

Category-theoretic methods for constructing universal Banach spaces

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Main goal

- Complementably universal Banach space with a basis
- Complementably universal Banach space with finite-dimensional Schauder decomposition
- Uncountable versions of the above spaces

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Categories with amalgamations

We say that \mathfrak{K} has the **amalgamation property** if

for every arrows $f: z \rightarrow x$, $g: z \rightarrow y$ there are arrows $f': x \rightarrow w$ and $g': y \rightarrow w$ such that $f' \circ f = g' \circ g$.

$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ g \uparrow & & \uparrow f' \\ z & \xrightarrow{f} & x \end{array}$$

The arrows $\langle f', g' \rangle$ provide a **pushout** of $\langle f, g \rangle$ if moreover for every \bar{f}, \bar{g} satisfying $\bar{f} \circ f = \bar{g} \circ g$ there exists a unique arrow h such that $h \circ f' = \bar{f}$ and $h \circ g' = \bar{g}$.

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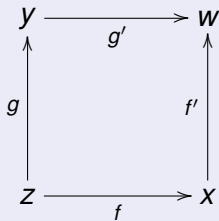
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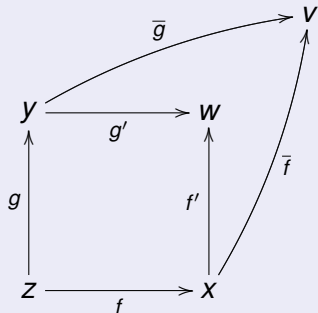
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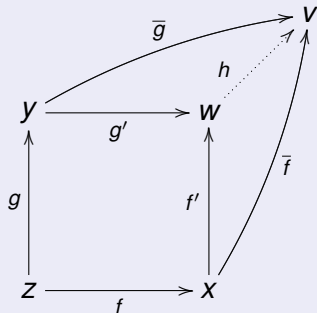
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Cofinality and homogeneity

- A family \mathcal{F} of objects of \mathfrak{K} is said to be **cofinal** in \mathfrak{K} if for every $x \in \mathfrak{K}$ there is $y \in \mathcal{F}$ such that $\mathfrak{K}(x, y) \neq \emptyset$.
- An object $u \in \mathfrak{K}$ is **cofinal** in \mathfrak{K} if for every $x \in \mathfrak{K}$ there is an arrow $f: x \rightarrow u$ in \mathfrak{K} .
- Let \mathfrak{L} be a subcategory of \mathfrak{K} . An object $u \in \mathfrak{K}$ is **\mathfrak{L} -homogeneous** if for every arrow $f: a \rightarrow b$ in \mathfrak{L} and for every arrows $i: a \rightarrow u$, $j: b \rightarrow u$ in \mathfrak{K} there exists an isomorphism $h: u \rightarrow u$ such that the diagram

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Sequences

By a **sequence** in a category \mathfrak{K} we mean a functor \vec{x} from an ordinal λ into \mathfrak{K} . A sequence \vec{x} of length λ can be described as a sequence $\{x_\alpha\}_{\alpha < \lambda}$ together with arrows $x_\alpha^\beta: x_\alpha \rightarrow x_\beta$ for $\alpha \leq \beta < \lambda$, such that

- 1 $x_\alpha^\alpha = \text{id}_{x_\alpha}$,
- 2 $\alpha < \beta < \gamma \implies x_\alpha^\gamma = x_\beta^\gamma \circ x_\alpha^\beta$.

Let \vec{x} and \vec{y} be sequences in \mathfrak{K} .

A **transformation** of \vec{x} into \vec{y} is a pair $\langle \varphi, \vec{f} \rangle$ such that

- 1 $\varphi: \lambda \rightarrow \delta$ is increasing;
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Arrows between sequences

- Let \vec{x}, \vec{y} be sequences in \mathfrak{K} and let $\langle \varphi, \vec{f} \rangle, \langle \psi, \vec{g} \rangle$ be transformations between them. We say that they are **equivalent** if all diagrams like

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Y_{\varphi(\alpha)} & \longrightarrow & Y_{\psi(\alpha)} & \longrightarrow & \cdots & \longrightarrow & Y_{\psi(\beta)} & \longrightarrow & Y_{\varphi(\beta)} & \longrightarrow & \cdots \\ & & \swarrow f_{\alpha} & & \uparrow g_{\alpha} & & & & \uparrow g_{\beta} & & \swarrow f_{\beta} & & \\ \cdots & \longrightarrow & & \longrightarrow & X_{\alpha} & \longrightarrow & \cdots & \longrightarrow & X_{\beta} & \longrightarrow & & \longrightarrow & \cdots \end{array}$$

are commutative.

- An **arrow of sequences** $\vec{x} \rightarrow \vec{y}$ is an equivalence class of this relation. We write $\vec{f}: \vec{x} \rightarrow \vec{y}$, having in mind the equivalence class of the transformation $\vec{f} = \{f_{\alpha}\}_{\alpha < \lambda}$.

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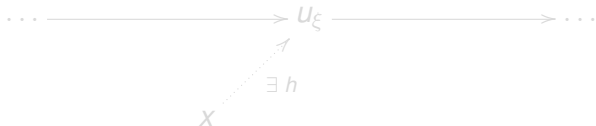
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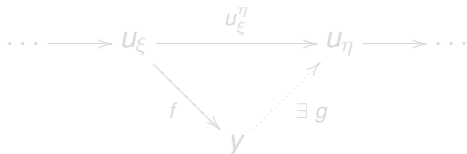
Definition:

Let \mathfrak{K} be a fixed category. A κ -**Fraïssé sequence** in \mathfrak{K} is a sequence \vec{u} satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.



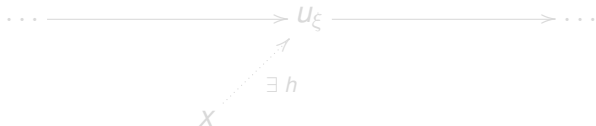
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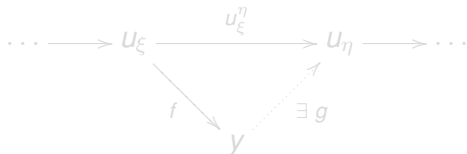
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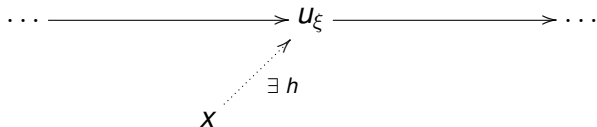
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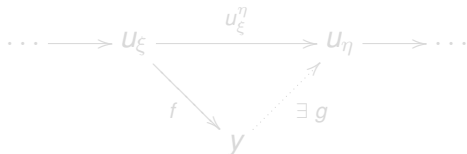
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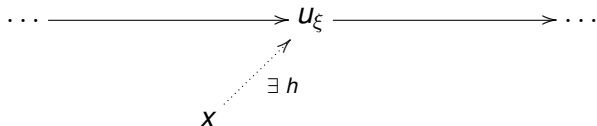
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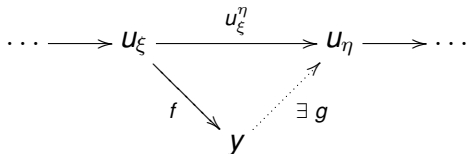
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The existence

$\mathfrak{S}_{<\kappa}(\mathfrak{K})$ = the category of all sequences in \mathfrak{K} of length $< \kappa$.

A category \mathfrak{K} is **κ -bounded** if for every sequence $\vec{x} \in \mathfrak{S}_{<\kappa}(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $F: \vec{x} \rightarrow a$.

Theorem

Let $\kappa > 1$ be a regular cardinal and let \mathfrak{K} be a κ -bounded category which has the amalgamation property and the joint embedding property. Assume further that \mathfrak{K} has at most κ isomorphic types of arrows.

Then there exists a Fraïssé sequence $\vec{u}: \kappa \rightarrow \mathfrak{K}$.

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Countable Fraïssé sequences

Theorem (Countable Cofinality)

Assume \vec{u} is a Fraïssé sequence in a category with amalgamation \mathfrak{K} . Then for every countable sequence \vec{x} in \mathfrak{K} there exists an arrow $\vec{f}: \vec{x} \rightarrow \vec{u}$.

Corollary

Let \vec{u} be a countable Fraïssé sequence in a category \mathfrak{K} . If \mathfrak{K} has the amalgamation property then \vec{u} is cofinal in $\mathfrak{S}_\omega(\mathfrak{K})$.

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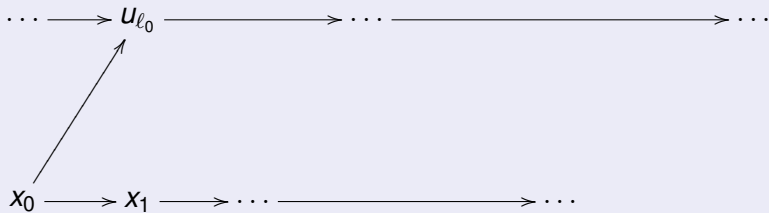
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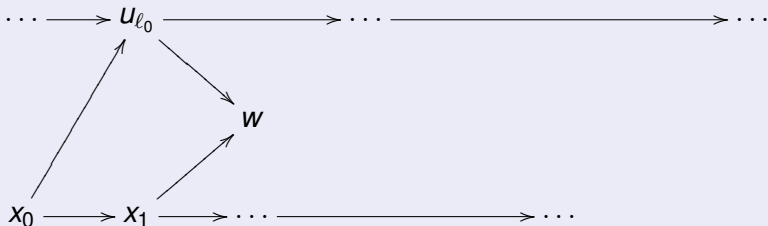
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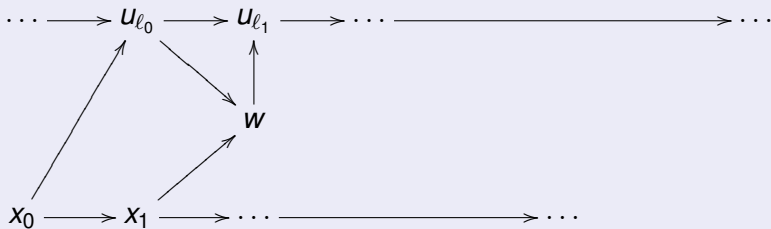
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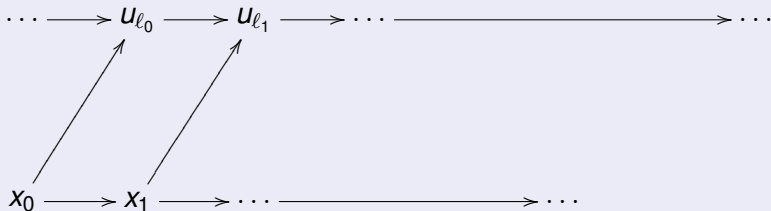
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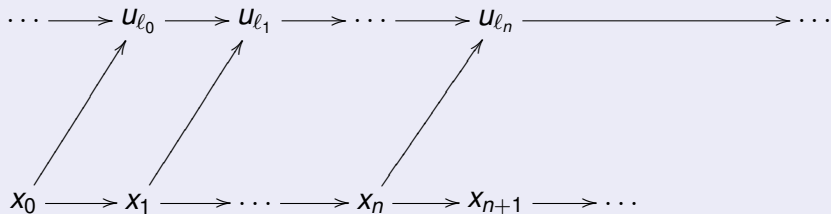
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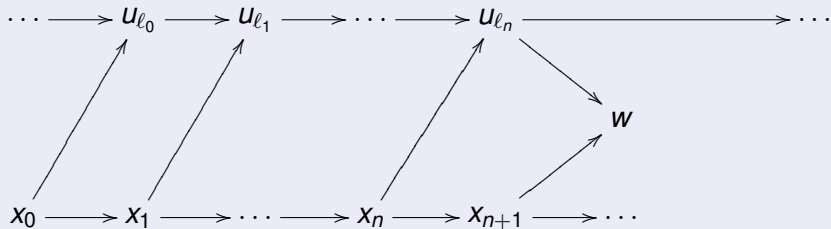
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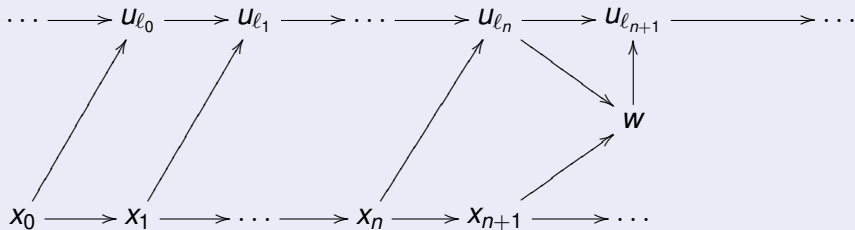
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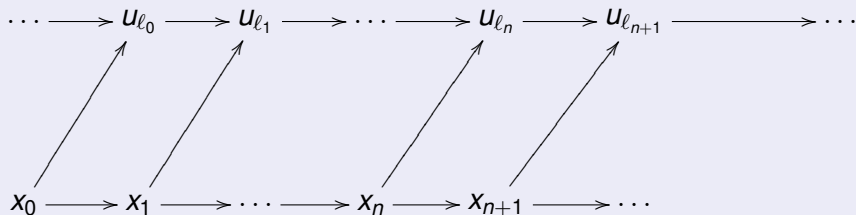
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Homogeneity & Uniqueness

Theorem

Assume that \vec{u}, \vec{v} are ω -Fraïssé sequences in a fixed category \mathfrak{K} .

- (a) Let $f: u_k \rightarrow v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ such that $F \circ u_k = v_\ell \circ f$. In particular $\vec{u} \approx \vec{v}$.
- (b) Assume \mathfrak{K} has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f: a \rightarrow b$, $i: a \rightarrow \vec{u}$, $j: b \rightarrow \vec{v}$ there exists an isomorphism $F: \vec{u} \rightarrow \vec{v}$ such that $F \circ i = j \circ f$.

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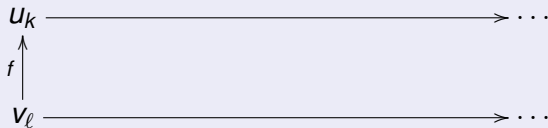
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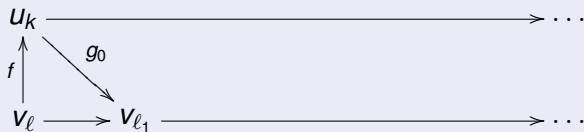
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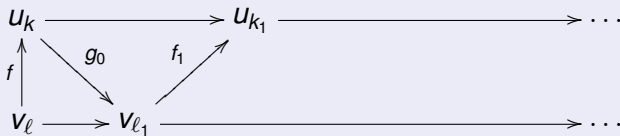
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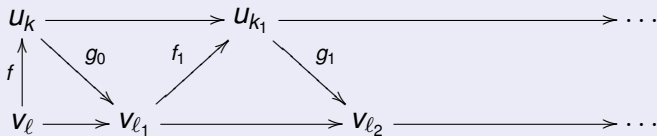
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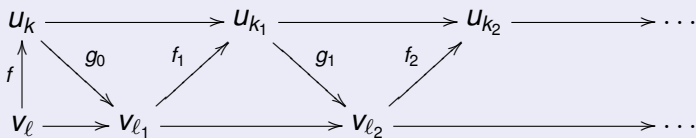
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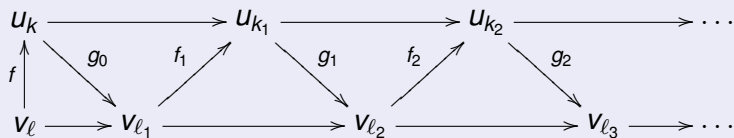
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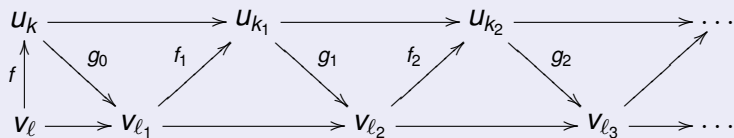
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The Gurarii space

Let \mathfrak{B}_{fd} denote the category of finite-dimensional Banach spaces with linear operators of norm ≤ 1 .

Claim

The subcategory of \mathfrak{B}_{fd} consisting of isometric embeddings has the amalgamation property.

The problem is that there are too many arrows...

Theorem (Gurarii, 1966)

There exists a separable Banach space \mathbb{G} with the following property:

(UD) *Given finite-dimensional spaces $X \subseteq Y$, given an isometric embedding $f: X \rightarrow \mathbb{G}$, given $\varepsilon > 0$, there exists an ε -isometric embedding $\bar{f}: Y \rightarrow \mathbb{G}$ such that $\bar{f} \upharpoonright X = f$.*

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Retractive pairs

Given a category \mathfrak{K} , define a new category $\dagger\mathfrak{K}$ as follows.

- The objects of $\dagger\mathfrak{K}$ are the objects of \mathfrak{K} .
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There are two natural functors $e: \dagger\mathfrak{K} \rightarrow \mathfrak{K}$ and $r: \dagger\mathfrak{K} \rightarrow \mathfrak{K}$.

A sequence \vec{x} in $\dagger\mathfrak{K}$ will be called **semicontinuous** if $e[\vec{x}]$ is continuous in \mathfrak{K} .

Example

Let \mathfrak{B} be the category of Banach spaces with linear transformations of norm ≤ 1 . A semicontinuous sequence \vec{x} in $\dagger\mathfrak{B}$ corresponds to a PRI in X , where X is the colimit of $e[\vec{x}]$ in the category \mathfrak{B} .

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Let \mathfrak{K} be a category and let \vec{u} and \vec{v} be semicontinuous Fraïssé sequences in $\ddagger\mathfrak{K}$ of the same regular length κ . Then for every arrow $f: u_0 \rightarrow \vec{v}$ there exists an isomorphism of sequences $\vec{f}: \vec{u} \rightarrow \vec{v}$ such that $\vec{f} \circ u_0^\infty = f$.

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Let $f: z \rightarrow x$, $g: z \rightarrow y$ be arrows in $\dagger\mathfrak{K}$. If $\langle e(f), e(g) \rangle$ has a pushout in \mathfrak{K} , then $\langle f, g \rangle$ has a **proper** amalgamation in $\dagger\mathfrak{K}$. That is, there exist arrows $h: x \rightarrow w$, $k: y \rightarrow w$ in $\dagger\mathfrak{K}$ such that the following diagrams commute in \mathfrak{K} .

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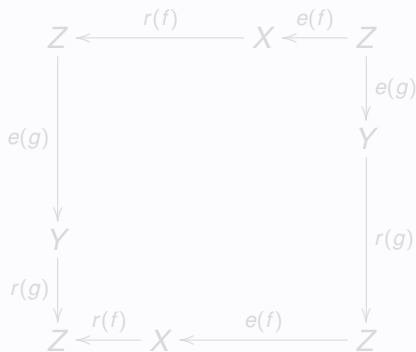
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Claim

If \mathfrak{K} has pullbacks or pushouts then $\ddagger\mathfrak{K}$ has proper amalgamations.

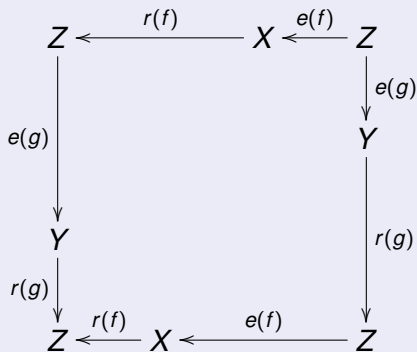
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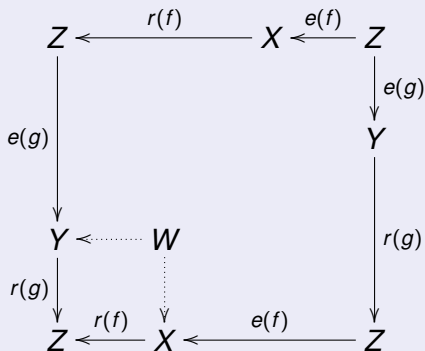
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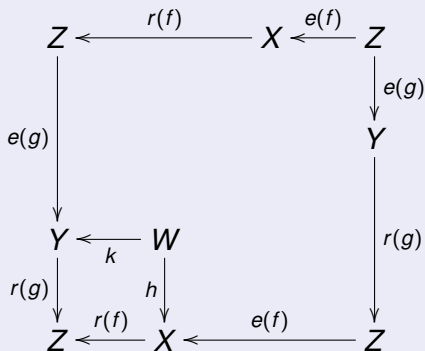
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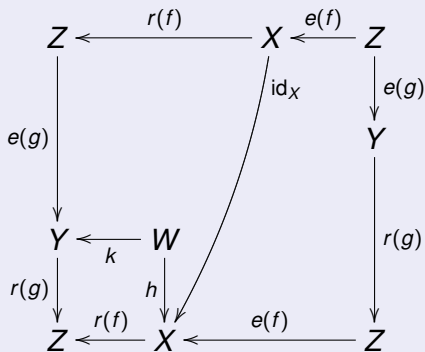
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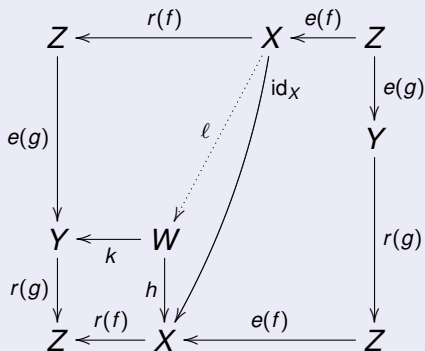
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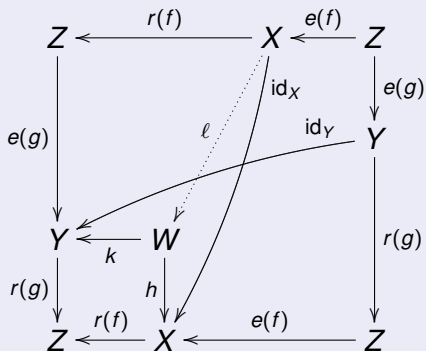
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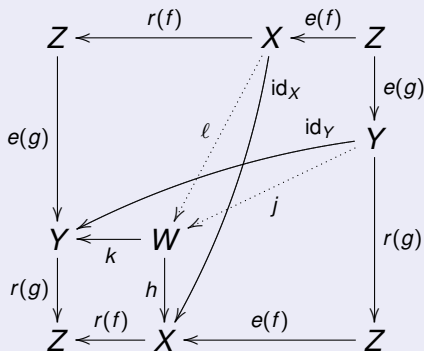
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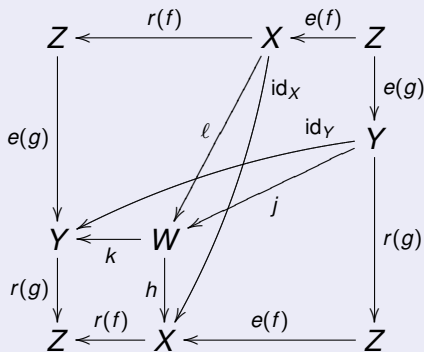
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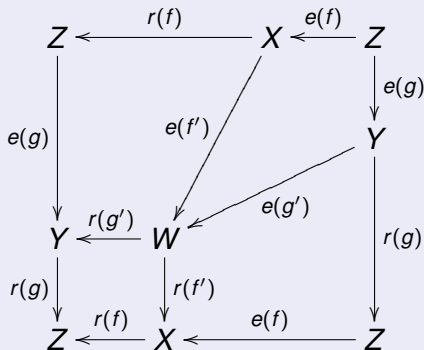
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Theorem

Let \mathfrak{K} be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Assume \vec{u} is a semi-continuous κ -Fraïssé sequence in $\ddagger\mathfrak{K}$.

Then for every semi-continuous sequence $\vec{x} \in \mathfrak{G}_{\leq \kappa}(\ddagger\mathfrak{K})$ there exists an arrow of sequences $\vec{f}: \vec{x} \rightarrow \vec{u}$ in $\ddagger\mathfrak{K}$.

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Let $\mathfrak{B}_{\text{sep}}$ be the category of all separable Banach spaces with linear transformations of norm ≤ 1 .

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Left-invertible arrows have pushouts in $\mathfrak{B}_{\text{sep}}$.

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The category $\mathfrak{B}_{\text{sep}}$ has 2^{\aleph_0} many isomorphic types of arrows.

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Assume $2^{\aleph_0} = \aleph_1$. Then there exists a semicontinuous ω_1 -Fraïssé sequence in $\mathfrak{B}_{\text{sep}}$.

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Assume $2^{\aleph_0} = \aleph_1$.

There exists a Banach space E with a PRI $\{P_\alpha\}_{\alpha < \omega_1}$ and of density \aleph_1 , which has the following properties:

- (a) The family $\{X \subseteq E : X \text{ is 1-complemented in } E\}$ is, modulo linear isometries, the class of all Banach spaces of density $\leq \aleph_1$ with a PRI.
- (b) Given separable subspaces $X, Y \subseteq E$, norm one projections $P: E \rightarrow X, Q: E \rightarrow Y$, both compatible with $\{P_\alpha\}_{\alpha < \omega_1}$, and given a linear isometry $T: X \rightarrow Y$, there exist a linear isometry $H: E \rightarrow E$ extending T and satisfying $H \circ P = Q \circ H$.

Moreover, the above properties describe the space E uniquely, up to a linear isometry.

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