Precise FEM solution of corner singularity using adjusted mesh applied to 2D flow

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Introduction

We present an alternative approach to the adaptive mesh refinement. It is based on the knowledge of singularity near the corner. For steady Navier-Stokes equations we proved in [1] that for nonconvex internal angles the velocities near the corners possess an expansion $u(\rho, \vartheta) = \rho^{\gamma}\varphi(\vartheta) + \ldots$ (+ smoother terms), where ρ, ϑ are local spherical coordinates. The local behaviour of the solution near the singular point is used to design a mesh which is adjusted to the shape of the solution. We show an example of 2D mesh with quadratic polynomials for velocity. Then we use this adjusted mesh for the numerical solution of flow in the channel with corners.

Model problem

We consider two-dimensional flow of viscous, incompressible fluid described by Navier-Stokes equations in a domain with corner singularity, cf. Fig. 1.

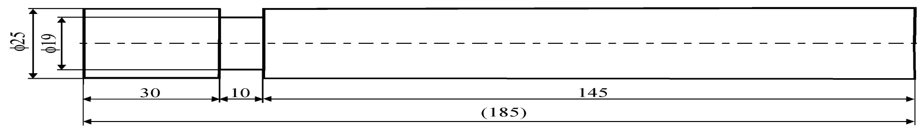

Fig. 1: *Geometry of the channel*

Due to symmetry, we solve the problem only on the upper half of the channel.

Let us denote this domain $\Omega \subset \mathbb{R}^2$. The steady Navier-Stokes problem for the incompressible fluid consists in finding the velocity $\mathbf{v} = (v_1, v_2)$, and pressure p defined in Ω and satisfying

$$(\mathbf{v} \cdot \nabla)\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} \tag{1}$$

$$\text{div } \mathbf{v} = 0 \tag{2}$$

together with boundary conditions on disjoint parts of the boundary Γ_{in}, Γ_{wall} and Γ_{out} (meaning, in turn, the inlet, the wall, and the outlet part)

$$\mathbf{v} = \mathbf{g} \text{ on } \Gamma_{in} \cup \Gamma_{wall} \tag{3}$$

$$\nu\frac{\partial \mathbf{v}}{\partial \mathbf{n}} - p\mathbf{n} = 0 \text{ on } \Gamma_{out} \qquad (\text{'do nothing'} \text{ boundary condition}). \tag{4}$$

We consider kinematic viscosity $\nu = 0.000025$ m^2/s and $v_{in\ max} = 1$ m/s, which give maximum Reynolds number around 760. We don't consider volumetric loads $\mathbf{f} = (f_1, f_2)$.

Algorithm derivation

In [1] we proved for Stokes flow that for internal angle $\alpha = \frac{3}{2}\pi$, the leading term of the expansion of the solution for the velocity components is

$$v_l(\rho, \vartheta) = \rho^{0.54448374} \varphi_l(\vartheta) + \dots, \; l = 1, 2, \tag{5}$$

where ρ is the distance from the corner, ϑ the angle.

Similar results have been proved for the Navier-Stokes equations.

Differentiating by ρ we get $\frac{\partial v_i(\rho,\vartheta)}{\partial \rho} \;\rightarrow\; \infty$ for $\rho \;\rightarrow\; 0$.

A priori estimate of the finite element error is (cf. [3])

$$\|\nabla(\mathbf{v} - \mathbf{v_h})\|_0 + \|p - p_h\|_0 \leq C\Big[\Big(\sum_T h_T^{2k} \mid \mathbf{v} \mid^2_{H^{k+1}(T)}\Big)^{1/2} + \Big(\sum_T h_T^{2k} \mid p \mid^2_{H^k(T)}\Big)^{1/2}\Big],$$

where $k = 2$. Taking into account the expansion (5), we derived in [1] the estimate

$$\mid \mathbf{v} \mid^2_{H^{k+1}(T)} \approx C \int\limits_{r_T - h_T}^{r_T} \rho^{2(\gamma-k-1)} \; \rho \; d\rho \approx C \; r_T^{2(\gamma-k)}, \tag{6}$$

where h_T is the diameter of the triangle T of a triangulation $\mathcal{T}_h$, and r_T is the distance of the element T from the corner.

Putting (6) into the a priori error estimate, we derive that we should guarantee

$$h_T^{2k} \left[-r_T^{2(\gamma-k)} + (r_T - h_T)^{2(\gamma-k)} \right] \approx h^{2k} \tag{7}$$

where h_T is the diameter of the triangle T of a triangulation $\mathcal{T}_h$, and r_T is the distance of the element T from the corner, in order to get the error estimate of the order $O(h^k)$ and uniformly distributed on elements.

After simplifications we used aproximate expression

$$h_T^{2k} \; r_T^{2(\gamma-k)} \approx h^{2k}. \tag{8}$$

This lead us in [1] to an algorithm for generating the mesh near the corner in recurrent form

$$\begin{aligned} h_i &= h \cdot (r_i)^{1-\frac{\gamma}{k}}, && (9) \\ r_{i+1} &= r_i - h_i, && (10) \\ i &= 1, 2, \dots, N, \end{aligned}$$

where r_1 is the distance of the large element from the corner.

Using this algorithm we obtained satisfactory results. Some of them were presented in [4].

At present time, we developed a programme for computing the element sizes directly from expression (7) using Newton's method.

This algorithm for mesh refinement is applied to the corner where the channel or tube suddenly decreases the diameter (forward step in Fig. 1).

We start with $r_1 = 0,25\,mm$, $h = 0,1732\,mm$, $k = 2$, $\gamma = 0,5444837$. This corresponds to the contribution cca 3 % of individual elements to the global error. This way we get fourteen diameters of elements, cf. Tab. 1.

i	$r_i(mm)$	$h_i(mm)$
1	0.25000	0.0600369
2	0.18996	0.0480779
3	0.14189	0.0379492
4	0.10394	0.0294666
5	0.07447	0.0224535
6	0.05202	0.0167410
7	0.03527	0.0121680
8	0.02311	0.0085813
9	0.01453	0.0058366
10	0.00869	0.0037980
11	0.00489	0.0023392
12	0.00255	0.0013434
13	0.00121	0.0007042
14	0.00050	0.0005042

Tab. 1. *Resulting refinement*

Design of the mesh

In [4] we showed that the best way how to use data given by the algorithm is design of the mesh corresponding to polar coordinate system due to its usage in estimates. We continue this idea and design two dimensional mesh near the corner with singularity as could be seen on Figure 2.

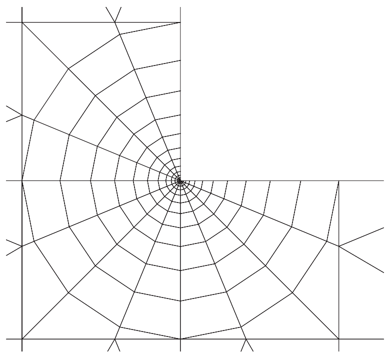

Fig. 2: *Refined detail of mesh*

This detail is connected to the whole computational mesh, cf. Figure 3 and 4.

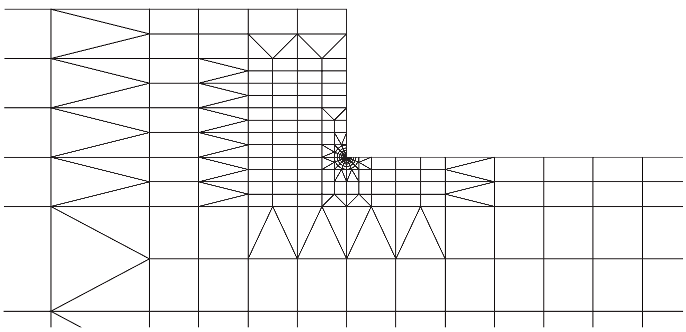

Fig. 3: *Computational mesh near the corner*

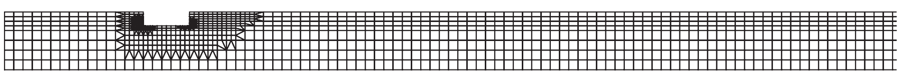

Fig. 4: *Whole computational mesh*

Evaluation of the error

To evaluate the error on elements we use now the modified absolute error computed using a posteriori error estimates, defined as

$$\mathcal{A}_m^2(v_1^h, v_2^h, p^h, \Omega_l) = \frac{|\Omega|\mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega_l)}{|\overline{\Omega}_l| \|(v_1^h, v_2^h, p^h)\|_{V,\Omega}^2}, \tag{11}$$

where $\mathcal{E}^2(v_1^h, v_2^h, p^h, \Omega_l)$ is estimate of error on element l, $|\Omega|$ is the area of the whole domain and $|\overline{\Omega}_l|$ is the mean area of elements obtained as $|\overline{\Omega}_l| = \frac{|\Omega|}{n}$. Here n means the number of all elements in the domain. More about the evaluation of error in [2].

Numerical results

On Figures 5-8 we present the graphical output of entities that chartacterize the flow in the channel. On Figure 5 there are the streamlines in the channel. Figure 6 with contours of velocity v_y shows that the solution is satisfactory smooth on refined area. On Figs. 7-8 we observe how strong the singularity is, both for velocity and pressure (note that here the flow is from the right to the left, to have better view).

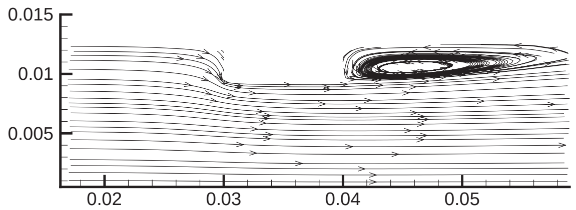

Fig. 5: *Streamlines*

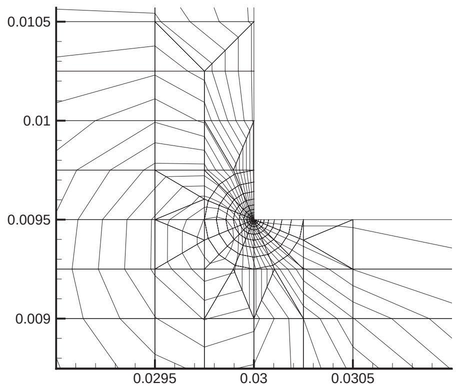

Fig. 6: *Isolines of velocity v_y*

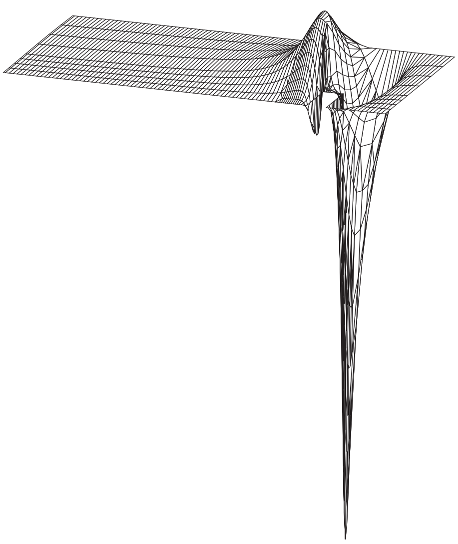

Fig. 7: *Velocity component v_y*

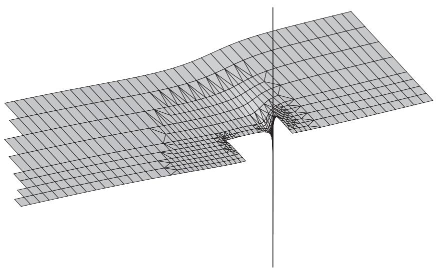

Fig. 8: *Pressure near the corner*

On Figs. 9-11 we show the errors on elements.

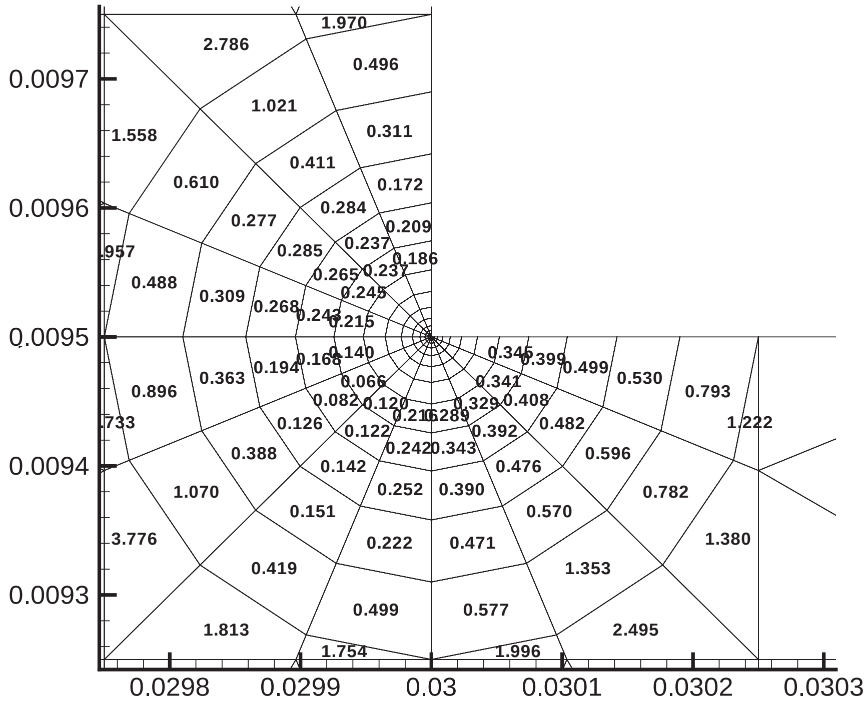

Fig. 9: *Errors on elements - whole refined area*

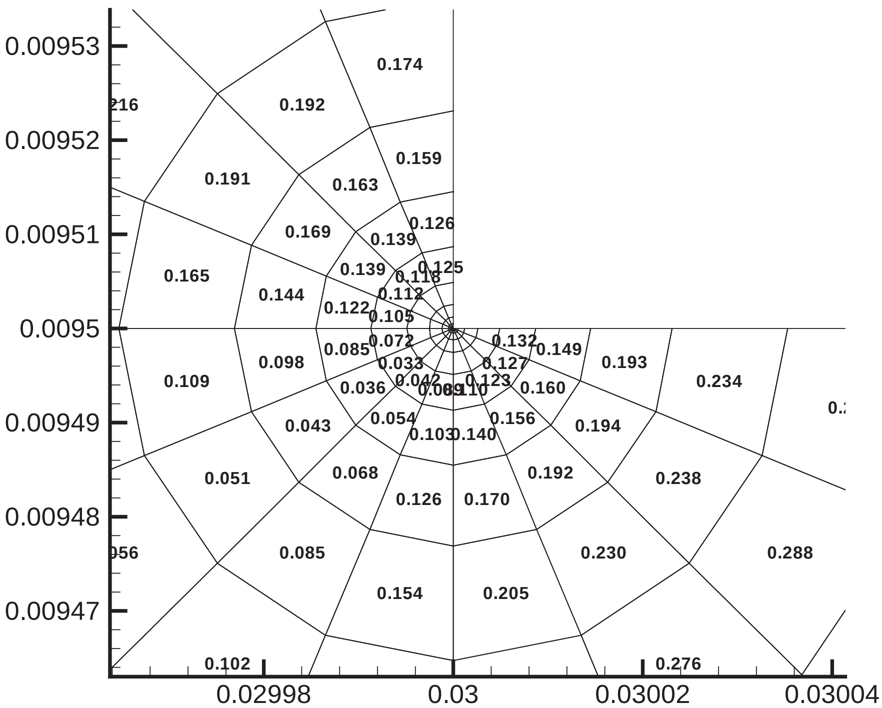

Fig. 10: *Errors on elements - detail*

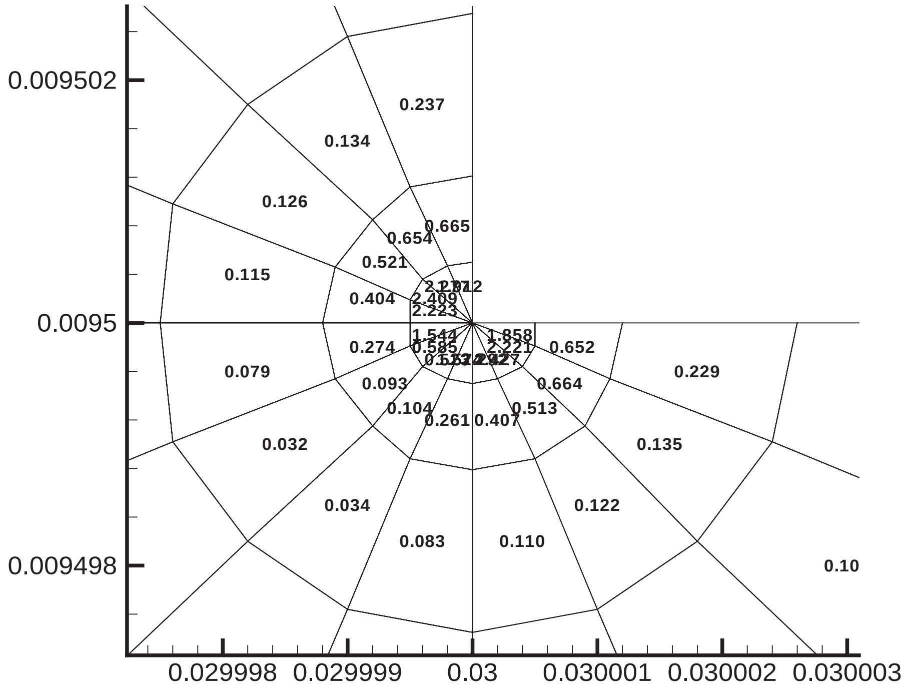

Fig. 11: *Errors on elements - with smallest elements*

Conclusions

Presented results give satisfactory confirmation of developed algorithm. Application of a priori error estimates of finite element method for mesh refinement near singularity is very efficient for our problem what can be seen especially from obtained errors on elements which are very uniformly distributed.

Derived algorithm is universal for design of the mesh close to internal angle $\frac{3}{2}\pi$. But it affords a way how to generate mesh for different angles as well, in accordance to parameter γ which is necesary to find for each angle. This approach is an alternative to 'classical' one using adaptive mesh refinement, which is still much more robust.

Reference

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