Let $f: X \rightarrow Z$, $g: Y \rightarrow Z$ be continuous surjections between compact spaces and assume f is a retraction. Then there are a compact W and continuous surjections $f': W \rightarrow X$, $g': W \rightarrow Y$ such that the diagram

$$\begin{array}{ccc} Y & \xleftarrow{g'} & W \\ g \downarrow & & \downarrow f' \\ Z & \xleftarrow{f} & X \end{array}$$

commutes and moreover g' is a retraction.

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Let $f: X \rightarrow Z$, $g: Y \rightarrow Z$ be continuous surjections between compact spaces and assume f is a retraction. Then there are a compact W and continuous surjections $f': W \rightarrow X$, $g': W \rightarrow Y$ such that the diagram

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- Under CH, there exists an \aleph_1 -Fraïssé sequence \vec{u} in $\mathfrak{A}\mathfrak{I}\mathfrak{d}$ such that $F[\vec{u}]$ is continuous.
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The above claim says that:

- The functor $F: \mathfrak{A}l\mathfrak{d} \rightarrow \mathfrak{C}o\mathfrak{m}p^{\leftarrow}$ has the **amalgamation property**, i.e. for every $x, y, z \in \mathfrak{A}l\mathfrak{d}$ for every arrow $f: z \rightarrow x$ in $\mathfrak{A}l\mathfrak{d}$ and for every arrow $g: F(z) \rightarrow F(y)$ in $\mathfrak{C}o\mathfrak{m}p^{\leftarrow}$ there exist $w \in \mathfrak{A}l\mathfrak{d}$, an arrow $h: F(x) \rightarrow F(w)$ in $\mathfrak{C}o\mathfrak{m}p^{\leftarrow}$ and an arrow $k: y \rightarrow w$ in $\mathfrak{A}l\mathfrak{d}$ such that the diagram

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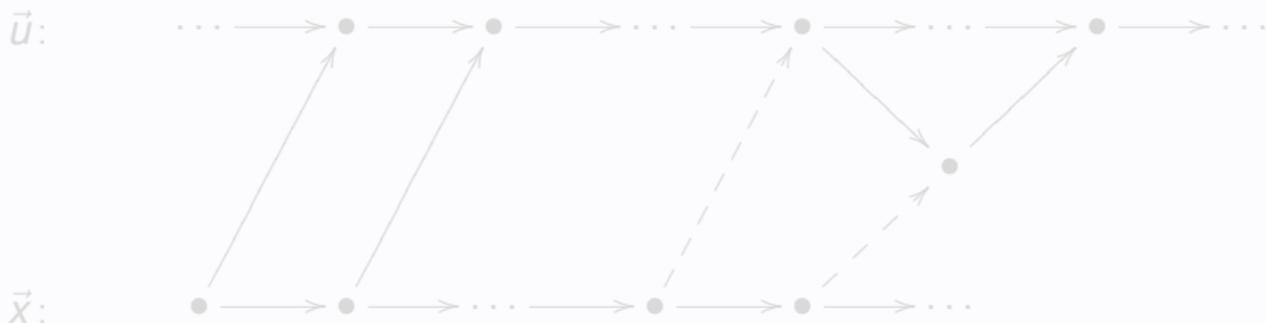


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Let $\kappa > \aleph_0$ be a regular cardinal and let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a covariant functor with amalgamation. Assume that

- \vec{u} is a κ -Fraïssé sequence in \mathfrak{K} ;
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Then for every sequence $\vec{x} \in \mathfrak{S}_{\kappa^+}(\mathfrak{K})$ such that $\Phi[\vec{x}]$ is continuous in \mathfrak{L} , there exists an arrow $\vec{f}: \Phi[\vec{x}] \rightarrow \Phi[\vec{u}]$ in $\mathfrak{S}_{\kappa^+}(\mathfrak{L})$.

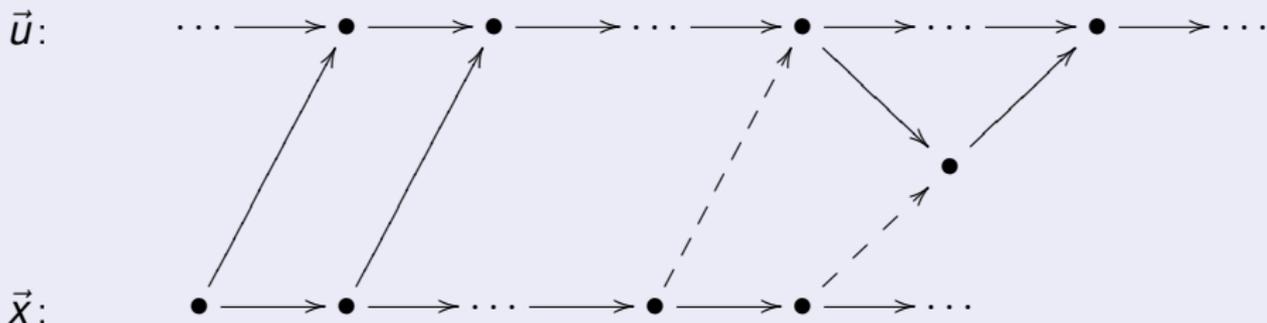


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We say that a functor $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ **does not add isomorphisms** if for every isomorphism $h: \Phi(a) \rightarrow \Phi(b)$ in \mathfrak{L} there is an isomorphism h' in \mathfrak{K} such that $h = \Phi(h')$.

Theorem

Let $\Phi: \mathfrak{K} \rightarrow \mathfrak{L}$ be a faithful covariant functor which does not add isomorphisms. Further, let \vec{u} and \vec{v} be κ -Fraïssé sequences in \mathfrak{K} such that $\Phi[\vec{u}]$ and $\Phi[\vec{v}]$ are continuous in \mathfrak{L} . Then for every arrows $f: a \rightarrow b$, $i: a \rightarrow \vec{u}$ and $j: b \rightarrow \vec{v}$ in \mathfrak{K} there exists an isomorphism of sequences $\vec{h}: \vec{u} \rightarrow \vec{v}$ in $\mathfrak{S}_{\kappa^+}(\mathfrak{K})$ for which the diagram

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Corollary

Assume CH. There exists a Valdivia compact K such that:

- The weight of K is \aleph_1 .
- Every metric compact is a retract of K .
- Every nonempty Valdivia compact of weight $\leq \aleph_1$ is a continuous image of K .
- For every retractions $r: X \rightarrow Y$, $k: K \rightarrow X$ and $\ell: K \rightarrow Y$, where X, Y are metric compacta, there exists a homeomorphism $h: K \rightarrow K$ such that

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Remark

If \mathfrak{N}_0 has an \aleph_1 -Fraïssé sequence then CH holds.

Question

Assume CH and let K be the Valdivia compact from the above corollary. Is every Valdivia compact of weight \aleph_1 a retract of K ?



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If \mathfrak{N}_1 has an \aleph_1 -Fraïssé sequence then CH holds.

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Retractive pairs

Fix a category \mathcal{K} . Denote by $\dagger\mathcal{K}$ the following category:

- The objects of $\dagger\mathcal{K}$ are the same as the objects of \mathcal{K} .
- Given $a, b \in \dagger\mathcal{K}$, an arrow $f: a \rightarrow b$ in $\dagger\mathcal{K}$ is a pair $f = \langle r, e \rangle$, where $r: b \rightarrow a$ and $e: a \rightarrow b$ are arrows of \mathcal{K} such that $re = \text{id}_a$. We shall write $r(f) = r$, $e(f) = e$.
- Given compatible arrows f, g in $\dagger\mathcal{K}$, their composition is

$$gf = \langle r(f)r(g), e(g)e(f) \rangle.$$

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If \mathcal{K} has pullbacks or pushouts then $\dagger\mathcal{K}$ has the amalgamation property.



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Given a category \mathfrak{K} , let $\Phi: \dagger\mathfrak{K} \rightarrow \mathfrak{K}$ be the contravariant “forgetful” functor, i.e. $\Phi(f) = r(f)$ for every arrow f in $\dagger\mathfrak{K}$.

We shall say that a sequence $\vec{x} \in \mathfrak{S}_\lambda(\dagger\mathfrak{K})$ is **semi-continuous** if $\Phi[\vec{x}]$ is continuous.

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Let \mathfrak{K} be a category and let \vec{u} and \vec{v} be semicontinuous Fraïssé sequences in $\dagger\mathfrak{K}$ of the same regular length κ . Then for every arrow $f: u_0 \rightarrow \vec{v}$ there exists an isomorphism of sequences $\vec{f}: \vec{u} \rightarrow \vec{v}$ such that $\vec{f}i_0 = f$.

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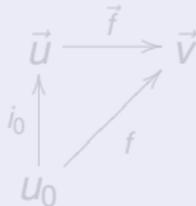


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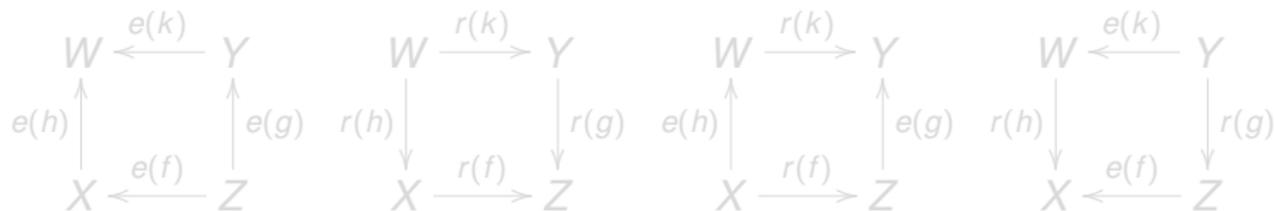
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Proper amalgamations

Let $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ be arrows in \mathfrak{A} .

We say that arrows $h: X \rightarrow W$, $k: Y \rightarrow W$ provide a **proper amalgamation** of f, g if $hf = kg$ and moreover $i(g)r(f) = r(k)i(h)$, $i(f)r(g) = r(h)i(k)$ hold.



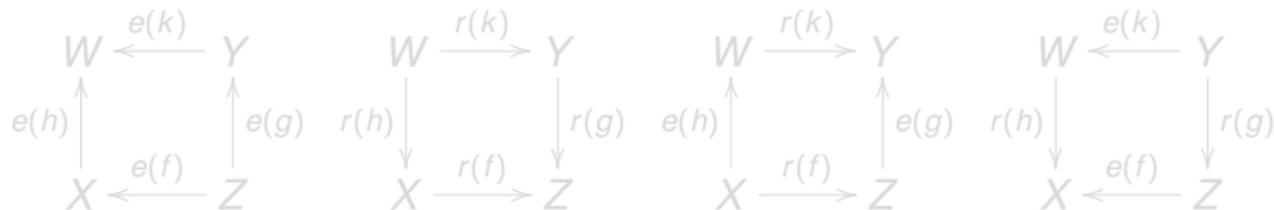
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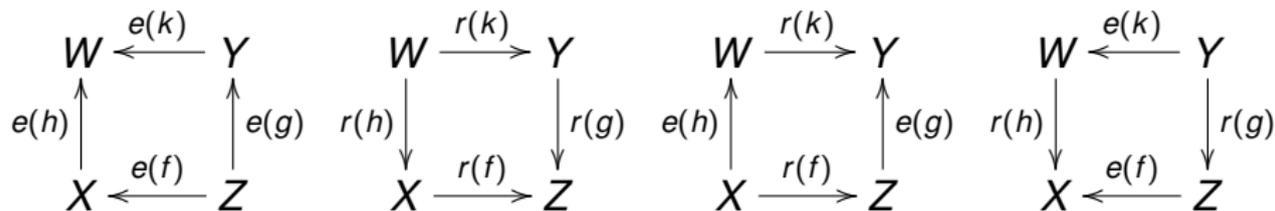
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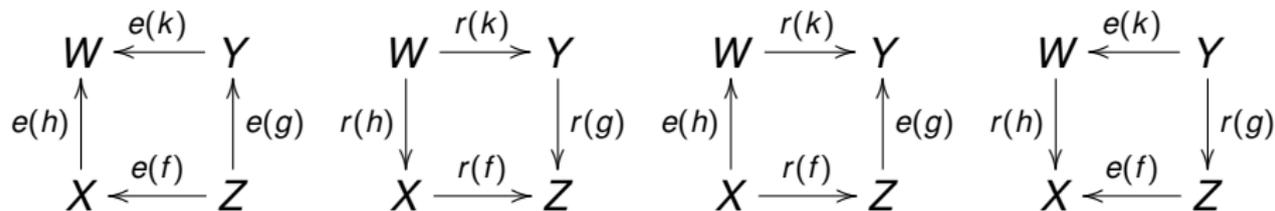
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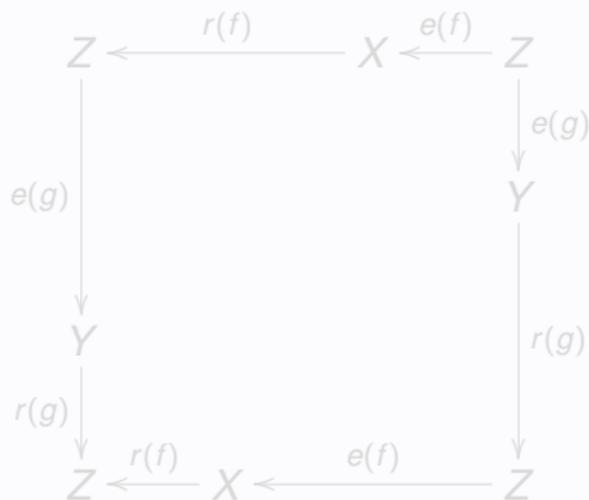
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If \mathfrak{K} has pullbacks or pushouts then $\ddagger\mathfrak{K}$ has proper amalgamations.

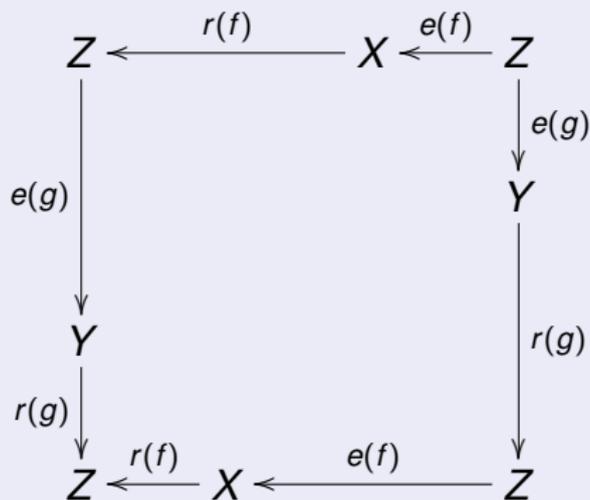
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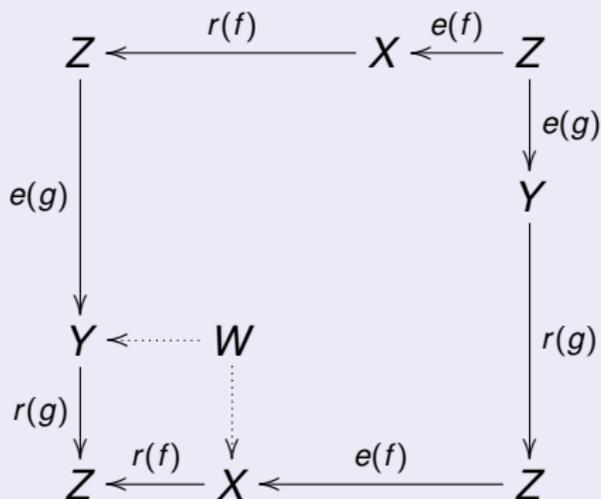
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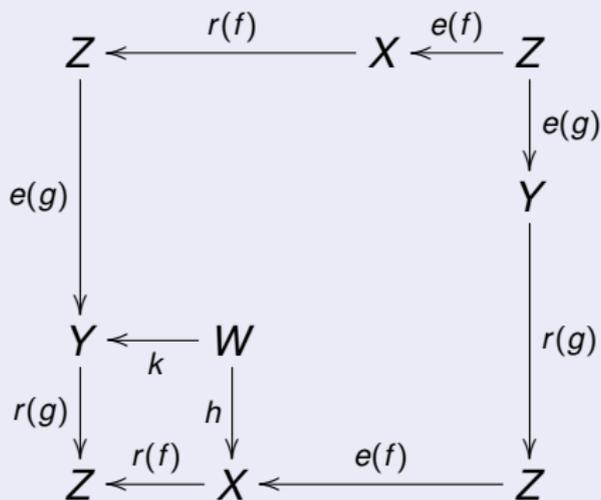
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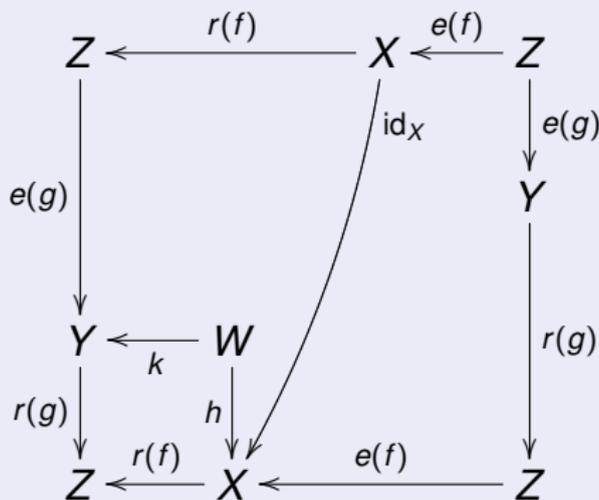
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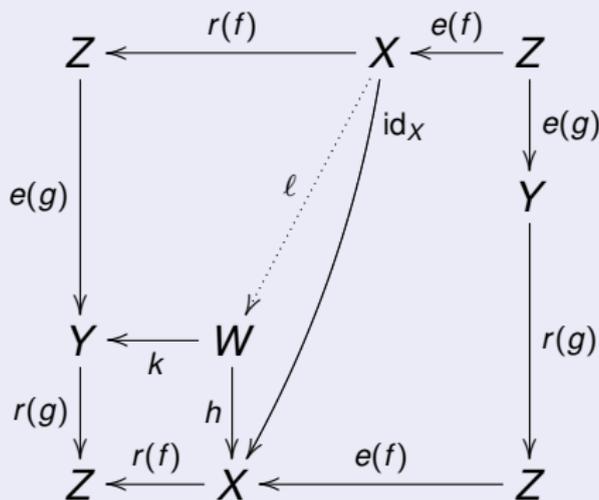
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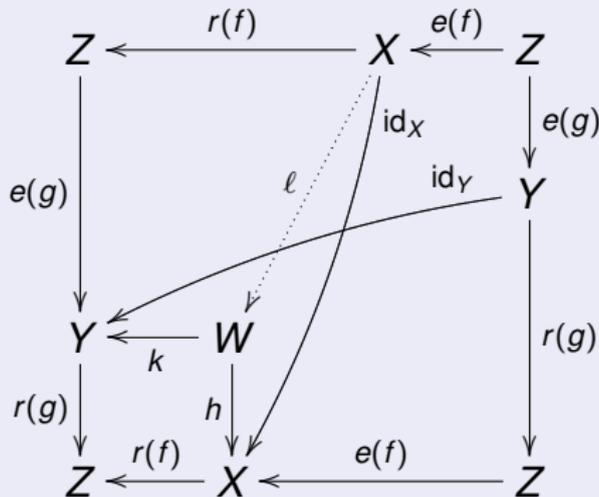
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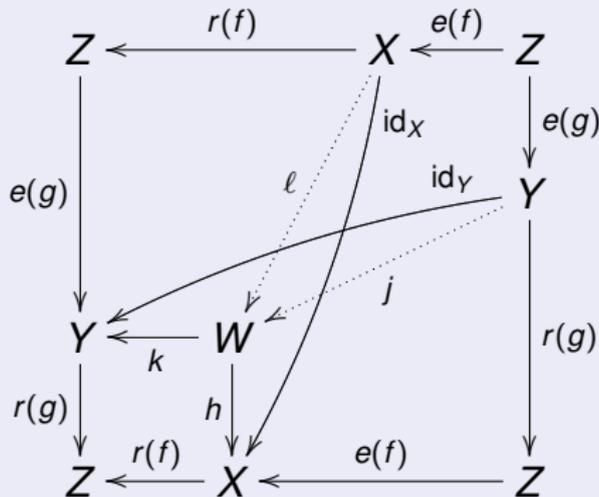
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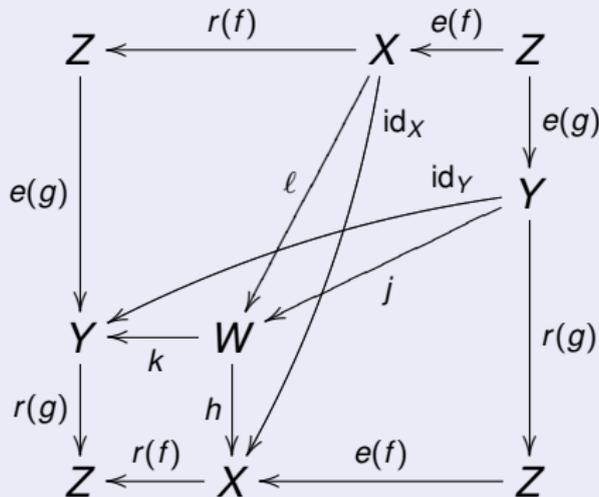
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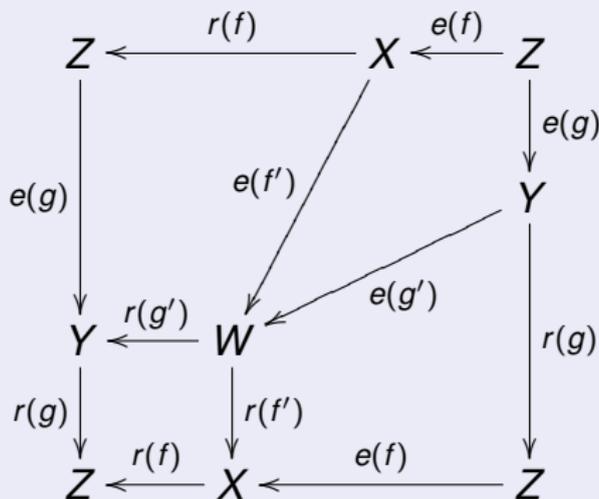
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Claim

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Theorem

Let \mathfrak{K} be a category such that \mathfrak{K} has proper amalgamations. Assume \vec{u} is a semi-continuous κ -Fraïssé sequence in \mathfrak{K} .

Then for every semi-continuous sequence $\vec{x} \in \mathfrak{S}_{\kappa^+}(\mathfrak{K})$ there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$ in \mathfrak{K} .

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Assume CH. Then there exists a Valdivia compact K of weight \aleph_1 such that every nonempty Valdivia compact of weight $\leq \aleph_1$ is a retract of K .



Theorem

Let \mathfrak{K} be a category such that $\dagger\mathfrak{K}$ has proper amalgamations. Assume \vec{u} is a semi-continuous κ -Fraïssé sequence in $\dagger\mathfrak{K}$.

Then for every semi-continuous sequence $\vec{x} \in \mathfrak{S}_{\kappa^+}(\dagger\mathfrak{K})$ there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$ in $\dagger\mathfrak{K}$.

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Banach spaces

Let \mathfrak{B}_{\aleph_0} be the category of separable Banach spaces, with arrows being linear transformations of norm ≤ 1 .

Claim

\mathfrak{B}_{\aleph_0} has pushouts and is \aleph_1 -closed.

Theorem

Under CH there exists a Banach space E of density \aleph_1 such that

- E has a projectional resolution of the identity (PRI);*
- every Banach space of density $\leq \aleph_1$ and with a PRI is linearly isometric to a one-complemented subspace of E .*



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