# Covering an uncountable square by countably many continuous functions

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# Motivations

## Theorem (Sierpiński)

*Let $S$ be a set of cardinality $\aleph_1$. Then there exists a sequence of functions $\{f_n \colon S \to S\}_{n\in\omega}$, such that*

$$S \times S = \bigcup_{n\in\omega} (f_n \cup f_n^{-1}).$$

## Proof.

- We assume that $S = \omega_1$.
- For each $\beta \in S$ fix a surjection $g_\beta \colon \omega \to \beta + 1$.
- Define $f_n(\beta) = g_\beta(n)$.

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## Remark (Sierpiński)

If $S$ has the above property then $|S| \leqslant \aleph_1$.

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- Fix $A \in [S]^{\aleph_1}$.
- For each $x \in A$ let $F_x = \{f_n(x) \colon n \in \omega\}$.
- The set $\bigcup_{x \in A} F_x$ has cardinality $\leqslant \aleph_1$.
- Suppose $p \in S$ is such that $p \notin F_x$ for $x \in A$.
- For each $a \in A$ there is $n(a) \in \omega$ such that $a = f_{n(a)}(p)$.
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Is it possible that the square of some uncountable subset of $\mathbb{R}$ is covered by countably many continuous real functions and their inverses?

In other words:

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Does there exist a family $\{f_n \colon \mathbb{R} \to \mathbb{R}\}_{n\in\omega}$ consisting of continuous functions such that

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How about covering by (continuous) non-decreasing functions?

- Suppose $S \times S \subseteq \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$, where each $f_n \colon S \to S$ is a non-decreasing function.
- Then both $f_n$ and $f_n^{-1}$ are chains in $S \times S$.
- Thus, if $|S| > \aleph_0$ then $S$ is a Countryman type!

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## Proposition

*There exists a compact line $K$ and a family $\{f_n \colon K \to K\}_{n\in\omega}$ consisting of continuous non-decreasing functions such that*

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# Another motivation

## Proposition (Shelah [5])

*There exists an $F_\sigma$ set $A \subseteq \mathbb{R}^2$ with the following properties.*

- *$S \times S \subseteq A$ for some uncountable set $S$.*
- *$X \times Y \not\subseteq A$ whenever $X, Y \in [\mathbb{R}]^{\aleph_2}$.*
- *$X \times Y \not\subseteq A$ whenever $X, Y$ are perfect subsets of $\mathbb{R}$.*

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Is it possible that $A = \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$, where each $f_n$ is a continuous real function?

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*Let* $\{f_n\colon \mathbb{R} \to \mathbb{R}\}_{n\in\omega}$ *be a family of continuous functions.*
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# Main result

## Theorem

*There exists a ccc forcing which introduces a family of 1-Lipschitz functions $\{f_n\colon 2^\omega \to 2^\omega\}_{n\in\omega}$ such that*

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*for some uncountable set $S \subseteq 2^\omega$.*

## The forcing:

$p \in \mathbb{P}$ iff $p = \langle n^p, s^p, v^p, f^p, \gamma^p, \varrho^p \rangle$, where

(1) $n^p \in \omega$, $s^p \in [\omega]^{<\omega}$ and $v^p \in [\omega_1]^{<\omega}$;

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(1) $n^p \in \omega$, $s^p \in [\omega]^{<\omega}$ and $v^p \in [\omega_1]^{<\omega}$;

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(4) $\gamma^p(\alpha) = f^p_{\varrho^p(\alpha,\beta)}(\gamma^p(\beta))$ whenever $\alpha < \beta$ and $\alpha, \beta \in v^p$.

# Main result

## Theorem

*There exists a ccc forcing which introduces a family of* 1*-Lipschitz functions* $\{f_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ *such that*

$$S \times S \subseteq \bigcup_{n\in\omega} (f_n \cup f_n^{-1})$$

*for some uncountable set* $S \subseteq 2^\omega$*.*

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# Corollaries

**Theorem (ZFC)**

*There exist a family of 1-Lipschitz functions $\{f_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ and an uncountable set $S \subseteq 2^\omega$ such that*

$$S \times S \subseteq \bigcup_{n\in\omega} (f_n \cup f_n^{-1}).$$

**Proof.**

By Keisler's absoluteness theorem [2] for the language $L_{\omega_1,\omega}(Q)$. □

**Theorem (ZFC)**

*There exist an $\aleph_1$-dense set $X \subseteq \mathbb{R}$ and a family of continuous functions $\{f_n \colon \mathbb{R} \to \mathbb{R}\}_{n\in\omega}$ such that $X \times X \subseteq \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$.*

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*It is relatively consistent with ZFC that for every set $X \in [\mathbb{R}]^{\aleph_1}$ there exists a sequence of continuous functions $f_n \colon \mathbb{R} \to \mathbb{R}$ with $X \times X \subseteq \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$.*

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*There exists a family of continuous functions* $\{u_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ *with the following properties:*

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2. *Some sort of homogeneity.*

*The above properties describe the family* $\{u_n\}_{n\in\omega}$ *uniquely.*

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*Let $\{u_n\}_{n\in\omega}$ be the universal homogeneous family of functions from the previous theorem. Then*

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# References


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