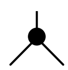
Deformations of algebras and their diagrams

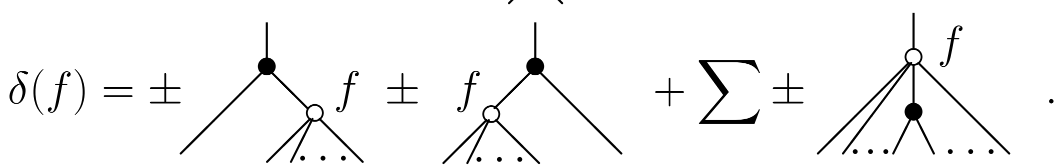
We will work over a fixed characteristic zero field $\mathbf{k}$. Everyone knows that deformations of an associative algebra (A, μ) are controlled by the Hochschild cohomology $H^*(A, A)$, which is the cohomology of

$$0 \xrightarrow{\delta} C^0(A,A) \xrightarrow{\delta} C^1(A,A) \xrightarrow{\delta} \cdots \xrightarrow{\delta} C^n(A,A) \xrightarrow{\delta} \cdots$$

where $C^n(A,A) := Lin(A^{\otimes n+1}, A)$ and the coboundary δ given by

$$\delta f(a_0 \otimes \cdots \otimes a_n) := (-1)^{n+1} a_0 f(a_1 \otimes \cdots \otimes a_n) + f(a_0 \otimes \cdots \otimes a_{n-1}) a_n$$
$$+ \sum_{i=0}^{n-1} (-1)^{i+n} f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n).$$

Graphically, modulo signs, with  symbolizing the multiplication,



By 'controlled by' we usually mean that

– $H^1(A, A)$ classifies infinitesimal deformations and
– $H^2(A, A)$ contains obstructions for their extensions.

More precisely, $C^*(A, A)$ carries the Gerstenhaber bracket $[-,-]$ which turns it into a dg-Lie algebra

$$\mathfrak{g} := (C^*(A,A), [-,-], \delta).$$

Let L be the Lie group $\mathfrak{g} \otimes (t) \subset \mathfrak{g} \otimes \mathbf{k}[[t]]$. Consider the solutions of the Maurer-Cartan equation in L

$$\mathrm{MC}(\mathfrak{g}) := \{s \in L^1;\ \delta s + \frac{1}{2}[s,s] = 0\},\quad \mathrm{G}(\mathfrak{g}) := \exp(L^0).$$

The moduli space of formal deformations of μ equals the quotient $\mathfrak{Def}(\mathfrak{g}) = \mathrm{MC}(\mathfrak{g})/\mathrm{G}(\mathfrak{g})$.

Our aim is to show that the same scheme holds for a wide class of algebras and their diagrams, though instead of dg-Lie one sometimes needs an L_∞-algebra. We will show how to construct, for a (diagram of) algebra(s) A belonging to a specified class of structures, an L_∞-algebra $\mathfrak{g} = (C^*(A,A), \delta = l_1, l_2, \ldots)$ governing its deformations.

We will focus on explicit calculations and examples. We, in particular, show that deformations of morphisms are controlled by a fully-fledged L_∞-structure. We give an example where a 'curved' (= with l_0-term) L_∞-algebra occurs. We also show that L_∞-deformation algebras are required for deformations of exotic structures.

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By a 'class of structures' we mean algebras over a (colored, in the case of diagrams) $\mathbf{k}$-vector space operad $\mathcal{P}$. We assume that operads are familiar; yet we we recall that:

$\triangleright$ An operad $\mathcal{P}$ is a collection $\{\mathcal{P}(n)\}_{n\geq 1}$ of vector spaces together with composition operations

$$\circ_i : \mathcal{P}(m) \otimes \mathcal{P}(n) \to \mathcal{P}(m+n-1), \quad 1 \leq i \leq m.$$

Each $\mathcal{P}(n)$ has moreover a right Σ_n-action. n is called the arity.

These data satisfy axioms evident in the most important example of the endomorphism operad of V, $\mathcal{E}nd_V = \{\mathcal{E}nd_V(n)\}_{n\geq 1}$,

$$\mathcal{E}nd_V(n) := Lin(V^{\otimes n}, V),$$

with

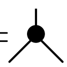
$$(f \circ_i g)(v_1 \otimes \cdots \otimes v_{n+m-1}) :=$$
$$:= f(v_1 \otimes \cdots \otimes v_{i-1}, g(v_i \otimes \cdots \otimes v_{i+m-1}), v_{i+m} \otimes \cdots \otimes v_{m+n-1})$$

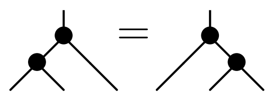
and Σ_n permuting the arguments.

▷ An algebra over $\mathcal{P}$ (or a $\mathcal{P}$-algebra) is an operadic morphism $a : \mathcal{P} \to \mathcal{E}nd_V$. Examples:

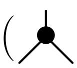
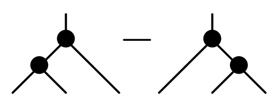
1. An associative algebra is a vector space A with an associative multiplication $\mu : A \otimes A \to A$:

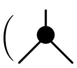
$$\mu(\mu(a,b),c) = \mu(a,\mu(b,c)).$$

If we 'visualize' μ as an 'operation' with two inputs and one output, $\mu =$ , the associativity is depicted as



The operad $\mathcal{Ass}$ describing associative algebras is the quotient

$\mathcal{Ass} := \mathbb{F}\,($  $)\,/\,($  $)$

of the free operad $\mathbb{F}($  $)$[1] modulo the operadic ideal generated by the associativity. Formally,

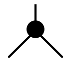
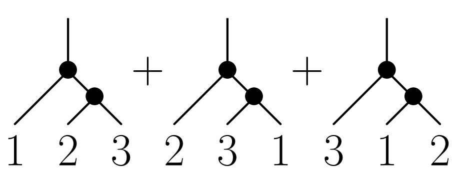
$$\mathcal{Ass} = \mathbb{F}(\mu)/(\mu \circ_1 \mu = \mu \circ_2 \mu).$$

2. The operad $\mathcal{Com}$ for commutative associative algebras is obtained from $\mathcal{Ass}$ by further assuming that the multiplication μ is symmetric, $\mu(a,b) = \mu(b,a)$.

3. A Lie algebra is a vector space L with an antisymmetric product $[-,-] : L \otimes L \to L$ (the 'bracket'), satisfying the Jacobi identity:

$$[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0.$$

The operad $\mathcal{Lie}$ for Lie algebras is the quotient

$\mathcal{Lie} := \mathbb{F}\,($  $)\,/\,($  $)$

[1] Explain.

To describe diagrams of algebras, we need colored operads.

▷ Fix a set of colors $\mathbf{C}$ (= nodes of the diagram). We modify the definition of an operad by assuming that each $\mathcal{P}(n)$ decomposes as

$$\mathcal{P}(n) = \bigoplus_{c,c_1,\ldots,c_n \in \mathbf{C}} \mathcal{P}\left(\begin{smallmatrix} & c & \\ c_1, & \ldots, & c_n \end{smallmatrix}\right).$$

Let $f \in \mathcal{P}\left(\begin{smallmatrix} & & c & & \\ c_1, \ldots, & c_{i-1}, & c_i, & c_{i+1}, & \ldots, c_m \end{smallmatrix}\right)$ and $g \in \mathcal{P}\left(\begin{smallmatrix} & d & \\ d_1, & \ldots, & d_k \end{smallmatrix}\right)$.

We require that $f \circ_i g \neq 0$ implies $d = c_i$, in which case

$$f \circ_i g \in \mathcal{P}\left(\begin{smallmatrix} & & c & & \\ c_1, \ldots, c_{i-1}, & d_1, & \ldots, & d_k, & c_{i+1}, \ldots, c_m \end{smallmatrix}\right).$$

Thus one may plug g into the i-th slot of f only if the colors match, otherwise the result is zero. If $\mathbf{C} = \{\mathrm{Pt}\}$, we get ordinary operads.

The main example is provided by the colored endomorphism operad $\mathcal{E}nd_{\mathbf{U}}$ on a 'colored' vector space $\mathbf{U} = \bigoplus_{c \in \mathbf{C}} \mathbf{U}_c$ given by

$$\mathcal{E}nd_{\mathbf{U}}\left(\begin{smallmatrix} & c & \\ c_1, & \ldots, & c_n \end{smallmatrix}\right) := \mathit{Lin}(U_{c_1} \otimes \cdots \otimes U_{c_n}, U_c).$$

▷ An algebra over a colored operad $\mathcal{P}$ is an morphism of colored operads $a : \mathcal{P} \to \mathcal{E}nd_{\mathbf{U}}$. Examples:

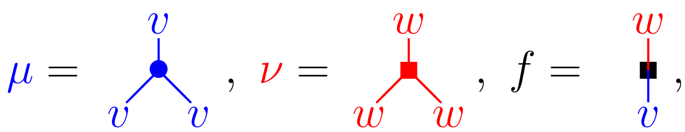
4. The two-colored operad $\mathcal{A}ss_{\bullet\to\bullet}$, with $\mathbf{C} := \{v, w\}$, describing morphisms $(V, \mu) \xrightarrow{f} (W, \nu)$ of associative algebras is the quotient

$$\frac{\mathbb{F}(\mu, \nu, f)}{(\mu(\mu \otimes \mathbb{1}) = \mu(\mathbb{1} \otimes \mu),\ \nu(\nu \otimes \mathbb{1}) = \nu(\mathbb{1} \otimes \nu),\ f\mu = \nu(f \otimes f))},$$

where

$$\mu \in \mathcal{A}ss_{\bullet\to\bullet}\left(\begin{smallmatrix} v \\ v, v \end{smallmatrix}\right),\ \nu \in \mathcal{A}ss_{\bullet\to\bullet}\left(\begin{smallmatrix} w \\ w, w \end{smallmatrix}\right) \text{ and } f \in \mathcal{A}ss_{\bullet\to\bullet}\left(\begin{smallmatrix} w \\ v \end{smallmatrix}\right).$$

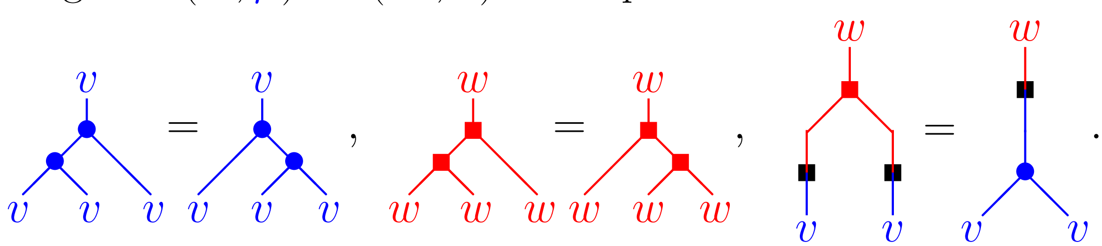
A morphism $(V,\mu) \xrightarrow{f} (W,\nu)$ is clearly the same as an 'algebra' $a : \mathcal{A}ss_{\bullet\to\bullet} \to \mathcal{E}nd_{\mathbf{U}}$, where $\mathbf{U} := U_v \oplus U_w$ with $U_v := V$ and $U_w := W$. If

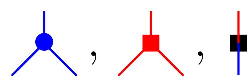


then the axioms

$$(v'v'')v''' = v'(v''v'''),\ (w'w'')w''' = w'(w''w'''),\ f(v'v'') = f(v')f(v'')$$

of the 'diagram' $(V,\mu) \xrightarrow{f} (W,\nu)$ are depicted as

(1)


Pictorially, $\mathcal{A}ss_{\bullet\to\bullet}$ is the free colored operad $\mathbb{F}($  $)$ modulo the operadic ideal generated by (1). Similarly one defines $\mathcal{C}om_{\bullet\to\bullet}$, $\mathcal{L}ie_{\bullet\to\bullet}$ and $\mathcal{P}_{\bullet\to\bullet}$ for a 'non-colored' $\mathcal{P}$.

5. Let again $\mathbf{C} := \{v, w\}$, $f : v \to w$, $g : w \to v$ be two arity 1 generators and denote

$$\mathcal{I}so := \frac{\mathbb{F}(f,g)}{(fg = 1\!\!1_W\ \ , gf = 1\!\!1_V)}.$$

An algebra $a : \mathcal{I}so \to \mathcal{E}nd_{\mathbf{U}}$ consists of two maps $f : V \to W$, $g : W \to V$ that are inverse to each other:

$$\begin{tikzcd} V \arrow[r, bend left, "f"] & W \arrow[l, bend left, "g"] \end{tikzcd},\ fg = 1\!\!1_W \text{ and } gf = 1\!\!1_V.$$

We abuse the notation by using the same symbols for operad generators and the corresponding operations. It is clear now how to construct the colored operad $\mathcal{P}_{\mathcal{D}}$ for $\mathcal{D}$-diagrams of $\mathcal{P}$-algebras.

▷ The construction of the L_∞-deformation complex

$$\mathfrak{g} = (C^*(A, A), \delta = l_1, l_2, \dots)$$

goes in two steps:

Step (1): Finding (if exists) the minimal model $\alpha : (\mathbb{F}(E), \partial) \to (\mathcal{P}, 0)$. By definition, α is a homology isomorphism, $\mathbb{F}(E)$ the free operad on a collection E, and the minimality means that $\partial(E)$ consists of decomposable elements of $\mathbb{F}(E)$. This step is nontrivial. Rich theory of minimal models is available, but will not be discussed here.

Step (2): The minimal model determines $\mathfrak{g}$ via a straightforward procedure. We illustrate everything on the example of the

▷ Hochschild cohomology. Step (1): Recall that

$$\mathcal{A}ss := \mathbb{F}(\mu)/(\mu(\mu \otimes 1\!\!1) - \mu(1\!\!1 \otimes \mu)).$$

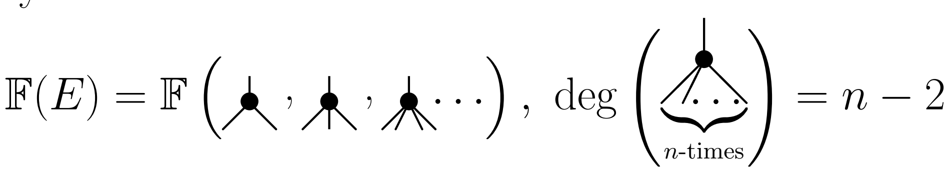
The minimal model for the operad $\mathcal{A}ss$ is well known to be

$$\mathcal{A}ss \xleftarrow{\alpha} (\mathbb{F}(\mu_2, \mu_3, \mu_4, \dots), \partial)\,,\ \deg(\mu_n) = n - 2,$$

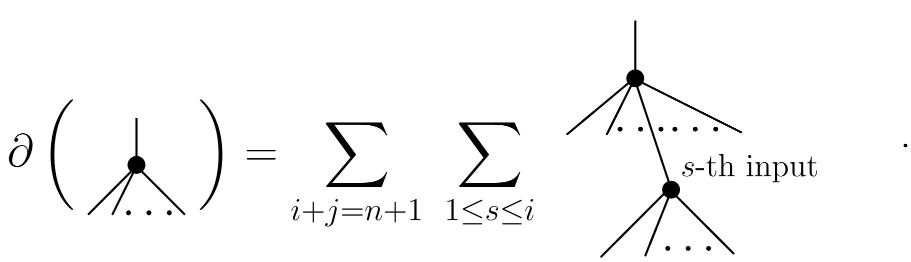
with $\alpha(\mu_2) = \mu$ while α is trivial on the remaining generators. The differential ∂ is given by

$$\partial(\mu_n) = \sum_{i+j=n+1} \sum_{0 \le s \le i-1} \pm \mu_i(1\!\!1^{\otimes s} \otimes \mu_j \otimes 1\!\!1^{\otimes i-s-1}).$$

Pictorially



with the differential given on generators by



Algebras over $(\mathbb{F}(\mu_2, \mu_3, \mu_4, \ldots), \partial)$ are Stasheff's A_∞-algebras. This principle is general – strongly homotopy $\mathcal{P}$-algebras are algebras over the minimal model of $\mathcal{P}$.

Step (2): The underlying vector space $C^*(A,A)$ of $\mathfrak{g}$ is determined by the generators of the minimal model as

$$C^n(A,A) := \mathit{Lin}_\Sigma(E_{n-1}, \mathcal{E}nd_A).$$

In our particular case, $E_{n-1} = \Sigma_{n+1}[\mu_{n+1}]$, so

$$\begin{aligned} C^n(A,A) &= \mathit{Lin}_{\Sigma_{n+1}}(\Sigma_{n+1}[\mu_{n+1}], \mathcal{E}nd_A(n+1)) \\ &= \mathit{Lin}_{\Sigma_{n+1}}(\Sigma_{n+1}[\mu_{n+1}], \mathit{Lin}(A^{\otimes n+1}, A)) = \mathit{Lin}(A^{\otimes n+1}, A). \end{aligned}$$

We recover the Hochschild cochains as expected.

The construction of δ and higher l_k, $k \geq 2$, uses the algebra structure $a : \mathcal{P} \to \mathcal{E}nd_A$. Let $\beta := a \circ \alpha : \mathbb{F}(E) \xrightarrow{\alpha} \mathcal{P} \xrightarrow{a} \mathcal{E}nd_A$.

Let T be an E-decorated tree representing an element of $\mathbb{F}(E)$. We denote by $e_v \in E$ the decoration of a vertex $v \in \mathit{Vert}(T)$.

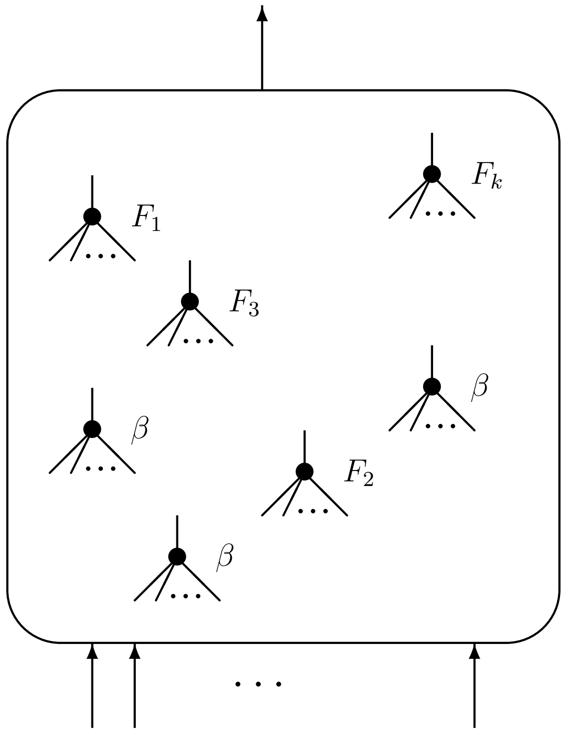
For cochains $F_1, \ldots, F_k \in \mathit{Lin}_\Sigma(E, \mathcal{E}nd_A) = C^*(A,A)$, homomorphism $\beta : \mathbb{F}(E) \to \mathcal{E}nd_A$ and distinct vertices $v_1, \ldots, v_k \in \mathit{Vert}(T)$, denote by

$$T^{\{v_1, \ldots, v_k\}}_{\{\beta\}}[F_1, \ldots, F_k]$$

the $\mathcal{E}nd_A$-decorated tree whose vertices v_i, $1 \leq i \leq k$, are decorated by $F_i(e_{v_i}) \in \mathcal{E}nd_A$ and the remaining vertices $v \notin \{v_1, \ldots, v_k\}$ by $\beta(e_v) \in \mathcal{E}nd_A$. We finally denote by

$$\mathit{comp}(T^{\{v_1, \ldots, v_k\}}_{\{\beta\}}[F_1, \ldots, F_k]) \in \mathcal{E}nd_A$$

the composition of the decorations along the tree. See:



For a generator $\xi \in E$, $\partial(\xi) \in \mathbb{F}(E)$ is a sum of E-decorated trees,

$$\partial(\xi) = \sum_{s \in S_\xi} T_s,$$

over a finite set S_ξ. Define $l_k(F_1, \ldots, F_k)(\xi) \in \mathcal{E}nd_A$ by

$$l_k(F_1, \ldots, F_k)(\xi) := \sum_{s \in S_\xi} \sum_{v_1, \ldots, v_k} \pm comp(T_{s,\{\beta\}}^{\{v_1, \ldots, v_k\}}[F_1, \ldots, F_k]).^2$$

The equivariant map $E \ni \xi \mapsto l_k(F_1, \ldots, F_k)(\xi) \in \mathcal{E}nd_A$ determines an element $l_k(F_1, \ldots, F_k) \in C^*(A; A)$. The assignment $F_1, \ldots, F_k \mapsto l_k(F_1, \ldots, F_k)$ is the requisite L_∞-structure map.

Theorem. *The object $(C^*(A, A), \delta = l_1, l_2, \ldots)$ is an L_∞-algebra. Formal deformations of the $\mathcal{P}$-algebra A are parametrized by elements $\kappa \in C^1(A; A)$ that satisfy the L_∞-Master Equation:*

$$0 = \delta(\kappa) + \frac{1}{2!} l_2(\kappa, \kappa) + \frac{1}{3!} l_3(\kappa, \kappa, \kappa) + \frac{1}{4!} l_4(\kappa, \kappa, \kappa, \kappa) + \cdots .$$

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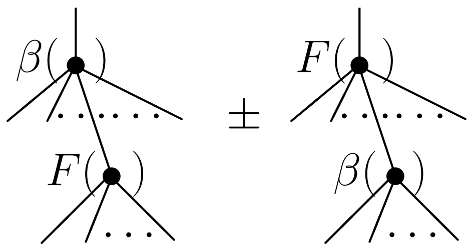
[2] A drawing on board would help.

Let us continue analyzing the associative algebra case. We start by describing $\delta = l_1$. Let $f : A^{\otimes n+1} \to A \in C^n(A, A)$ and $F : E \to \mathcal{E}nd_A$ be the corresponding map of collections given by

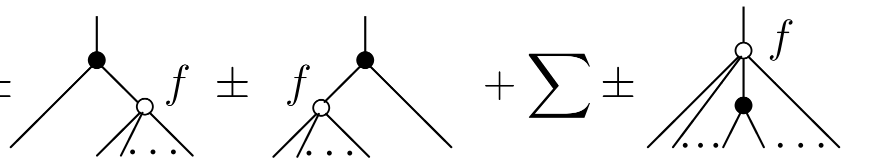
$$F(\mu_k) = \begin{cases} f, & \text{if } k = n+1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

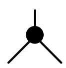
We easily see that $\delta(F)(\mu_k) = 0$ if $k \neq n+2$, while

$$\delta(F)\,(\mu_{n+2}) = \sum \pm$$

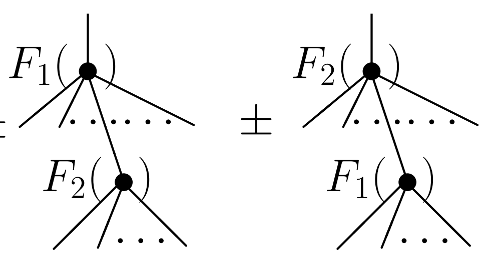


$$= \pm$$

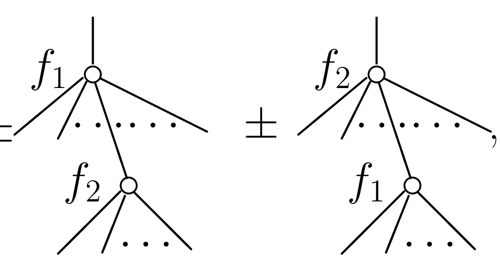


In the second line, $\mu =$  . We recognize the pictorial form of the Hochschild differential from page 1. Analogously, assuming that $F_i : E \to \mathcal{E}nd_A$ are determined by multilinear maps $f_i \in C^{n_i}(A, A)$, $i = 1, 2$, as F was determined by f above, one gets

$$l_2(F_1, F_2)\,(\,\mu_{n_1+n_2+1}) = \sum \pm$$



$$= \sum \pm$$



which is the graphical form of the Gerstenhaber bracket. The higher l_n's are trivial since the differential in the minimal model is quadratic i.e. given by sum over trees with two vertices. We clearly have:

Fact. *The L_∞-alebra $\mathfrak{g} = (C^*(A,A), [-,-], \delta = l_1, l_2, \ldots)$ is dg-Lie if and only if the minimal model of $\mathcal{P}$ is quadratic.*

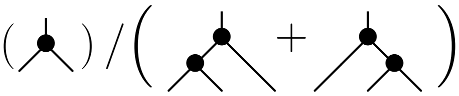
Non-quadratic minimal models are typical for non-Koszul operads. All reasonable cases are Koszul. This explains why we do not see L_∞-proper very often in Nature.

$\triangleright$ An anti-associative algebra is a vector space A with an anti-associative multiplication $\mu : A \otimes A \to A$:

$$\mu(\mu(a,b),c) + \mu(a,\mu(b,c)) = 0.$$

The operad $\widetilde{\mathcal{A}ss}$ describing anti-associative algebras,

$$\widetilde{\mathcal{A}ss} := \mathbb{F}(\quad)/(\quad + \quad)$$



is not Koszul. Its minimal model of $\widetilde{\mathcal{A}ss}$ is of the form

$$(\widetilde{\mathcal{A}ss}, 0) \xleftarrow{\alpha} (T(\mu_2, \mu_3, \mu_5^1, \mu_5^2, \mu_5^3, \mu_5^4, \ldots), \partial),$$

where the subscripts denote the arity. Notice the gap in the arity 4 generators! From this one easily sees that in the relevant part

$$C^0(A;A) \xrightarrow{\delta} C^1(A;A) \xrightarrow{\delta} C^2(A;A) \xrightarrow{\delta} C^3(A;A) \xrightarrow{\delta} \cdots$$

of the deformation complex one has

$$\begin{aligned}
C^0(A;A) &= Lin(A,A)\\
C^1(A;A) &= Lin(A^{\otimes 2},A)\\
C^2(A;A) &= Lin(A^{\otimes 3},A), \text{ and}\\
C^3(A;A) &= Lin(A^{\otimes 5},A) \oplus Lin(A^{\otimes 5},A)\\
&\quad \oplus Lin(A^{\otimes 5},A) \oplus Lin(A^{\otimes 5},A).
\end{aligned}$$

Observe that $C^3(A;A)$ consists, unlike the Hochschild case, of 5-linear maps!

One calculates the initial part of the differential as:

$$\begin{aligned}
\partial(\mu_2) &:= 0,\\
\partial(\mu_3) &:= \mu_2 \circ_1 \mu_2 + \mu_2 \circ_2 \mu_2,\\
\partial(\mu_5^1) &:= (\mu_2 \circ_2 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_3 \mu_2) \circ_4 \mu_2 + (\mu_2 \circ_1 \mu_2) \circ_3 \mu_3\\
&\quad - (\mu_3 \circ_1 \mu_2) \circ_3 \mu_2 + (\mu_2 \circ_1 \mu_3) \circ_1 \mu_2 - (\mu_3 \circ_1 \mu_2) \circ_1 \mu_2\\
&\quad + (\mu_2 \circ_1 \mu_3) \circ_4 \mu_2 - (\mu_3 \circ_2 \mu_2) \circ_4 \mu_2;
\end{aligned}$$

the formulas for $\partial(\mu_5^2)$, $\partial(\mu_5^3)$ and $\partial(\mu_5^4)$ are similar. The cubicity of $\partial(\mu_5^1)$ implies that $C^*(A, A)$ carries a nontrivial

$$l_3 : C^1(A; A) \otimes C^1(A; A) \otimes C^2(A; A) \to C^3(A; A).$$

While in the single-algebra case one need exotic structures to get a fully-fledged L_∞, nontrivial higher l_n's are typical for diagrams.

$\triangleright$ Deformations of a morphism of associative algebras. The construction of the L_∞-deformation complex can be easily modified to the colored case.

We start by describing the minimal model of the two-colored operad $\mathcal{A}ss_{\bullet\to\bullet}$. Let E be the $\{v, w\}$-colored Σ-module with the generators:

$\mu_n \colon v^{\otimes n} \to v$ of degree $n-2$ and biarity $(1, n)$ $(n \geq 2)$,

$\nu_n \colon w^{\otimes n} \to w$ of degree $n-2$ and biarity $(1, n)$ $(n \geq 2)$, and

$f_n \colon v^{\otimes n} \to w$ of degree $n-1$ and biarity $(1, n)$ $(n \geq 1)$.

Then the minimal model for $\mathcal{A}ss_{\bullet\to\bullet}$ is

$$(\mathbb{F}(E), \partial) \xrightarrow{\alpha} \mathcal{A}ss_{\bullet\to\bullet},$$

where

$$\alpha(\mu_n) = \begin{cases} \mu & \text{if } n = 2,\\ 0 & \text{otherwise} \end{cases}, \ \alpha(\nu_n) = \begin{cases} \nu & \text{if } n = 2,\\ 0 & \text{otherwise} \end{cases},$$

$$\alpha(f_n) = \begin{cases} f & \text{if } n = 1, \\ 0 & \text{otherwise} \end{cases}.$$

The differential ∂ is given by:

$$\partial(\mu_n) = \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{s=0}^{n-j} \pm \mu_i \circ_{s+1} \mu_j,$$

$$\partial(\nu_n) = \sum_{\substack{i+j=n+1 \\ i,j \geq 2}} \sum_{s=0}^{n-j} \pm \nu_i \circ_{s+1} \nu_j,$$

$$\partial(f_n) = \sum_{l=2}^{n} \sum_{r_1+\cdots+r_l=n} \pm \nu_l(f_{r_1} \otimes \cdots \otimes f_{r_l}) + \sum_{\substack{i+j=n+1 \\ i \geq 1, j \geq 2}} \sum_{s=0}^{n-j} \pm f_i \circ_{s+1} \mu_j.$$

Since $\partial(f_n)$ contains terms of arbitrary homogeneity, in the L_∞-deformation complex for a morphism, l_n is nontrivial for all $n \geq 1$! In the last example we show that diagrams with loops lead to curved L_∞-algebras.

▷ Deformations of an isomorphism. A small cofibrant resolution of $\mathcal{I}so$ is a $\{v, w\}$-colored operad

$$\mathcal{R}_{\text{iso}} := (\mathbb{F}(f_0, f_1, \ldots; g_0, g_1, \ldots), \partial),$$

with generators of two types,

$$\text{(i) generators } \{f_n\}_{n \geq 0},\ \deg(f_n) = n, \begin{cases} f_n : v \to w & \text{if } n \text{ is even}, \\ f_n : v \to v & \text{if } n \text{ is odd}, \end{cases}$$

$$\text{(ii) generators } \{g_n\}_{n \geq 0},\ \deg(g_n) = n, \begin{cases} g_n : w \to v & \text{if } n \text{ is even}, \\ g_n : w \to w & \text{if } n \text{ is odd}. \end{cases}$$

The differential ∂ is given by

$$\begin{aligned} \partial f_0 &:= 0, & \partial g_0 &:= 0, \\ \partial f_1 &:= g_0 f_0 - 1, & \partial g_1 &:= f_0 g_0 - 1 \end{aligned}$$

and, on the remaining generators, by the formula

$$\begin{aligned}
\partial f_{2m} &:= \sum_{0 \leq i < m} (f_{2i} f_{2(m-i)-1} - g_{2(m-i)-1} f_{2i}), \ m \geq 0, \\
\partial f_{2m+1} &:= \sum_{0 \leq j \leq m} g_{2j} f_{2(m-j)} - \sum_{0 \leq j < m} f_{2j+1} f_{2(m-j)-1}, \ m \geq 1, \\
\partial g_{2m} &:= \sum_{0 \leq i < m} (g_{2i} g_{2(m-i)-1} - f_{2(m-i)-1} g_{2i}), \ m \geq 0, \\
\partial g_{2m+1} &:= \sum_{0 \leq j \leq m} f_{2j} g_{2(m-j)} - \sum_{0 \leq j < m} g_{2j+1} g_{2(m-j)-1}, \ m \geq 1.
\end{aligned}$$

One easily gets the underlying cochain complex (notice its 2-periodicity!)

$$C^n(A; A) = \begin{cases} Lin(V, W) \oplus Lin(W, V) & \text{for } n \geq 1 \text{ odd, and} \\ Lin(V, V) \oplus Lin(W, W) & \text{for } n \geq 1 \text{ even.} \end{cases}$$

The occurrence of 1's in ∂f_1 and ∂g_1 indicates the existence of a curvature. Our recipe makes sense also for $k = 0$ and describes $l_0 \in C^2(A; A)$ as the direct sum of the identity maps

$$\mathbb{1}_V \oplus \mathbb{1}_W \in Lin(V, V) \oplus Lin(W, W) = C^2(A; A).$$

If $\kappa = f \oplus g \in C^1(A; A) = Lin(V, W) \oplus Lin(W, V)$, then the 'curved' Maurer-Cartan equation

$$-l_0 + \frac{1}{2} l_2(\kappa, \kappa) = 0$$

expands into

$$-(\mathbb{1}_V \oplus \mathbb{1}_W) + \frac{1}{2}(2gf \oplus 2fg) = 0 \in Lin(V, V) \oplus Lin(W, W),$$

which says that f and g are mutually inverse isomorphisms.

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The deformation cohomology based on a resolution of the corresponding operad was first considered in the proceedings [2] of the Winter School 'Geometry and Physics,' Zdíkov, Bohemia, January 1993. The L_∞-deformation complex was constructed by van der Laan in [5]. The explicit description used in the talk was obtained in [3], its colored version then in [1]. The minimal model of the anti-associative operad was studied in [4].

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