Abstract Evolution Systems

Wiesław Kubiś

Institute of Mathematics, CAS



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Joint work with Paulina Radecka.



A rewriting system is a pair (X, \rightarrow) , where \rightarrow is a binary (typically irreflexive) relation on X. The elements of X are called states and the elements of \rightarrow are called transitions. A sequence of transitions

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Definition

A rewriting system (X, \rightarrow) is locally confluent if for every state z, for every transitions $z \rightarrow x$, $z \rightarrow y$ there exist a state w and paths $x \rightarrow \cdots \rightarrow w$, $y \rightarrow \cdots \rightarrow w$.

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Remark

In general, local confluence does not imply confluence.

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A rewriting system (X, \rightarrow) is terminating if there is no infinite path

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Corollary

A locally confluent terminating rewriting system with an origin has a unique normalized state.

The setup

Let \mathcal{F} be a class of finite structures, closed under isomorphisms. We say that \mathcal{F} has the amalgamation property if for every $Z \in \mathcal{F}$, for every embeddings $f: Z \to X$, $g: Z \to Y$ with $X, Y \in \mathcal{F}$, there exist $W \in \mathcal{F}$ and embeddings $f': X \to W$, $g': Y \to W$ such that $f' \circ f = g' \circ g$. Let us assume \mathcal{F} has the origin, namely, a "trivial" structure that embeds into any other one.

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Theorem

Assume \mathcal{F} is as above, and there is a natural number N such that $|X| \leq N$ for every $X \in \mathcal{F}$.

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Theorem

Assume \mathcal{F} is as above, and there is a natural number N such that $|X| \leq N$ for every $X \in \mathcal{F}$. Then there exists a unique, up to isomorphism, structure $V \in \mathcal{F}$

satisfying

- (1) Every $X \in \mathcal{F}$ embeds into V.
- (2) Every isomorphism between \mathcal{F} -substructures of V extends to an automorphism of V.

Claim

Confluent terminating rewriting systems and finite homogeneous structures are two faces of the same theory.

An evolution system is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where \mathfrak{V} is a category, Θ is a fixed \mathfrak{V} -object, called the origin, and \mathcal{T} is a class of \mathfrak{V} -arrows, called transitions.

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The only requirements are:

- **1** All identities are in \mathcal{T} .
- **2** $h \circ t \in \mathcal{T}$, whenever $t \in \mathcal{T}$ and h is an isomorphism.

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where each of the arrows above is a transition.

A path is a finite composition of transitions.

An object X is finite if there exists a path from the origin Θ to X.

We say that \mathcal{E} is confluent if for every finite object Z, for every **paths** $f: Z \to X$, $g: Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

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We say that \mathcal{E} is locally confluent if for every finite object Z, for every **transitions** $f: Z \to X$, $g: Z \to Y$ there exist paths f', g' such that $f' \circ f = g' \circ g$.

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Finite amalgamation property implies confluence.

An evolution system ${\ensuremath{\mathcal E}}$ is terminating if there is no evolution consisting of nontrivial transitions.

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An object N is normalized if it is finite and every path starting from N consists of trivial transitions.

Proposition

Assume \mathcal{E} is a confluent terminating evolution system. Then there exists a unique, up to isomorphism normalized object N. Furthermore:

- **I** Every finite object has a path into N.
- **2** For every paths $f_0, f_1 : A \to N$ with A finite, there exists an automorphism $h : N \to N$ such that $f_1 = h \circ f_0$.

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Given a rewriting system (X, \rightarrow) , define $x \leq y$ if there is a path

$$x = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n = y.$$

Fix $\Theta \in X$. Then $\mathbb{X} = \langle (X, \leq), \rightarrow, \Theta \rangle$ becomes an evolution system.

Theorem (P. Nowakowski & W.K. 2021)

The smallest projective planes $\mathbb{Z}_2 \mathbf{P}^2$, $\mathbb{Z}_3 \mathbf{P}^2$ are homogeneous. No other projective plane is homogeneous.

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with the absorption property, namely, for every n, for every transition $f: U_n \to Y$ there are m > n and a path $g: Y \to U_m$ with $g \circ f = u_n^m$, where u_n^m is the path from U_n to U_m extracted from \vec{u} .

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Theorem

Assume \mathcal{E} is an essentially countable evolution system with the finite amalgamation property. Then:

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- **2** The colimit $U_{\infty} = \lim \vec{u}$ is defined uniquely (up to isomorphism) by the absorption property.
- **3** U_{∞} is homogeneous with respect to paths between finite objects.

Fraïssé theory revisited

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We can convert \mathcal{F} to an evolution system, where the origin is \emptyset and transitions are one-point extensions. The category \mathfrak{V} consists of all structures in the language \mathcal{L} , where the arrows are all homomorphisms.

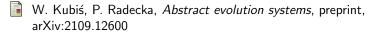
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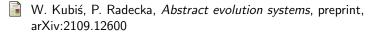
We can convert \mathcal{F} to an evolution system, where the origin is \emptyset and transitions are one-point extensions. The category \mathfrak{V} consists of all structures in the language \mathcal{L} , where the arrows are all homomorphisms. Assume \mathcal{F} has the amalgamation property. Then the generic evolution leads to the Fraïssé limit of \mathcal{F} , the unique homogeneous countable structure U_{∞} whose age is \mathcal{F} .



W. Kubiś, P. Radecka, *Abstract evolution systems*, preprint, arXiv:2109.12600

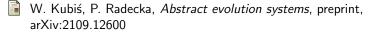


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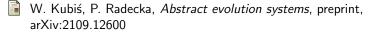
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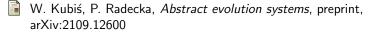
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THE END