# Disconnected rational homotopy theory

Andrey Lazarev and Martin Markl

**Abstract.** We construct two algebraic versions of homotopy theory of rational disconnected topological spaces, one based on differential graded commutative associative algebras (cdga) and the other one on complete differential graded Lie algebras (dgla). As an application we obtain results on the structure of Maurer-Cartan (MC) spaces of complete differential graded Lie algebras.

In 1991–93 I published 3 joint papers with Ştefan Papadima on rational homotopy theory (in 2004 another one). Therefore I try to emphasize the homotopy point of view, but everything can also be interpreted as the Maurer-Cartan approach to deformation theory.

Recal that a degree $-1$ element $x \in \mathfrak{g}_{-1}$ of a dgla $\mathfrak{g}$ is *Maurer-Cartan* if it satisfies the *Maurer-Cartan* equation

$$d(x) + \frac{1}{2}[x, x] = 0.$$

Given $\mathfrak{g}$, the *simplicial MC-space* $\mathrm{MC}_\bullet(\mathfrak{g})$ is the simplicial space of MC elements in $\mathfrak{g} \otimes \Omega(\Delta^\bullet)$. The set $\pi_0 \mathrm{MC}_\bullet(\mathfrak{g})$ is the *MC moduli set* of $\mathfrak{g}$ and will be denoted by $\mathscr{MC}(\mathfrak{g})$.

**Principle.** *For each type of structure $X$ there exists a dgla $\mathfrak{g}_X$ which governs deformations of $X$ in the sense that*

$$\textit{moduli space of deformations of } X \cong \mathscr{MC}(\mathfrak{g}_X).$$

The geometric realization of the simplicial MC space gives a functor

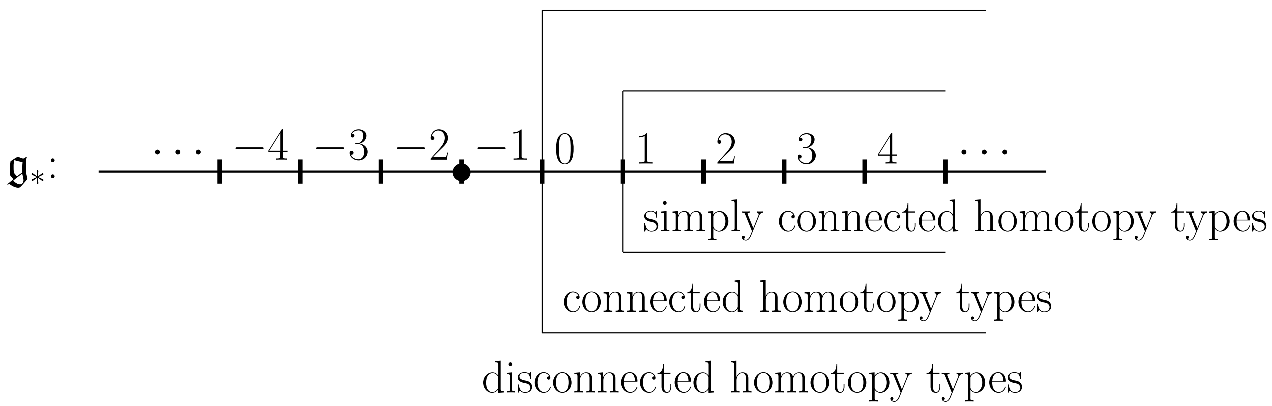
$$\mathfrak{g} \longmapsto |\mathrm{MC}_\bullet(\mathfrak{g})|$$

from dglas to topological spaces. If the ground field is $\mathbb{Q}$, and restricting to dglas with $\mathfrak{g}_n = 0$ for $n \le 0$, we see the functor that in rational homotopy theory relates positively graded dglas a to simply connected rational homotopy types.

Dglas satisfying the weaker condition $\mathfrak{g}_n = 0$ for $n \le -1$, i.e. non-negatively graded ones, should correspond to connected homotopy types.[1] The MC moduli space functor acting on dg-Lie algebras with

[1]Made precise by e.g. Neisendorfer.

no degree restriction can be conceptually interpreted as *disconnected rational homotopy theory*, see the following picture where $\bullet$ marks the degree of MC elements.




Therefore, ideologically,

$$\text{disconnected homotopy theory} = \text{deformation theory}$$

# The cdga side

Recall the classical result by e.g. Bousfield and Gugenheim:

**Theorem.** *The homotopy category of*

*connected rational nilpotent spaces of finite type*

*is equivalent to the homotopy category of*

*non-negatively graded cdgas of finite type.*

We proved the following generalization:

**Theorem A.** *The following two categories are equivalent.*

- *The homotopy category* $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}^{dc}$ *of simplicial sets with finitely many components that are rational, nilpotent and of finite type, and*
- *the homotopy category* $\mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}^{dc}$ *of homologically disconnected* $\mathbb{Z}$*-graded cdgas of finite type over* $\mathbb{Q}$*.*

The meaning of $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}^{dc}$ clear. $\mathrm{f}\mathbb{Q}\text{-ho}\mathscr{A}^{dc}$ is defined as follows.

A $\mathbb{Z}$-graded cdga $A$ is *homologically disconnected* if $H^n(A) = 0$ for $n < 0$ and $H^0(A)$ is isomorphic to $\prod_{i \in J} \mathbf{k}$ for some *finite* set $J$.

Let $\mathscr{A}^{dc}$ be the full subcategory of homologically disconnected cdgas of the category $\mathscr{A}$ of $\mathbb{Z}$-graded cdgas.

A dg ideal $I$ in $A$ is an *augmentation ideal* if $A/I \cong \mathbf{k}$. $A$ is of *finite type* if for any augmentation ideal, $H(I/I^2)$ is finite dimensional in every degree.

Assume $\mathbf{k} = \mathbb{Q}$ and denote by $\mathrm{f}\mathbb{Q}\text{-}\mathscr{A}^{dc}$ the subcategory of $\mathscr{A}^{dc}$ of algebras with a cofibrant replacement of finite type. $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{A}^{dc}$ is the corresponding homotopy category.

Above, 'homotopy,' 'cofibrant,' &c., refers to the standard Hinich's CMC structure – *fibrations* are epis and *WE's* are cohomology isomorphisms.

**Sketch of proof.** One proves first that $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}^{dc}$ is equivalent to $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{A}^{dc}_{\geq 0}$, the homotopy category of algebras that are finite (cartesian) products of non-negatively graded cdgas of finite type over $\mathbb{Q}$.

On the level of objects almost evident: each $X \in \mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}^{dc}$ is a finite disjoint union of connected components, and:

$$\Omega(X_1 \sqcup \cdots \sqcup X_n) \cong \Omega(X_1) \times \cdots \times \Omega(X_n).$$

For homotopy classes more involved and based on the following theorem whose proof is surprisingly involved:

**Theorem.** *Let $A, D \in \mathscr{A}$ be cdgas and $u \in A$ a cocycle. Denote*

$$[A, D]^u := \big\{\chi \in [A, D] \mid \chi_*([u]) \in H(D) \text{ is invertible}\big\}.$$

*There is a natural isomorphism $\big[A[u^{-1}], D\big] \cong [A, D]^u$.*

The theorem implies e.g. that

$$[A_1 \times A_2, D] \cong [A_1, D] \sqcup [A_2, D]$$

for $A_1, A_2, D$ homologically connected [hint: localize at the representatives of $(1, 0)$ resp. $(0, 1)$ in $H(A_1 \times A_2) = H(A_1) \times H(A_2)$].

The proof is then finished using:

**Theorem.** *The inclusion $\mathscr{A}_{\geq 0} \subset \mathscr{A}$ induces an equivalence of the homotopy categories $\mathrm{ho}\mathscr{A}^{dc}_{\geq 0}$ and $\mathrm{ho}\mathscr{A}^{dc}$.*

A related question is whether $\mathrm{ho}\mathscr{A}_{\geq 0}$ is a full subcategory of $\mathrm{ho}\mathscr{A}$. We see no compelling reason for the homotopy classes of maps in both categories to be the same.

# The dgla side

**Theorem B.** *The following categories are equivalent:*

- *the homotopy category* $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^{dc}$ *of pointed simplicial sets with finitely many components that are rational, nilpotent and of finite type, and*
- *the homotopy category* $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ *of disconnected complete dglas of finite type.*

Above, $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}_+^{dc}$ is an obvious pointed version of $\mathrm{fN}\mathbb{Q}\text{-ho}\mathscr{S}^{dc}$. $\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$defined as follows.

Let $\hat{\mathscr{L}}$ be the category of $\mathbb{Z}$-graded *complete* dglas, i.e. inverse limits of finite-dimensional nilpotent dglas. *Finite type* means finite-dimensional homology in each degree.

Explaining disconnected more subtle. Need the some auxiliary but important notions.

Denote by $\mathfrak{s}$ the dgla spanned by $x$ and $[x,x]$ with $|x| = -1$ and $d(x) := -\frac{1}{2}[x,x]$. The dgla $\mathfrak{s}$ models $S^0$. It is the smallest dgla generated by a non-trivial *Maurer-Cartan* element, i.e. one satisfying

$$d(x) + \frac{1}{2}[x,x] = 0.$$

An MC element $\xi \in \mathfrak{h}_{-1}$ allows to *twist* the differential $d$ in $\mathfrak{h}$ by

$$d^\xi(?) = d(?) + [?,\xi].$$

The dgla $\mathfrak{h}$ with the twisted differential will be denoted by $\mathfrak{h}^\xi$.

For a complete dgla $\mathfrak{g}$ we form the complete dgla $\mathfrak{g} * \mathfrak{s}$ where $*$ is the coproduct in the category $\hat{\mathscr{L}}$. Clearly $x$ is an MC element in $\mathfrak{g} * \mathfrak{s}$. The twisted dgla $(\mathfrak{g} * \mathfrak{s})^x$ is analogous to adjoining a base point to a topological space.

The *disjoint product* of complete dglas $\mathfrak{g}$ and $\mathfrak{h}$ is the complete dgla

$$\mathfrak{g} \sqcup \mathfrak{h} := (\mathfrak{g} * \mathfrak{s})^x * \mathfrak{h}.$$

Generalizes to arbitrary finite number of complete dglas.

$\mathrm{f}\mathbb{Q}\text{-ho}\hat{\mathscr{L}}^{dc}$ is the subcategory in the homotopy category of $\hat{\mathscr{L}}$ formed by dglas weakly equivalent to the disjoint products of finitely many non-negatively graded complete dglas whose homology are finite-dimensional in each degree.

'Homotopy' etc., refer to the CMC of $\hat{\mathscr{L}}$ given by postulating that a morphism $f : \mathfrak{g} \to \mathfrak{h}$ in $\hat{\mathscr{L}}$ is

(1) a *weak equivalence* if $\mathcal{C}(f) : \mathcal{C}(\mathfrak{h}) \to \mathcal{C}(\mathfrak{g})$ is a quasi-isomorphism in $\mathscr{A}_+$;
(2) a *fibration* if $f$ is surjective; if, in addition, $f$ is a weak equivalence then $f$ is called an *acyclic fibration*;
(3) a *cofibration* if $f$ has the left lifting property with respect to all acyclic fibrations.

Here $\mathcal{C}(\mathfrak{g})$ denotes the CE-complex of a complete dgla $\mathfrak{g}$:

$$\mathcal{C}(\mathfrak{g}) = \big(S(\uparrow \mathfrak{g}^*), d_1 + d_2\big).$$

$\uparrow$ is the suspension, $\mathfrak{g}^*$ is the *continuous* dual, $d_1$ is the dual of the differential in $\mathfrak{g}$ and $d_2$ is the dual of $[-,-] : \mathfrak{g}\widehat{\otimes}\mathfrak{g} \to \mathfrak{g}$, which is a map $\mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^*$, extended as a derivation.

**Remark.** The completeness of $\mathfrak{g}$ is crutial, the CE-complex $\mathcal{C}(\mathfrak{g})$ *may not* exist for a general $\mathbb{Z}$-graded dgla $\mathfrak{g}$; the dual of the bracket $[-,-] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ is a map $\mathfrak{g}^* \to (\mathfrak{g}\otimes\mathfrak{g})^*$, but $(\mathfrak{g}\otimes\mathfrak{g})^* \not\subset \mathfrak{g}^*\otimes \mathfrak{g}^*$ if $\mathfrak{g}$ is infinite-dimensional. So the dual of the bracket is in general *not* a map $\mathfrak{g}^* \to \mathfrak{g}^*\otimes \mathfrak{g}^*$.

**Remark.** The CMC above has the property that *cofibrant objects are $C_\infty$-algebras*. Hint: cofibrant objects are retracts of free ones, a retract of a complete free dgla is a free complete dgla, i.e. a dgla of the form $\big(\hat{\mathbb{L}}(X), d\big)$, which is a $C_\infty$-algebra with the underlying space $X^*$.

Neisendorfer proved that the subcategory $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}_+^c$ of $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}_+^{dc}$ consisting of *connected* spaces is equivalent to the homotopy category $\mathrm{ho}(nDGLA)$ of non-negatively graded (discrete) dglas $L$ whose homology $H(L)$ is of finite type and nilpotent. Theorem B gives another description of $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}_+^c$.

Denote by $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}_{\geq 0}$ the full subcategory of $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}^{dc}$ of complete non-negatively graded dglas with a finite-type homology. We obtain

**Corollary.** *The functor* $\mathrm{MC}_\bullet$ *induces an equivalence between the categories* $\mathrm{f}\mathbb{Q}\text{-}\mathrm{ho}\hat{\mathscr{L}}_{\geq 0}$ *and* $\mathrm{fN}\mathbb{Q}\text{-}\mathrm{ho}\mathscr{S}_+^c$.

Proof of Theorem B based on

**Theorem.** *The category $\hat{\mathscr{L}}$ is a CMC with fibrations, cofibrations and weak equivalences as above. It is Quillen equivalent to the CMC $\mathscr{A}_+$ via the adjunctions $\hat{\mathcal{L}}$ and $\mathcal{C}$.*

*Here $\hat{\mathcal{L}}$ denotes the completed Harrison complex of a cdga $A$:*

$$\hat{\mathcal{L}}(A) = \big(\hat{\mathbb{L}}(\downarrow A^*), d_1 + d_2\big).$$

The above Quillen equivalence induces an equivalence between the category f$\mathbb{Q}$-ho$\hat{\mathscr{L}}^{dc}$ and the augmented version of f$\mathbb{Q}$-ho$\mathscr{A}^{dc}$ which is in turn equivalent to fN$\mathbb{Q}$-ho$\mathscr{S}_+^{dc}$ by an augmented version of Theorem A.

# Maurer-Cartan spaces

Recall that the *simplicial MC-space* $\mathrm{MC}_\bullet(\mathfrak{g})$ is the simplicial space of MC elements in $\mathfrak{g} \otimes \Omega(\Delta^\bullet)$. The set $\pi_0 \mathrm{MC}_\bullet(\mathfrak{g})$ is the *MC moduli set* of $\mathfrak{g}$ and will be denoted by $\mathscr{MC}(\mathfrak{g})$.

**Theorem C.** *Let $\mathfrak{g}_i$, $i \in J$, be complete dglas indexed by a finite set $J$. Then the simplicial set $\mathrm{MC}_\bullet\big(\bigsqcup_{i\in J} \mathfrak{g}_i\big)$ is weakly equivalent to the disjoint union $\bigcup_{i\in J} \mathrm{MC}_\bullet(\mathfrak{g}_i)$.*

The theorem can be seen using

**Theorem.** *For complete dglas $\mathfrak{g}$, $\mathfrak{h}$ there is a* quasi-*isomorphism*

$$\mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h}) \simeq \mathcal{C}(\mathfrak{g}) \times \mathcal{C}(\mathfrak{h}).$$

So the cdga coresponding to $\mathcal{C}(\mathfrak{g} \sqcup \mathfrak{h})$ is, in the homotopy category, the same as the product of cdgas corresponding to $\mathfrak{g}$ and $\mathfrak{h}$, resp. And the *product* of cdgas corresponds to the *union* of the corresponding spaces. So

$$\mathrm{MC}_\bullet(\mathfrak{g} \sqcup \mathfrak{h}) \simeq \mathrm{MC}_\bullet(\mathfrak{g}) \cup \mathrm{MC}_\bullet(\mathfrak{h}).$$

Theorem C is an obvious generalization to a finite number of factors.

**Corollary.** *Let $\mathfrak{g}_i, i \in J$, be a collection of complete dglas indexed by a finite set $J$. Then there is a bijection*

$$\mathscr{MC}\Big(\bigsqcup_{i\in J} \mathfrak{g}_i\Big) \cong \bigcup_{i\in J} \mathscr{MC}(\mathfrak{g}_i)$$

*of the MC moduli sets.*

One has the following interesting *decomposition theorem* for MC spaces. Define a *connected cover* $\overline{\mathfrak{h}}$ of a dgla $\mathfrak{h}$ as the sub-dgla of $\mathfrak{h}$ given by

$$\overline{\mathfrak{h}}_n := \begin{cases} \mathfrak{h}_n & \text{for } n < 0, \\ \mathrm{Ker}(\partial : \mathfrak{h}_0 \to \mathfrak{h}_1) & \text{for } n = 0, \text{and} \\ 0 & \text{for } n > 0. \end{cases} \tag{0.1}$$

Observe that $\overline{\mathfrak{h}}$ has precisely one MC element - 0.

**Theorem D.** *For a complete dgla $\mathfrak{g}$, one has a weak equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \sim \bigcup_{[\xi]\in\mathscr{MC}(\mathfrak{g})} \mathrm{MC}_\bullet(\overline{\mathfrak{g}^\xi}) \tag{0.2}$$

*where the disjoint union in the right hand side runs over chosen representatives of the isomorphism classes in $\mathscr{MC}(\mathfrak{g})$. If $\mathscr{MC}(\mathfrak{g})$ is finite, one furthermore has a weak equivalence*

$$\mathrm{MC}_\bullet(\mathfrak{g}) \sim \mathrm{MC}_\bullet\Big(\bigsqcup_{[\xi]\in\mathscr{MC}(\mathfrak{g})} \overline{\mathfrak{g}^\xi}\Big)$$

*of simplicial sets.*

As a consequence of the apparatus we developed, we get the following *homotopy invariance property* of the simplicial MC space:

**Theorem.** *Let $f : \mathfrak{g}' \to \mathfrak{g}''$ be a* continuous *morphism of* complete *dglas such that the induced map*

$$\mathcal{C}(f) : \mathcal{C}(\mathfrak{g}'') \to \mathcal{C}(\mathfrak{g}')$$

*is a homology isomorphism. Then*

$$\mathrm{MC}_\bullet(f) : \mathrm{MC}_\bullet(\mathfrak{g}') \to \mathrm{MC}_\bullet(\mathfrak{g}'')$$

*is a homotopy equivalence of simplicial sets. In particular,*

$$\mathscr{MC}(\mathfrak{g}') \cong \mathscr{MC}(\mathfrak{g}'').$$

Notice that the completeness of $\mathfrak{g}'$, $\mathfrak{g}''$ and the continuity of $f$ are necessary, as $\mathcal{C}(-)$'s need not be defined in general.

Some results about MC spaces of the similar flavour were formulated by Buijs and Murillo in [1]. They work with 'ordinary,' i.e. *non-complete*, dglas. So they needed to impose some additional conditions on dglas and their maps that would guarrant that $\mathcal{C}(-)$'s exists. We were not able to verify some of their proofs.

The results mentioned in the talk and lot more available in [2].

## References

[1] U. Buijs and A. Murillo. Algebraic models of non-connected spaces and homotopy theory of $L_\infty$ algebras. *Adv. Math.*, 236:60–91, 2013.

[2] A. Lazarev and M. Markl. Disconnected rational homotopy theory. *Adv. Math.*, 283:303–361, 2015.