Category-theoretic methods for constructing universal Banach spaces

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Outline

Main goal

- Complementably universal Banach space with a basis
- Complementably universal Banach space with finite-dimensional Schauder decomposition
- Uncountable versions of the above spaces

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Categories with amalgamations

We say that $\mathfrak{K}$ has the amalgamation property if

for every arrows $f\colon z \to x$, $g\colon z \to y$ there are arrows $f'\colon x \to w$ and $g'\colon y \to w$ such that $f' \circ f = g' \circ g$.

$$\begin{tikzcd} y \arrow[r, "g'"] & w \\ z \arrow[u, "g"] \arrow[r, "f"'] & x \arrow[u, "f'"'] \end{tikzcd}$$

The arrows $\langle f', g' \rangle$ provide a pushout of $\langle f, g \rangle$ if moreover for every $\overline{f}, \overline{g}$ satisfying $\overline{f} \circ f = \overline{g} \circ g$ there exists a unique arrow h such that $h \circ f' = \overline{f}$ and $h \circ g' = \overline{g}$.

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$$\begin{array}{ccc} y & \xrightarrow{g'} & w \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f'} \\ z & \xrightarrow[f]{} & x \end{array}$$

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The pushout of $\langle f, g \rangle$

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\begin{tikzcd}
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Cofinality and homogeneity

- A family $\mathcal{F}$ of objects of $\mathfrak{K}$ is said to be cofinal in $\mathfrak{K}$ if for every $x \in \mathfrak{K}$ there is $y \in \mathcal{F}$ such that $\mathfrak{K}(x, y) \neq \emptyset$.
- An object $u \in \mathfrak{K}$ is cofinal in $\mathfrak{K}$ if for every $x \in \mathfrak{K}$ there is an arrow $f \colon x \to u$ in $\mathfrak{K}$.
- Let $\mathfrak{L}$ be a subcategory of $\mathfrak{K}$. An object $u \in \mathfrak{K}$ is $\mathfrak{L}$-homogeneous if for every arrow $f \colon a \to b$ in $\mathfrak{L}$ and for every arrows $i \colon a \to u$, $j \colon b \to u$ in $\mathfrak{K}$ there exists an isomorphism $h \colon u \to u$ such that the diagram

$$\begin{tikzcd} u \arrow[r, "h"] & u \\ a \arrow[u, "i"] \arrow[r, "f"'] & b \arrow[u, "j"'] \end{tikzcd}$$

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Sequences

By a sequence in a category $\mathfrak{K}$ we mean a functor $\vec{x}$ from an ordinal λ into $\mathfrak{K}$. A sequence $\vec{x}$ of length λ can be described as a sequence $\{x_\alpha\}_{\alpha<\lambda}$ together with arrows $x_\alpha^\beta \colon x_\alpha \to x_\beta$ for $\alpha \leqslant \beta < \lambda$, such that

1. $x_\alpha^\alpha = \mathrm{id}_{x_\alpha}$,
2. $\alpha < \beta < \gamma \implies x_\alpha^\gamma = x_\beta^\gamma \circ x_\alpha^\beta$.

Let $\vec{x}$ and $\vec{y}$ be sequences in $\mathfrak{K}$.
A transformation of $\vec{x}$ into $\vec{y}$ is a pair $\langle \varphi, \vec{f} \rangle$ such that

1. $\varphi \colon \lambda \to \delta$ is increasing;
2. $\vec{f} = \{f_\alpha\}_{\alpha<\lambda}$, where $f_\alpha \colon x_\alpha \to y_{\varphi(\alpha)}$;
3. $\alpha < \beta \implies f_\beta \circ x_\alpha^\beta = y_{\varphi(\alpha)}^{\varphi(\beta)} \circ f_\alpha$.

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Arrows between sequences

- Let $\vec{x}$, $\vec{y}$ be sequences in $\mathfrak{K}$ and let $\langle \varphi, \vec{f} \rangle$, $\langle \psi, \vec{g} \rangle$ be transformations between them. We say that they are equivalent if all diagrams like

$$
\begin{tikzcd}
\cdots \arrow[r] & y_{\varphi(\alpha)} \arrow[r] & y_{\psi(\alpha)} \arrow[r] & \cdots \arrow[r] & y_{\psi(\beta)} \arrow[r] & y_{\varphi(\beta)} \arrow[r] & \cdots \\
\cdots \arrow[rr] & & x_\alpha \arrow[ul, "f_\alpha"] \arrow[u, "g_\alpha"'] \arrow[r] & \cdots \arrow[r] & x_\beta \arrow[u, "g_\beta"] \arrow[ur, "f_\beta"'] \arrow[rr] & & \cdots
\end{tikzcd}
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  are commutative.
- An arrow of sequences $\vec{x} \to \vec{y}$ is an equivalence class of this relation. We write $\vec{f} \colon \vec{x} \to \vec{y}$, having in mind the equivalence class of the transformation $\vec{f} = \{f_\alpha\}_{\alpha<\lambda}$.

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Definition:

Let $\mathfrak{K}$ be a fixed category. A κ-Fraïssé sequence in $\mathfrak{K}$ is a sequence $\vec{u}$ satisfying the following conditions:

(U) For every $x \in \mathfrak{K}$ there exists $\xi < \kappa$ such that $\mathfrak{K}(x, u_\xi) \neq \emptyset$.

$$\begin{tikzcd}
\cdots \arrow[rr] & & u_\xi \arrow[rr] & & \cdots \\
& x \arrow[ur, dotted, "\exists\, h"'] & & &
\end{tikzcd}$$

(A) For every $\xi < \kappa$ and for every arrow $f \in \mathfrak{K}(u_\xi, y)$, where $y \in \mathfrak{K}$, there exist $\eta \geqslant \xi$ and $g \in \mathfrak{K}(y, u_\eta)$ such that $u_\xi^\eta = g \circ f$.

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The existence

$\mathfrak{S}_{<\kappa}(\mathfrak{K})$ = the category of all sequences in $\mathfrak{K}$ of length $< \kappa$.

A category $\mathfrak{K}$ is κ-**bounded** if for every sequence $\vec{x} \in \mathfrak{S}_{<\kappa}(\mathfrak{K})$ there are $a \in \mathfrak{K}$ and an arrow of sequences $F \colon \vec{x} \to a$.

Theorem

Let $\kappa > 1$ be a regular cardinal and let $\mathfrak{K}$ be a κ-bounded category which has the amalgamation property and the joint embedding property. Assume further that $\mathfrak{K}$ has at most κ isomorphic types of arrows.

Then there exists a Fraïssé sequence $\vec{u} \colon \kappa \to \mathfrak{K}$.

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Countable Fraïssé sequences

Theorem (Countable Cofinality)

Assume $\vec{u}$ is a Fraïssé sequence in a category with amalgamation $\mathfrak{K}$. Then for every countable sequence $\vec{x}$ in $\mathfrak{K}$ there exists an arrow $\vec{f}\colon \vec{x} \to \vec{u}$.

Corollary

Let $\vec{u}$ be a countable Fraïssé sequence in a category $\mathfrak{K}$. If $\mathfrak{K}$ has the amalgamation property then $\vec{u}$ is cofinal in $\mathfrak{S}_\omega(\mathfrak{K})$.

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Cofinality

$$\cdots \longrightarrow u_{\ell_0} \longrightarrow \cdots \longrightarrow \cdots$$

$$x_0 \longrightarrow x_1 \longrightarrow \cdots \longrightarrow \cdots$$

Cofinality

$$\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[rr] & & \cdots \arrow[rr] & & \cdots \\
x_0 \arrow[ur] \arrow[r] & x_1 \arrow[r] & \cdots \arrow[r] & \cdots & &
\end{tikzcd}$$

Cofinality

$$\begin{tikzcd}
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& & w & & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & \cdots &
\end{tikzcd}$$

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Cofinality

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\begin{tikzcd}
\cdots \arrow[r] & u_{\ell_0} \arrow[r] & u_{\ell_1} \arrow[r] & \cdots \arrow[r] & u_{\ell_n} \arrow[r] & \cdots \\
x_0 \arrow[r] \arrow[ur] & x_1 \arrow[r] \arrow[ur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[ur] & x_{n+1} \arrow[r] & \cdots
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 & & & & & w & \\
x_0 \arrow[r] \arrow[uur] & x_1 \arrow[r] \arrow[uur] & \cdots \arrow[r] & x_n \arrow[r] \arrow[uur] & x_{n+1} \arrow[r] \arrow[ur] & \cdots &
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Homogeneity & Uniqueness

Theorem

Assume that $\vec{u}$, $\vec{v}$ are ω-Fraïssé sequences in a fixed category $\mathfrak{K}$.

(a) *Let $f\colon u_k \to v_\ell$, where $k, \ell < \omega$. Then there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $F \circ u_k = v_\ell \circ f$. In particular $\vec{u} \approx \vec{v}$.*

(b) *Assume $\mathfrak{K}$ has the amalgamation property. Then for every $a, b \in \mathfrak{K}$ and for every arrows $f\colon a \to b$, $i\colon a \to \vec{u}$, $j\colon b \to \vec{v}$ there exists an isomorphism $F\colon \vec{u} \to \vec{v}$ such that $F \circ i = j \circ f$.*

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Back-and-forth method

$$
\begin{tikzcd}
u_k \arrow[rrrr] & & & & \cdots \\
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$$\begin{tikzcd}
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The Gurarii space

Let $\mathfrak{B}_{\mathrm{fd}}$ denote the category of finite-dimensional Banach spaces with linear operators of norm $\leqslant 1$.

Claim

The subcategory of $\mathfrak{B}_{\mathrm{fd}}$ consisting of isometric embeddings has the amalgamation property.

The problem is that there are too many arrows...

Theorem (Gurarii, 1966)

There exists a separable Banach space $\mathbb{G}$ with the following property:

(UD) *Given finite-dimensional spaces $X \subseteq Y$, given an isometric embedding $f\colon X \to \mathbb{G}$, given $\varepsilon > 0$, there exists an ε-isometric embedding $\bar{f}\colon Y \to \mathbb{G}$ such that $\bar{f} \restriction X = f$.*

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Retractive pairs

Given a category $\mathfrak{K}$, define a new category $\ddagger\mathfrak{K}$ as follows.

- The objects of $\ddagger\mathfrak{K}$ are the objects of $\mathfrak{K}$.
- An arrow from x into y in $\ddagger\mathfrak{K}$ is a pair $\langle e, r \rangle$ such that $e \colon x \to y$, $r \colon y \to x$ are arrows of $\mathfrak{K}$ and $r \circ e = \mathrm{id}_x$.

There are two natural functors $e \colon \ddagger\mathfrak{K} \to \mathfrak{K}$ and $r \colon \ddagger\mathfrak{K} \to \mathfrak{K}$.
A sequence $\vec{x}$ in $\ddagger\mathfrak{K}$ will be called semicontinuous if $e[\vec{x}]$ is continuous in $\mathfrak{K}$.

Example

Let $\mathfrak{B}$ be the category of Banach spaces with linear transformations of norm $\leqslant 1$. A semicontinuous sequence $\vec{x}$ in $\ddagger\mathfrak{B}$ corresponds to a PRI in X, where X is the colimit of $e[\vec{x}]$ in the category $\mathfrak{B}$.

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Theorem

Let $\mathfrak{K}$ be a category and let $\vec{u}$ and $\vec{v}$ be semicontinuous Fraïssé sequences in $\ddagger\mathfrak{K}$ of the same regular length κ. Then for every arrow $f \colon u_0 \to \vec{v}$ there exists an isomorphism of sequences $\vec{f} \colon \vec{u} \to \vec{v}$ such that $\vec{f} \circ u_0^\infty = f$.

$$\begin{tikzcd} \vec{u} \arrow[r, "\vec{f}"] & \vec{v} \\ u_0 \arrow[u, "u_0^\infty"] \arrow[ur, "f"'] & \end{tikzcd}$$

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Let $f\colon z \to x$, $g\colon z \to y$ be arrows in $\ddagger\mathfrak{K}$. If $\langle e(f), e(g)\rangle$ has a pushout in $\mathfrak{K}$, then $\langle f, g\rangle$ has a proper *amalgamation in $\ddagger\mathfrak{K}$. That is, there exist arrows $h\colon x \to w$, $k\colon y \to w$ in $\ddagger\mathfrak{K}$ such that the following diagrams commute in $\mathfrak{K}$.*

$$\begin{tikzcd} w & y \arrow[l, "e(k)"'] \\ x \arrow[u, "e(h)"] & z \arrow[l, "e(f)"'] \arrow[u, "e(g)"'] \end{tikzcd} \quad \begin{tikzcd} w \arrow[r, "r(k)"] \arrow[d, "r(h)"'] & y \arrow[d, "r(g)"] \\ x \arrow[r, "r(f)"] & z \end{tikzcd} \quad \begin{tikzcd} w \arrow[r, "r(k)"] & y \\ x \arrow[u, "e(h)"] \arrow[r, "r(f)"] & z \arrow[u, "e(g)"'] \end{tikzcd} \quad \begin{tikzcd} w \arrow[d, "r(h)"'] & y \arrow[l, "e(k)"'] \arrow[d, "r(g)"] \\ x & z \arrow[l, "e(f)"'] \end{tikzcd}$$

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Claim

If $\mathfrak{K}$ has pullbacks or pushouts then $\ddagger\mathfrak{K}$ has proper amalgamations.

Proof.

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\begin{tikzcd}
Z \arrow[dd, "e(g)"'] & & X \arrow[ll, "r(f)"'] & Z \arrow[l, "e(f)"'] \arrow[d, "e(g)"] \\
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Z \arrow[dd, "e(g)"'] & & X \arrow[ll, "r(f)"'] \arrow[ddl, "e(f')"'] & Z \arrow[l, "e(f)"'] \arrow[d, "e(g)"] \\
& & & Y \arrow[dll, "e(g')"] \arrow[dd, "r(g)"] \\
Y \arrow[d, "r(g)"'] & W \arrow[l, "r(g')"'] \arrow[d, "r(f')"] & & \\
Z & X \arrow[l, "r(f)"'] & & Z \arrow[ll, "e(f)"']
\end{tikzcd}$$

□

Theorem

Let $\mathfrak{K}$ be a category such that $\ddagger\mathfrak{K}$ has proper amalgamations. Assume $\vec{u}$ is a semi-continuous κ-Fraïssé sequence in $\ddagger\mathfrak{K}$.

Then for every semi-continuous sequence $\vec{x} \in \mathfrak{S}_{\leqslant\kappa}(\ddagger\mathfrak{K})$ there exists an arrow of sequences $\vec{f} \colon \vec{x} \to \vec{u}$ in $\ddagger\mathfrak{K}$.

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Let $\mathfrak{B}_{\text{sep}}$ be the category of all separable Banach spaces with linear transformations of norm $\leqslant 1$.

Claim

Left-invertible arrows have pushouts in $\mathfrak{B}_{\text{sep}}$.

Claim

The category $\mathfrak{B}_{\text{sep}}$ has $2^{\aleph_0}$ many isomorphic types of arrows.

Theorem

Assume $2^{\aleph_0} = \aleph_1$. Then there exists a semicontinuous ω_1-Fraïssé sequence in $\ddagger\mathfrak{B}_{\text{sep}}$.

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Assume $2^{\aleph_0} = \aleph_1$.
There exists a Banach space E *with a PRI* $\{P_\alpha\}_{\alpha<\omega_1}$ *and of density* $\aleph_1$, *which has the following properties:*

(a) *The family* $\{X \subseteq E \colon X \text{ is } 1\text{-complemented in } E\}$ *is, modulo linear isometries, the class of all Banach spaces of density* $\leqslant \aleph_1$ *with a PRI.*

(b) *Given separable subspaces* $X, Y \subseteq E$, *norm one projections* $P \colon E \to X$, $Q \colon E \to Y$, *both compatible with* $\{P_\alpha\}_{\alpha<\omega_1}$, *and given a linear isometry* $T \colon X \to Y$, *there exist a linear isometry* $H \colon E \to E$ *extending* T *and satisfying* $H \circ P = Q \circ H$.

Moreover, the above properties describe the space E *uniquely, up to a linear isometry.*

Selected bibliography

- DROSTE, M.; GÖBEL, R., *A categorical theorem on universal objects and its application in abelian group theory and computer science*, Proceedings of the International Conference on Algebra, Part 3 (Novosibirsk, 1989), 49–74, Contemp. Math., 131, Part 3, Amer. Math. Soc., Providence, RI, 1992
- FRAÏSSÉ, R., *Sur quelques classifications des systèmes de relations*, Publ. Sci. Univ. Alger. Sér. A. **1** (1954) 35–182
- IRWIN, T.; SOLECKI, S., *Projective Fraïssé limits and the pseudo-arc*, Trans. Amer. Math. Soc. **358**, no. 7 (2006) 3077–3096
- JÓNSSON, B., *Homogeneous universal relational systems*, Math. Scand. 8 (1960) 137–142
- KUBIŚ, W., Fraïssé sequences: category-theoretic approach to universal homogeneous structures, preprint `http://arxiv.org/abs/0711.1683`