

Universal Banach spaces with a projectional resolution of the identity: category-theoretic approach

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Background

Theorem (Pełczyński, 1969)

There exists a complementably universal Banach space for the class of Banach spaces with a Schauder basis.

A Banach space E is **complementably universal** for a class of spaces \mathcal{K} if

- $E \in \mathcal{K}$.
- Every $X \in \mathcal{K}$ is isomorphic to a complemented subspace of E .

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A **Schauder basis** in X is a sequence $\{x_n\}_{n \in \omega} \subseteq X$ such that for every $x \in X$ there are uniquely determined scalars $\{\lambda_n\}_{n \in \omega}$ such that the series

$$\sum_{n=0}^{\infty} \lambda_n x_n$$

converges to x in the norm.

Define $P_n: X \rightarrow X$ by

$$P_n \left(\sum_{i=0}^{\infty} \lambda_i x_i \right) = \sum_{i < n} \lambda_i x_i.$$

Then P_n is a bounded projection and $\lim_{n \rightarrow \infty} P_n x = x$ for $x \in X$.

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Proposition

Given a Banach space X , the following properties are equivalent.

- 1 X has a Schauder basis.
- 2 There exists a sequence $\{P_n\}_{n \in \omega}$ of bounded projections of X onto finite-dimensional subspaces, converging pointwise to the identity and such that

$$P_n P_m = P_{\min\{n,m\}}$$

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Markushevich bases

Let X be a Banach space of density \aleph_1 . A **Markushevich basis** in X is a bi-orthogonal system $\langle x_\alpha, y_\alpha \rangle$, $\alpha < \omega_1$, such that

- 1 $\{x_\alpha : \alpha < \omega_1\}$ is linearly dense in X ,
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Projectional resolutions

A **projectional resolution of the identity** (PRI for short) in a Banach space X is a sequence $\{P_\alpha\}_{\alpha < \delta}$ of bounded projections of X such that

- δ is a limit ordinal (a well-ordered set with no maximum),
- $P_\alpha P_\beta = P_{\min\{\alpha, \beta\}}$,
- $\lim_{\alpha < \delta} P_\alpha x = x$ for every $x \in X$,
- $\lim_{\alpha < \varrho} P_\alpha x = P_\varrho x$ for every limit ordinal $\varrho < \delta$.

We say that the PRI $\{P_\alpha\}_{\alpha < \delta}$ is

- **regular**, if

$$\text{dens}(P_\alpha X) < \text{dens}(X)$$

for every $\alpha < \delta$;

- **normalized**, if $\|P_\alpha\| = 1$ for each $\alpha < \delta$.

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Given a Banach space X of density \aleph_1 , the following conditions are equivalent:

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How to interpret a PRI in the language of category theory?

- Chain of closed subspaces is an inductive sequence in the category of Banach spaces.
- Projectional resolution is an inverse sequence of right-invertible arrows, “compatible” with the given inductive sequence.

Claim

Let X be a Banach space, represented as the closure of the union of a chain $\{X_n\}_{n \in \omega}$. TFAE:

- 1 There exists a normalized PRI $\{P_n\}_{n \in \omega}$ such that $X_n = P_n X$ for $n \in \omega$.
- 2 For each $n \in \omega$, X_n is 1-complemented in X_{n+1} .

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Given a category \mathcal{K} , define a new category $\ddagger\mathcal{K}$ as follows.

- The objects of $\ddagger\mathcal{K}$ are the objects of \mathcal{K} .
- An arrow from x into y in $\ddagger\mathcal{K}$ is a pair $\langle e, r \rangle$ such that $e: x \rightarrow y$, $r: y \rightarrow x$ are arrows of \mathcal{K} and $r \circ e = \text{id}_x$.

There are two natural functors $e: \ddagger\mathcal{K} \rightarrow \mathcal{K}$ and $r: \ddagger\mathcal{K} \rightarrow \mathcal{K}$.

A sequence \vec{x} in $\ddagger\mathcal{K}$ will be called **semicontinuous** if $e[\vec{x}]$ is continuous in \mathcal{K} .

Example

Let \mathfrak{B} be the category of Banach spaces with linear transformations of norm ≤ 1 . A semicontinuous sequence \vec{x} in $\ddagger\mathfrak{B}$ corresponds to a normalized PRI in X , where X is the colimit of $e[\vec{x}]$ in the category \mathfrak{B} .

Given a category \mathcal{K} , define a new category $\dagger\mathcal{K}$ as follows.

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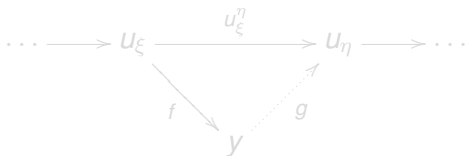
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Given a category \mathfrak{K} , denote by $\langle \mathfrak{K} \rangle$ the category of all inductive sequences in \mathfrak{K} .

$\vec{u} \in \langle \mathfrak{K} \rangle$ will be called a **Fraïssé sequence** if it satisfies the following two conditions.

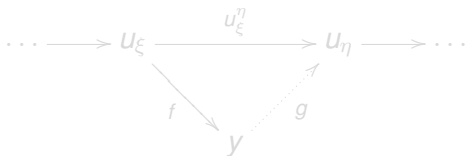
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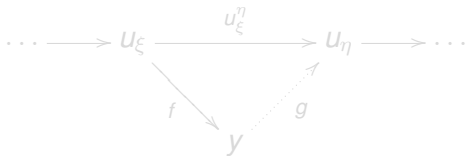
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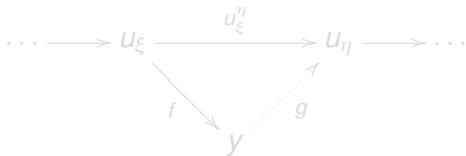
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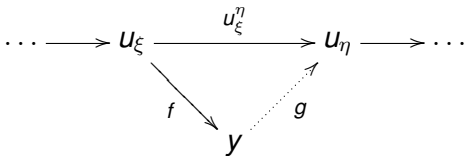
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We say that \mathfrak{K} has the **amalgamation property** if

for every arrows $f: z \rightarrow x$, $g: z \rightarrow y$ there are arrows $f': x \rightarrow w$ and $g': y \rightarrow w$ such that $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ z & \xrightarrow{f} & x \end{array}$$

The arrows $\langle f', g' \rangle$ provide a **pushout** of $\langle f, g \rangle$ if moreover for every \bar{f}, \bar{g} satisfying $\bar{f} \circ f = \bar{g} \circ g$ there exists a unique arrow h such that $h \circ f' = \bar{f}$ and $h \circ g' = \bar{g}$.

We say that \mathfrak{K} has the **amalgamation property** if

for every arrows $f: z \rightarrow x$, $g: z \rightarrow y$ there are arrows $f': x \rightarrow w$ and $g': y \rightarrow w$ such that $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ z & \xrightarrow{f} & x \end{array}$$

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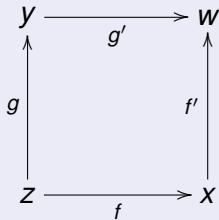
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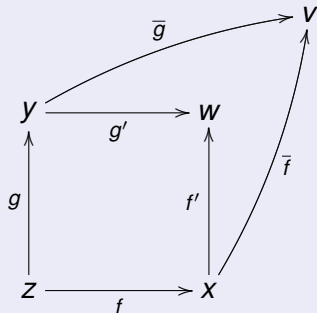
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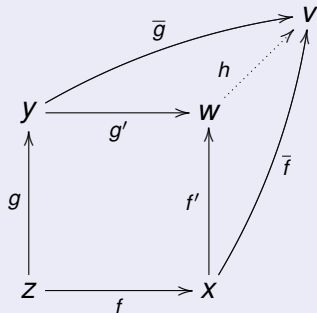
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Proposition

Let $f: z \rightarrow x$, $g: z \rightarrow y$ be arrows in $\ddagger\mathfrak{K}$. If $\langle e(f), e(g) \rangle$ has a pushout in \mathfrak{K} , then $\langle f, g \rangle$ has a **proper** amalgamation in $\ddagger\mathfrak{K}$. That is, there exist arrows $h: x \rightarrow w$, $k: y \rightarrow w$ in $\ddagger\mathfrak{K}$ such that the following diagrams commute in \mathfrak{K} .

$$\begin{array}{ccc}
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 w & \xleftarrow{e(k)} & y \\
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 x & \xleftarrow{e(f)} & z
 \end{array} &
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Fraïssé limits

Theorem

Let \mathfrak{K} be a category with the amalgamation property and with an initial object.

- 1 Assume that \mathfrak{K} has countably many isomorphic types of arrows. Then there exists a Fraïssé sequence \vec{u} of length ω in \mathfrak{K} .
- 2 (**Universality**) For every countable sequence $\vec{x} \in \langle \mathfrak{K} \rangle$ there is an arrow $\vec{F}: \vec{x} \rightarrow \vec{u}$.
- 3 (**Homogeneity**) Given arrows $f: a \rightarrow \vec{u}$, $g: a \rightarrow \vec{u}$ there exists an automorphism $\vec{H}: \vec{u} \rightarrow \vec{u}$ such that $g = \vec{H} \circ f$.
- 4 (**Uniqueness**) The sequence \vec{u} is unique, up to isomorphism.

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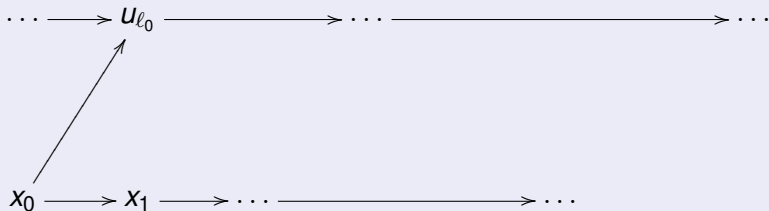
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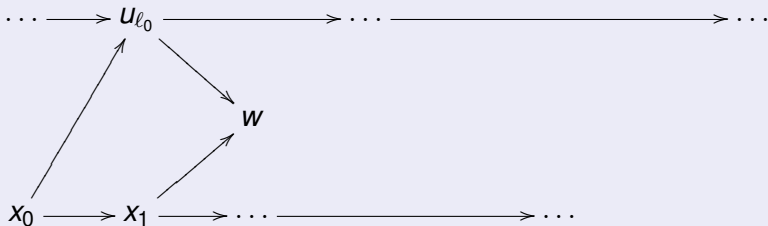
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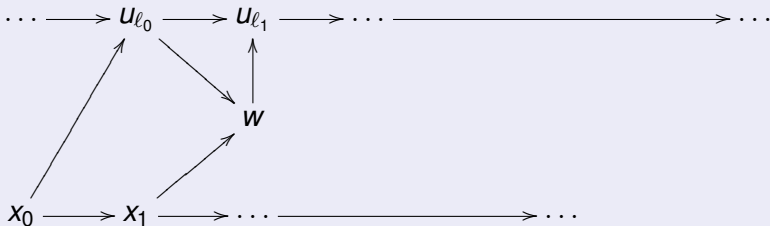
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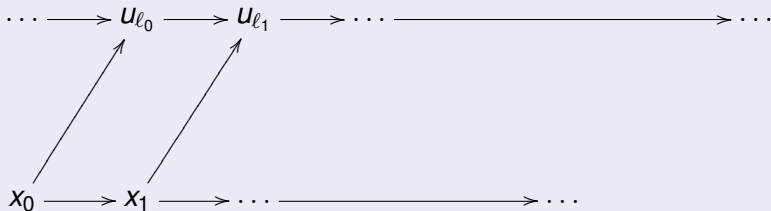
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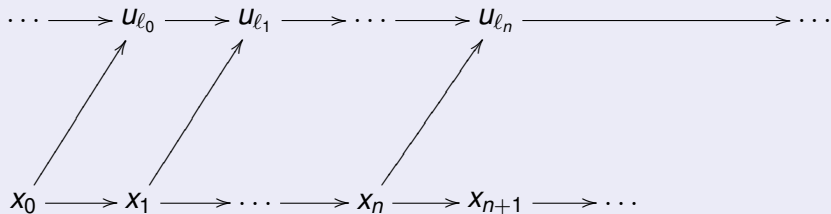
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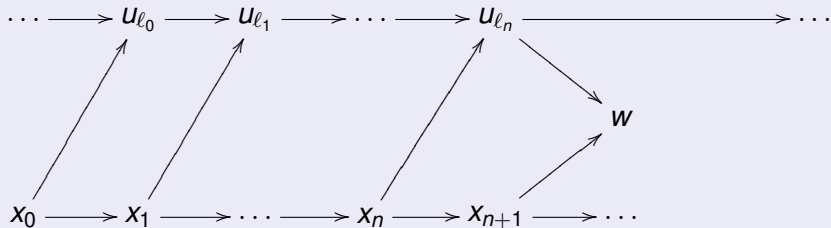
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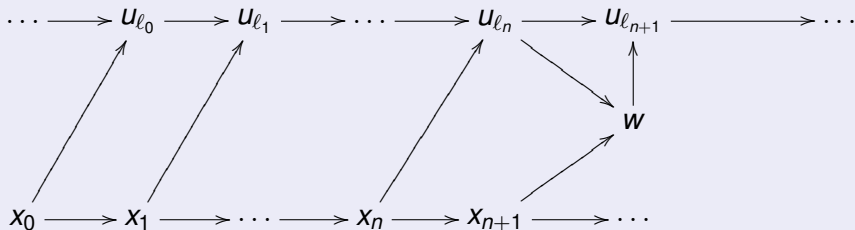
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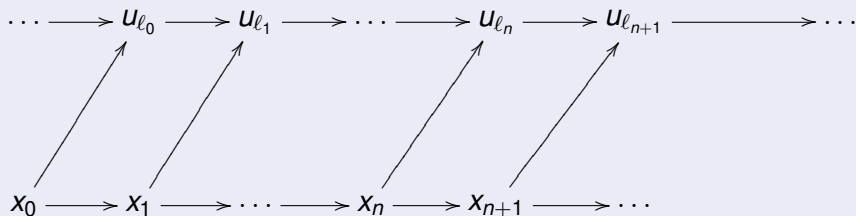
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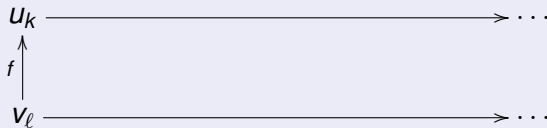
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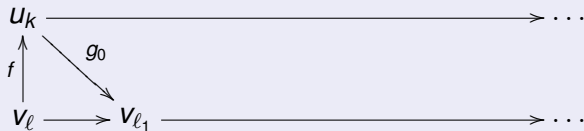
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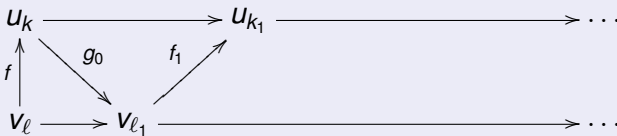
Back-and-forth method



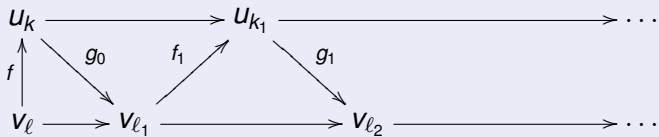
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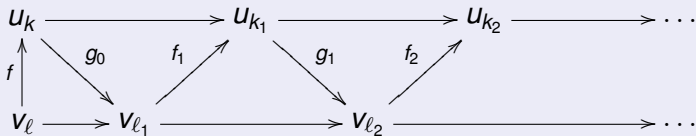
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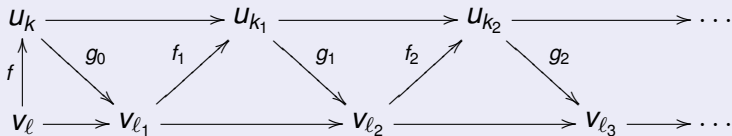
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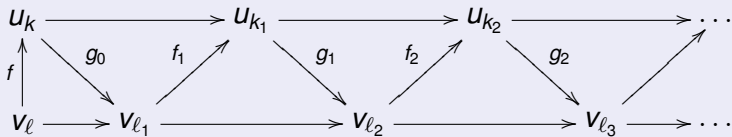
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Theorem

Let \mathfrak{K} be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Let \vec{u} be a κ -Fraïssé sequence in $\ddagger\mathfrak{K}$.

Let $\mathfrak{L} \supseteq \mathfrak{K}$ be such that

$$U_\infty = \lim e[\vec{u}]$$

exists in \mathfrak{L} .

Then for every $X \in \mathfrak{L}$ such that $X = \lim e[\vec{x}]$, where \vec{x} is a semicontinuous sequence of length $\leq \kappa$ in $\ddagger\mathfrak{K}$, there exists arrows $i: X \rightarrow U_\infty$ and $p: U_\infty \rightarrow X$ such that

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Application I: Pełczyński's result

Example

Call a Banach space **rational** if it is isometric to a space of the form $\langle \mathbb{R}^d, \|\cdot\| \rangle$, where $d \in \omega$ and the unit ball

$$B = \{x \in \mathbb{R}^d : \|x\| \leq 1\}$$

is the convex hull of a finite subset of \mathbb{Q}^d . Call a linear transformation $T: \mathbb{R}^d \rightarrow \mathbb{R}^k$ **rational** if $T\mathbb{Q}^d \subseteq \mathbb{Q}^k$.

Denote by \mathfrak{R} the category of all rational Banach spaces with rational linear transformations of norm ≤ 1 .

Claim

Left-invertible arrows have pushouts in \mathfrak{R} .

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Application II: A non-separable complementably universal Banach space

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Let $\mathfrak{B}_{\text{sep}}$ be the category of all separable Banach spaces with linear transformations of norm ≤ 1 .

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The category $\mathfrak{B}_{\text{sep}}$ has 2^{\aleph_0} many isomorphic types of arrows.

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Assume $2^{\aleph_0} = \aleph_1$. Then there exists a semicontinuous ω_1 -Fraïssé sequence in $\mathfrak{B}_{\text{sep}}$.

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Assume the continuum hypothesis.

Then there exists a Banach space U_{ω_1} which is complementably universal for the class of all Banach spaces of density $\leq \aleph_1$ with a countably norming Markushevich basis.

THE END
