# Universal Banach spaces with a projectional resolution of the identity: category-theoretic approach

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# Background

**Theorem (Pełczyński, 1969)**

*There exists a complementably universal Banach space for the class of Banach spaces with a Schauder basis.*

A Banach space $E$ is complementably universal for a class of spaces $\mathcal{K}$ if

- $E \in \mathcal{K}$.
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A Schauder basis in $X$ is a sequence $\{x_n\}_{n\in\omega} \subseteq X$ such that for every $x \in X$ there are uniquely determined scalars $\{\lambda_n\}_{n\in\omega}$ such that the series

$$\sum_{n=0}^{\infty} \lambda_n x_n$$

converges to $x$ in the norm.

Define $P_n \colon X \to X$ by

$$P_n\left(\sum_{i=0}^{\infty} \lambda_i x_i\right) = \sum_{i<n} \lambda_i x_i.$$

Then $P_n$ is a bounded projection and $\lim_{n\to\infty} P_n x = x$ for $x \in X$.

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## Proposition

*Given a Banach space $X$, the following properties are equivalent.*

1. *$X$ has a Schauder basis.*
2. *There exists a sequence $\{P_n\}_{n\in\omega}$ of bounded projections of $X$ onto finite-dimensional subspaces, converging pointwise to the identity and such that*
$$P_n P_m = P_{\min\{n,m\}}$$
*holds for every $n, m \in \omega$.*

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# Markushevich bases

Let $X$ be a Banach space of density $\aleph_1$. A Markushevich basis in $X$ is a bi-orthogonal system $\langle x_\alpha, y_\alpha \rangle$, $\alpha < \omega_1$, such that

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# Projectional resolutions

A projectional resolution of the identity (PRI for short) in a Banach space $X$ is a sequence $\{P_\alpha\}_{\alpha<\delta}$ of bounded projections of $X$ such that

- $\delta$ is a limit ordinal (a well-ordered set with no maximum),
- $P_\alpha P_\beta = P_{\min\{\alpha,\beta\}}$,
- $\lim_{\alpha<\delta} P_\alpha x = x$ for every $x \in X$,
- $\lim_{\alpha<\varrho} P_\alpha x = P_\varrho x$ for every limit ordinal $\varrho < \delta$.

We say that the PRI $\{P_\alpha\}_{\alpha<\delta}$ is

- regular, if
  $$\operatorname{dens}(P_\alpha X) < \operatorname{dens}(X)$$
  for every $\alpha < \delta$;
- normalized, if $\|P_\alpha\| = 1$ for each $\alpha < \delta$.

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*Given a Banach space $X$ of density $\aleph_1$, the following conditions are equivalent:*

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2. *$X$ has a countably norming Markushevich basis $\langle x_\alpha, y_\alpha \rangle_{\alpha<\omega_1}$, i.e. the space*

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- Chain of closed subspaces is an inductive sequence in the category of Banach spaces.
- Projectional resolution is an inverse sequence of right-invertible arrows, "compatible" with the given inductive sequence.

**Claim**

*Let $X$ be a Banach space, represented as the closure of the union of a chain $\{X_n\}_{n\in\omega}$. TFAE:*

1. *There exists a normalized PRI $\{P_n\}_{n\in\omega}$ such that $X_n = P_n X$ for $n \in \omega$.*
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Given a category $\mathfrak{K}$, define a new category $\ddagger\mathfrak{K}$ as follows.

- The objects of $\ddagger\mathfrak{K}$ are the objects of $\mathfrak{K}$.
- An arrow from $x$ into $y$ in $\ddagger\mathfrak{K}$ is a pair $\langle e, r\rangle$ such that $e\colon x \to y$, $r\colon y \to x$ are arrows of $\mathfrak{K}$ and $r \circ e = \mathrm{id}_x$.

There are two natural functors $e\colon \ddagger\mathfrak{K} \to \mathfrak{K}$ and $r\colon \ddagger\mathfrak{K} \to \mathfrak{K}$.
A sequence $\vec{x}$ in $\ddagger\mathfrak{K}$ will be called semicontinuous if $e[\vec{x}]$ is continuous in $\mathfrak{K}$.

**Example**

Let $\mathfrak{B}$ be the category of Banach spaces with linear transformations of norm $\leqslant 1$. A semicontinuous sequence $\vec{x}$ in $\ddagger\mathfrak{B}$ corresponds to a normalized PRI in $X$, where $X$ is the colimit of $e[\vec{x}]$ in the category $\mathfrak{B}$.

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Given a category $\mathfrak{K}$, denote by $\langle\mathfrak{K}\rangle$ the category of all inductive sequences in $\mathfrak{K}$.

$\vec{u} \in \langle\mathfrak{K}\rangle$ will be a called a Fraïssé sequence if it satisfies the following two conditions.

1. For every $x \in \mathfrak{K}$ there is an arrow from $x$ into some element of $\vec{u}$.
2. Given an arrow $f\colon u_\xi \to y$ in $\mathfrak{K}$, there are $\eta \geqslant \xi$ and an arrow $g\colon y \to u_\eta$ in $\mathfrak{K}$ such that $g \circ f = u_\xi^\eta$.

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\begin{tikzcd}
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We say that $\mathfrak{K}$ has the amalgamation property if

for every arrows $f\colon z \to x$, $g\colon z \to y$ there are arrows $f'\colon x \to w$ and $g'\colon y \to w$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd} y \arrow[r, "g'"] & w \\ z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"'] \end{tikzcd}$$

The arrows $\langle f', g' \rangle$ provide a pushout of $\langle f, g \rangle$ if moreover for every $\overline{f}, \overline{g}$ satisfying $\overline{f} \circ f = \overline{g} \circ g$ there exists a unique arrow $h$ such that $h \circ f' = \overline{f}$ and $h \circ g' = \overline{g}$.

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## The pushout of $\langle f, g \rangle$

$$\begin{tikzcd}
 & & v \\
y \arrow[r, "g'"'] \arrow[urr, bend left, "\bar{g}"] & w & \\
z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"] \arrow[uur, bend right, "\bar{f}"'] &
\end{tikzcd}$$

## The pushout of $\langle f, g \rangle$

$$\begin{tikzcd}
 & & v \\
y \arrow[r, "g'"'] \arrow[urr, bend left, "\overline{g}"] & w \arrow[ur, dotted, "h"] & \\
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## Proposition

*Let $f\colon z \to x$, $g\colon z \to y$ be arrows in $\ddagger\mathfrak{K}$. If $\langle e(f), e(g)\rangle$ has a pushout in $\mathfrak{K}$, then $\langle f, g\rangle$ has a* proper *amalgamation in $\ddagger\mathfrak{K}$. That is, there exist arrows $h\colon x \to w$, $k\colon y \to w$ in $\ddagger\mathfrak{K}$ such that the following diagrams commute in $\mathfrak{K}$.*

$$
\begin{tikzcd}
w & y \arrow[l, "e(k)"'] \\
x \arrow[u, "e(h)"] & z \arrow[l, "e(f)"] \arrow[u, "e(g)"']
\end{tikzcd}
\quad
\begin{tikzcd}
w \arrow[r, "r(k)"] \arrow[d, "r(h)"'] & y \arrow[d, "r(g)"] \\
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# Fraïssé limits

**Theorem**

*Let $\mathfrak{K}$ be a category with the amalgamation property and with an initial object.*

1. *Assume that $\mathfrak{K}$ has countably many isomorphic types of arrows. Then there exists a Fraïssé sequence $\vec{u}$ of length $\omega$ in $\mathfrak{K}$.*
2. *(Universality) For every countable sequence $\vec{x} \in \langle \mathfrak{K} \rangle$ there is an arrow $\vec{F} \colon \vec{x} \to \vec{u}$.*
3. *(Homogeneity) Given arrows $f \colon a \to \vec{u}$, $g \colon a \to \vec{u}$ there exists an automorphism $\vec{H} \colon \vec{u} \to \vec{u}$ such that $g = \vec{H} \circ f$.*
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## Universality

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[rr] & & \cdots \arrow[rr] & & \cdots \\
x_0 \arrow[r] \arrow[ur] & x_1 \arrow[r] & \cdots \arrow[rr] & & \cdots &
\end{tikzcd}
$$

## Universality

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[rr] \arrow[dr] & & \cdots \arrow[rr] & & \cdots \\
& & w & & & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[rr] & & \cdots &
\end{tikzcd}
$$

## Universality

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] \arrow[dr] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & \cdots \\
 & & w \arrow[u] & & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & \cdots &
\end{tikzcd}
$$

## Universality

$$\begin{array}{ccccccc} \cdots \longrightarrow & u_{\ell_0} & \longrightarrow & u_{\ell_1} & \longrightarrow \cdots \longrightarrow & \cdots \\ & \nearrow & & \nearrow & & \\ x_0 & \longrightarrow & x_1 & \longrightarrow \cdots \longrightarrow \cdots & & \end{array}$$

## Universality

$$
\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[rr] & & \cdots \\
x_0 \arrow[ur] \arrow[r] & x_1 \arrow[ur] \arrow[r] & \cdots \arrow[r] & x_n \arrow[ur] \arrow[r] & x_{n+1} \arrow[r] & \cdots &
\end{tikzcd}
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\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[rr] \arrow[dr] & & \cdots \\
& & & & & w & \\
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 & & & & & w \arrow[u] & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[uur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[uur] & x_{n+1} \arrow[r] \arrow[ur] & \cdots &
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X_0 \arrow[r] \arrow[ur] & X_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & X_n \arrow[r] \arrow[ur] & X_{n+1} \arrow[r] \arrow[ur] & \cdots &
\end{tikzcd}$$

## Back-and-forth method

$$\begin{tikzcd}
u_k \arrow[r] & \cdots \\
v_\ell \arrow[u, "f"] \arrow[r] & \cdots
\end{tikzcd}$$

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U_k \arrow[rr] \arrow[dr, "g_0"] & & \cdots \\
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## Theorem

*Let $\mathfrak{K}$ be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Let $\vec{u}$ be a $\kappa$-Fraïssé sequence in $\ddagger\mathfrak{K}$.*

*Let $\mathfrak{L} \supseteq \mathfrak{K}$ be such that*

$$U_\infty = \lim e[\vec{u}]$$

*exists in $\mathfrak{L}$.*

*Then for every $X \in \mathfrak{L}$ such that $X = \lim e[\vec{x}]$, where $\vec{x}$ is a semicontinuous sequence of length $\leqslant \kappa$ in $\ddagger\mathfrak{K}$, there exists arrows $i\colon X \to U_\infty$ and $p\colon U_\infty \to X$ such that*

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# Application I: Pełczyński's result

## Example

Call a Banach space rational if it is isometric to a space of the form $\langle \mathbb{R}^d, \|\cdot\|\rangle$, where $d \in \omega$ and the unit ball

$$B = \{x \in \mathbb{R}^d \colon \|x\| \leqslant 1\}$$

is the convex hull of a finite subset of $\mathbb{Q}^d$. Call a linear transformation $T \colon \mathbb{R}^d \to \mathbb{R}^k$ rational if $T\mathbb{Q}^d \subseteq \mathbb{Q}^k$.
Denote by $\mathfrak{R}$ the category of all rational Banach spaces with rational linear transformations of norm $\leqslant 1$.

## Claim

*Left-invertible arrows have pushouts in $\mathfrak{R}$.*

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*Let $\vec{u}$ be a Fraïssé sequence in $\ddagger\mathfrak{N}$. Then the colimit*

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*in the category of Banach spaces is a separable Banach space with a normalized PRI onto finite-dimensional subspaces.*

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*The space $U_\omega$ is complementably universal for the class of Banach spaces with a Schauder basis.*

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# Application II: A non-separable complementably universal Banach space

**Example**

Let $\mathfrak{B}_{\mathrm{sep}}$ be the category of all separable Banach spaces with linear transformations of norm $\leqslant 1$.

**Claim**

*Left-invertible arrows have pushouts in $\mathfrak{B}_{\mathrm{sep}}$.*

**Claim**

*The category $\mathfrak{B}_{\mathrm{sep}}$ has $2^{\aleph_0}$ many isomorphic types of arrows.*

**Theorem**

*Assume $2^{\aleph_0} = \aleph_1$. Then there exists a semicontinuous $\omega_1$-Fraïssé sequence in $\ddagger\mathfrak{B}_{\mathrm{sep}}$.*

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## Theorem

*Assume the continuum hypothesis.*
*Then there exists a Banach space $U_{\omega_1}$ which is complementably universal for the class of all Banach spaces of density $\leqslant \aleph_1$ with a countably norming Markushevich basis.*

***

THE END

***