Automorphism groups of homogeneous structures

Wiesław Kubiś

Institute of Mathematics, Czech Academy of Sciences

Applications of Algebra XXIII

Zakopane, 11–17 March 2019

Joint work with S. Shelah

Mathematical structures

Definition

A structure is a set endowed with some relations and algebraic operations.

Mathematical structures

Definition

A structure is a set endowed with some relations and algebraic operations.
A structure is relational if it has no algebraic operations.

Mathematical structures

Definition

A structure is a set endowed with some relations and algebraic operations.
A structure is relational if it has no algebraic operations.

Examples:

Sets, graphs, partially ordered sets, tournaments, cyclically ordered sets.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f \colon A \to B$ between finite substructures A, B of M extends to an automorphism $h \colon M \to M$.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
Aut(M) will denote the group of all automorphisms of M.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
$\mathrm{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
$\mathrm{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

- Any set.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
$\mathrm{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

- Any set.
- Any complete or discrete graph.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f \colon A \to B$ between finite substructures A, B of M extends to an automorphism $h \colon M \to M$.
$\mathrm{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

- Any set.
- Any complete or discrete graph.
- $\langle \mathbb{Q}, < \rangle$.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
$\mathrm{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

- Any set.
- Any complete or discrete graph.
- $\langle \mathbb{Q}, < \rangle$.
- The Rado graph.

Homogeneity

Definition

A structure M is homogeneous if every isomorphism $f\colon A \to B$ between finite substructures A, B of M extends to an automorphism $h\colon M \to M$.
$\operatorname{Aut}(M)$ will denote the group of all automorphisms of M.

Examples of homogeneous structures:

- Any set.
- Any complete or discrete graph.
- $\langle \mathbb{Q}, < \rangle$.
- The Rado graph.
- Any finite cyclic group.

Uniform homogeneity

Definition

A structure M is uniformly homogeneous if

1. M is homogeneous and

Uniform homogeneity

Definition

A structure M is uniformly homogeneous if

1. M is homogeneous and
2. for every finite substructure $A \subseteq M$ there exists an extension operator $e_A \colon \operatorname{Aut}(A) \to \operatorname{Aut}(M)$ such that

$$e_A(g \circ h) = e_A(g) \circ e_A(h)$$

for every $g, h \in \operatorname{Aut}(A)$.

Question (E. Jaligot, 2007)

Let M be a countable homogeneous structure. Is it always true that the group $\mathrm{Aut}(M)$ contains isomorphic copies of all groups of the form $\mathrm{Aut}(X)$, where X is a substructure of M?

Katětov functors

Definition

Let $\mathscr{F}$ be a class of finite structures of the same type and let M be a countable homogeneous structure such that every $A \in \mathscr{F}$ embeds into M and every finite substructure of M is isomorphic to some $A \in \mathscr{F}$.

Katětov functors

Definition

Let $\mathscr{F}$ be a class of finite structures of the same type and let M be a countable homogeneous structure such that every $A \in \mathscr{F}$ embeds into M and every finite substructure of M is isomorphic to some $A \in \mathscr{F}$. A Katětov functor is a pair $\langle K, \eta \rangle$ such that K assigns to each embedding $e \colon A \to B$ with $A, B \in \mathscr{F}$ an embedding $K(e) \colon M \to M$, η assigns to each $A \in \mathscr{F}$ an embedding $\eta_A \colon A \to M$.

Katětov functors

Definition

Let $\mathscr{F}$ be a class of finite structures of the same type and let M be a countable homogeneous structure such that every $A \in \mathscr{F}$ embeds into M and every finite substructure of M is isomorphic to some $A \in \mathscr{F}$. A Katětov functor is a pair $\langle K, \eta \rangle$ such that K assigns to each embedding $e\colon A \to B$ with $A, B \in \mathscr{F}$ an embedding $K(e)\colon M \to M$, η assigns to each $A \in \mathscr{F}$ an embedding $\eta_A\colon A \to M$. Furthermore, K is a functor, i.e., $K(\mathrm{id}_A) = \mathrm{id}_M$, $K(e \circ f) = K(e) \circ K(f)$, and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "e"'] & M \arrow[d, "K(e)"] \\ B \arrow[r, "\eta_B"'] & M \end{tikzcd}$$

for every embedding $e\colon A \to B$ with $A, B \in \mathscr{F}$.

Theorem (Mašulović & K.)

Assume $\langle \mathscr{F}, M \rangle$ admits a Katětov functor. Then for every substructure X of M there exists a topological group embedding

$$e_X \colon \operatorname{Aut}(X) \to \operatorname{Aut}(M).$$

Theorem (Mašulović & K.)

Assume $\langle \mathscr{F}, M\rangle$ admits a Katětov functor. Then for every substructure X of M there exists a topological group embedding

$$e_X \colon \operatorname{Aut}(X) \to \operatorname{Aut}(M).$$

Theorem

All well known homogeneous relational structures admit a Katětov functor.

Definition

A class of finite structures $\mathscr{F}$ is hereditary if for every $A \in \mathscr{F}$ it holds that

$$\{X\colon X \text{ is a substructure of } A\} \subseteq \mathscr{F}.$$

Definition

A class of finite structures $\mathscr{F}$ is hereditary if for every $A \in \mathscr{F}$ it holds that

$$\{X\colon X \text{ is a substructure of } A\} \subseteq \mathscr{F}.$$

Definition

A class of structures $\mathscr{F}$ has the amalgamation property if for every $C, A, B \in \mathscr{F}$, for every embeddings $f\colon C \to A$, $g\colon C \to B$ there exist $D \in \mathscr{F}$ and embeddings $f'\colon A \to D$, $g'\colon B \to D$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & D \end{tikzcd}$$

Theorem (Fraïssé 1954)

Let $\mathscr{F}$ be a countable hereditary class of relational structures with the amalgamation property. Then there exists a unique countable homogeneous structure M such that:

- *Every $A \in \mathscr{F}$ embeds into M.*
- *Every finite $B \subseteq M$ is isomorphic to some $B' \in \mathscr{F}$.*

Main results

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure E such that:

Main results

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,

Main results

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,
- S_∞ *does not embed into* $\mathrm{Aut}(E)$,

Main results

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,
- S_∞ *does not embed into* $\mathrm{Aut}(E)$,
- $S_\infty \approx \mathrm{Aut}(X)$ *for some* $X \subseteq E$.

Main results

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure E such that:

- *every finite group embeds into* $\mathrm{Aut}(E)$,
- S_∞ *does not embed into* $\mathrm{Aut}(E)$,
- $S_\infty \approx \mathrm{Aut}(X)$ *for some* $X \subseteq E$.

Furthermore, E is not uniformly homogeneous.

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure M such that:

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure M such that:

- $\mathrm{Aut}(M)$ *is torsion-free,*

Theorem (Shelah & K.)

There exists a countable homogeneous relational structure M such that:

- $\mathrm{Aut}(M)$ *is torsion-free,*
- *for every $n \in \mathbb{N}$ there is a finite $A \subseteq M$ with $S_n \approx \mathrm{Aut}(A)$.*

R. Fraïssé, *Sur lextension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954) 363–388

E. Jaligot, *On stabilizers of some moieties of the random tournament*, Combinatorica 27 (2007) 129–133

W. Kubiś, D. Mašulović, *Katětov functors*, Applied Categorical Structures 25 (2017) 569–602

W. Kubiś, S. Shelah, *Homogeneous structures with non-universal automorphism groups*, preprint, https://arxiv.org/abs/1811.09650

R. Fraïssé, *Sur lextension aux relations de quelques propriétés des ordres*, Ann. Sci. Ecole Norm. Sup. (3) 71 (1954) 363–388

E. Jaligot, *On stabilizers of some moieties of the random tournament*, Combinatorica 27 (2007) 129–133

W. Kubiś, D. Mašulović, *Katětov functors*, Applied Categorical Structures 25 (2017) 569–602

W. Kubiś, S. Shelah, *Homogeneous structures with non-universal automorphism groups*, preprint, `https://arxiv.org/abs/1811.09650`

Thank you for your attention!