A posteriori error estimates in accuracy analysis of stabilized FEM

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joint work with
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Motivation and goals of the research

- Solution of flows of incompressible viscous fluid with higher Reynolds numbers by FEM
- Stabilization techniques for FEM, Galerkin Least Squares method (GLS), semiGLS
- Accuracy of stabilized methods – a posteriori error estimates
Steady Navier-Stokes problem

Find flow velocity \( u(x) \in [C^2(\Omega)]^2 \) and pressure \( p(x) \in C^1(\Omega)/\mathbb{R} \) satisfying

\[
(u \cdot \nabla)u - \nu \Delta u + \nabla p = f \quad \text{in } \Omega, \\
\nabla \cdot u = 0 \quad \text{in } \Omega,
\]

with boundary conditions

\[
u \ (\nabla u) n + pn = 0 \quad \text{on } \partial \Omega_h.
\]

- \( \Omega \subset \mathbb{R}^2 \) ... domain with boundary \( \partial \Omega \) filled with incompressible viscous fluid
- \( \nu \) ... kinematic viscosity of the fluid
- \( f(x, t) \) ... vector of intensity of volume forces per mass unit
- \( \partial \Omega_g \) and \( \partial \Omega_h \) ... subsets of \( \partial \Omega \) satisfying \( \partial \Omega = \partial \Omega_g \cup \partial \Omega_h \)
- \( n \) ... unit outer normal vector to the boundary \( \partial \Omega \)
**Weak formulation**

Vector function spaces

\[ V_g = \left\{ \mathbf{v} = (v_1, v_2) \mid \mathbf{v} \in [H^1(\Omega)]^2; \text{Tr} v_i = g_i, i = 1, 2, \text{ on } \partial \Omega_g \right\} \]

\[ V = \left\{ \mathbf{v} = (v_1, v_2) \mid \mathbf{v} \in [H^1(\Omega)]^2; \text{Tr} v_i = 0, i = 1, 2, \text{ on } \partial \Omega_g \right\} \]

*Find* \( \mathbf{u}(x) \in V_g, \mathbf{u} - \mathbf{u}_g \in V \) and \( p(x) \in L_2(\Omega)/\mathbb{R} \) satisfying

\[
\int_\Omega (\mathbf{u} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} d\Omega + \nu \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} d\Omega - \int_\Omega p \nabla \cdot \mathbf{v} d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} d\Omega \\
\int_\Omega \psi \nabla \cdot \mathbf{u} d\Omega = 0
\]

for \( \mathbf{v} \in V \) and \( \psi \in L_2(\Omega) \).

▷ \( \mathbf{u}_g \in V_g \) satisfies the Dirichlet boundary condition \( \mathbf{g} \)
Approximation of the problem by FEM

Taylor–Hood finite elements – satisfying Babuška-Brezzi condition

\[ \exists C_B > 0, \text{const.} \quad \forall \psi_h \in Q_h \quad \sup_{\nabla \cdot v_h \in V_h} \frac{\psi_h, \nabla \cdot v_h}{\|v_h\|_1} \geq C_B \|\psi_h\|_0 \]

function spaces for approximation:

velocities

\[ V_{gh} = \left\{ v_h \in [C(\Omega)]^2; \ v_{hi} |_K \in R_2(K), \ i = 1, 2, \ v_h = g \text{ on } \partial \Omega_g \right\} \]

pressure and test functions for the continuity equation

\[ Q_h = \left\{ \psi_h \in C(\Omega); \ \psi_h |_K \in R_1(K) \right\} \]

test functions for momentum equations

\[ V_h = \left\{ v_h \in [C(\Omega)]^2; \ v_{hi} |_K \in R_2(K), \ i = 1, 2, \ v_h = 0 \text{ on } \partial \Omega_g \right\} \]

where

\[ R_m(K) = \begin{cases} P_m(K), & \text{if } K \text{ is a triangle} \\ Q_m(K), & \text{if } K \text{ is a quadrilateral} \end{cases} \]
Galerkin Least Squares stabilization technique

Basic scheme (Hughes, Franca, Hulbert, 1989):
Instead of solving the problem

1. Find $u_h$ satisfying the variational formulation

   \[ B(u_h, w_h) = L(w_h), \quad \forall w_h \in V_h \]

   derived from the classical formulation

   \[ A u = f \text{ in } \Omega. \]

2. Solve problem

   a. Find $u_h$ satisfying

      \[ B(u_h, w_h) + \tau (Au_h, Aw_h)_{\tilde{\Omega}} = L(w_h) + \tau (f, Aw_h)_{\tilde{\Omega}}, \quad \forall w_h \in V_h. \]

   Here $A$ represents the scalar advection-diffusion operator, $\tau$ is a stabilization parameter, and

   \[ \tilde{\Omega} = \bigcup_{K} \Omega_K \quad \text{(element interiors)} \]
GLS for Navier–Stokes equations (Franca, Madureira, '93)

\[
(\nabla \mathbf{u}) \mathbf{u} - 2\nu \nabla \cdot \varepsilon(\mathbf{u}) + \nabla p = f \quad \text{in } \Omega \\
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \\
\mathbf{u} = 0 \quad \text{on } \partial \Omega \\
\varepsilon(\mathbf{u})_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

Find \( \mathbf{u}_h \in V_{gh} \) and \( p_h \in Q_h \) satisfying in \( \Omega \)

\[
B_{GLS}(\mathbf{u}_h, p_h; \mathbf{v}_h, \psi_h) = L_{GLS}(\mathbf{v}_h, \psi_h), \quad \forall \mathbf{v}_h \in V_h, \quad \forall \psi_h \in Q_h
\]

where

\[
B_{GLS}(\mathbf{u}_h, p_h; \mathbf{v}_h, \psi_h) \equiv \\
\equiv ((\mathbf{u}_h \cdot \nabla) \mathbf{u}_h, \mathbf{v}_h)_0 + (2\nu \varepsilon(\mathbf{u}_h), \varepsilon(\mathbf{v}_h))_0 - (p_h, \nabla \cdot \mathbf{v}_h)_0 + \\
+ (\psi_h, \nabla \cdot \mathbf{u}_h)_0 + (\nabla \cdot \mathbf{u}_h, \delta \nabla \cdot \mathbf{v}_h)_0 + \\
+ \sum_K ((\mathbf{u}_h \cdot \nabla) \mathbf{u}_h + \nabla p_h - 2\nu \nabla \cdot \varepsilon(\mathbf{u}_h), \tau((\mathbf{u}_h \cdot \nabla) \mathbf{v}_h + \nabla \psi_h - 2\nu \nabla \cdot \varepsilon(\mathbf{v}_h)))_K
\]

\[
L_{GLS}(\mathbf{v}_h, \psi_h) \equiv (f, \mathbf{v}_h)_0 + \sum_K (f, \tau((\mathbf{u}_h \cdot \nabla) \mathbf{v}_h + \nabla \psi_h - 2\nu \nabla \cdot \varepsilon(\mathbf{v}_h)))_K
\]
SemiGLS method

- do not consider stabilization of the continuity equation ($\delta = 0$)
- formulation with Laplacian instead of $\varepsilon(u)_{ij}$

Find $u_h \in V_h$ and $p_h \in Q_h$ satisfying in $\Omega$

$$B_{sGLS}(u_h, p_h; v_h, \psi_h) = L_{sGLS}(v_h, \psi_h), \quad \forall v_h \in V_h, \quad \forall \psi_h \in Q_h$$

where

$$B_{sGLS}(u_h, p_h; v_h, \psi_h) \equiv \int_{\Omega} (u_h \cdot \nabla) u_h \cdot v_h d\Omega$$

$$+ \nu \int_{\Omega} \nabla u_h : \nabla v_h d\Omega - \int_{\Omega} p_h \nabla \cdot v_h d\Omega + \int_{\Omega} \psi_h \nabla \cdot u_h d\Omega +$$

$$+ \sum_{K=1}^{N} \int_{K} [(u_h \cdot \nabla) u_h - \nu \Delta u_h + \nabla p_h] \cdot \tau [(u_h \cdot \nabla) v_h - \nu \Delta v_h + \nabla \psi_h] d\Omega$$

$$L_{sGLS}(v_h, \psi_h) \equiv \int_{\Omega} f \cdot v_h d\Omega + \sum_{K=1}^{N} \int_{K} f \cdot \tau [(u_h \cdot \nabla) v_h - \nu \Delta v_h + \nabla \psi_h] d\Omega$$
Stabilization parameter $\tau$

$$\tau(x) = \frac{\xi(\text{Re}_K(x))}{\sqrt{\lambda_K} \ | u_h(x) |_2},$$

where

$$\text{Re}_K(x) = \frac{| u_h(x) |_2}{4\sqrt{\lambda_K \nu}},$$

$$\xi(\text{Re}_K(x)) = \begin{cases} 
\text{Re}_K(x), & 0 \leq \text{Re}_K(x) < 1 \\
1, & \text{Re}_K(x) \geq 1 
\end{cases},$$

$$\lambda_K = \max_{0 \neq v_h \in (R_2(K)/\mathbb{R})^2} \frac{\| \Delta v_h \|_{0,K}^2}{\| \nabla v_h \|_{0,K}^2}.$$

Parameter $\lambda_K$ is computed for each element as the largest eigenvalue of the problem

$$(\Delta w_h, \Delta v_h) = \lambda_K (\nabla w_h, \nabla v_h), \quad \forall v_h \in (R_2(K)/\mathbb{R})^2.$$
Accuracy investigation – distortion from stabilization

- difference between discrete solution with and without stabilization
- applicable in the range of Reynolds numbers we can solve with method without stabilization
- evaluated as

\[ \delta \eta = \sqrt{\frac{n}{\sum_{i=1}^{n} (\eta_{sGLS_i} - \eta_{Newton_i})^2}} \cdot \frac{\sum_{i=1}^{n} \eta_{Newton_i}^2}{\sum_{i=1}^{n} \eta_{Newton_i}^2} \cdot 100 \% \]

- \( \eta \) represents in turn \( u_{h1}, u_{h2} \) and \( p_h \)
a posteriori error estimates for Taylor–Hood elements

\[ R^2(u_1h, u_2h, p_h, T_K) = \frac{| \Omega | \, E^2(u_1h, u_2h, p_h, T_K)}{| T_K | \| (u_1h, u_2h, p_h) \|_{V, \Omega}^2} \]

\[ U^2(u_1 - u_1h, u_2 - u_2h, p - p_h) \leq E^2(u_1h, u_2h, p_h) \]

\[ U^2(u_1 - u_1h, u_2 - u_2h, p - p_h) = \| (e_{u_1}, e_{u_2}) \|_{1,K}^2 + \| e_p \|_{0,K}^2 \]

\[ E^2(u_1h, u_2h, p_h) = C \left[ h_K^2 \int_K \left( R_1^2(u_1h, u_2h, p_h) + R_2^2(u_1h, u_2h, p_h) \right) d\Omega + \right. \]

\[ + \left. \int_K R_3^2(u_1h, u_2h) d\Omega \right] \]

\[ R_1(u_1h, u_2h, p_h) = f_{x_1} - u_h \cdot \nabla u_1h + \nu \Delta u_1h - \frac{\partial p_h}{\partial x_1} \]

\[ R_2(u_1h, u_2h, p_h) = f_{x_1} - u_h \cdot \nabla u_2h + \nu \Delta u_2h - \frac{\partial p_h}{\partial x_2} \]

\[ R_3(u_1h, u_2h) = \nabla \cdot u_h \]

- constant \( C \) is determined from a numerical experiment
- applicable for any Reynolds number we can find solution
Results of numerical experiments

Test problems:
- lid driven cavity
- channel with sudden extension of diameter
- flow past NACA 0012 airfoil

Streamlines and pressure contours for $\text{Re} = 100,000$, mesh $128 \times 128$
Streamlines, Re = 10,000 mesh 32×32 without stabilization, 32×32, 64×64 and 128×128 by semiGLS
Differences between solutions obtained without stabilization and by semiGLS

<table>
<thead>
<tr>
<th>mesh</th>
<th>32×32</th>
<th>64×64</th>
<th>128×128</th>
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</thead>
<tbody>
<tr>
<td>$\delta_{u_{h1}}$ [%]</td>
<td>41.69</td>
<td>39.07</td>
<td>21.42</td>
</tr>
<tr>
<td>$\delta_{u_{h2}}$ [%]</td>
<td>70.81</td>
<td>49.12</td>
<td>22.24</td>
</tr>
<tr>
<td>$\delta_{p_{h}}$ [%]</td>
<td>197.90</td>
<td>137.10</td>
<td>42.82</td>
</tr>
</tbody>
</table>
a posteriori errors on elements, \( \text{Re} = 10,000 \), mesh 32\( \times \)32 without stabilization (left) and by semiGLS (right)
a posteriori errors on elements, Re = 10,000, mesh 64×64 without stabilization (left) and by semiGLS (right)
a posteriori errors on elements, Re = 10,000, mesh 128×128
without stabilization (left) and by semiGLS (right)
Channel with sudden extension of diameter

Geometry of the channel

Streamlines, Re = 1,000,
without stabilization (left) and by semiGLS (right)
Differences between solutions obtained with and without stabilization

<table>
<thead>
<tr>
<th>mesh</th>
<th>channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_{u_{h1}} \ [%]$</td>
<td>0.0718</td>
</tr>
<tr>
<td>$\delta_{u_{h2}} \ [%]$</td>
<td>2.7202</td>
</tr>
<tr>
<td>$\delta_{p_{h}} \ [%]$</td>
<td>0.5139</td>
</tr>
</tbody>
</table>
a posteriori error estimates, Re = 1,000, without stabilization (upper) and by semiGLS (lower)
Streamlines (upper) and velocity $u_{h1}$ (lower) by semiGLS algorithm, $Re = 80,000$
Plot of velocity $u_{h2}$ (upper) and pressure (lower) by semiGLS algorithm, $Re = 80,000$. 
Flow past NACA 0012 airfoil

- mesh consists of 6,220 elements, 18,478 nodes, 43,085 degrees of freedom

Computational mesh for NACA 0012 problem, angle of incidence of 34°
Computational mesh for NACA 0012 problem – detail
Streamlines (left) and pressure contours (right), Re = 100
a posteriori error on elements, Re = 100, without stabilization (left) and by semiGLS method (right)
Unsteady flow past NACA 0012 airfoil, $Re = 1,000$

Streamlines by semiGLS algorithm
and by Guermond, Quartapelle (1997), $t = 1.6s$, $Re = 1,000$
Unsteady flow past NACA 0012 airfoil, \( \text{Re} = 1,000 \)

Pressure contours by semiGLS algorithm and by Guermond, Quartapelle (1997), \( t = 1.6s, \text{Re} = 1,000 \)
Unsteady flow past NACA 0012 airfoil, $Re = 100,000$

Streamlines and pressure contours, $t = 1.6s$, $Re = 100,000$
Unsteady flow past NACA 0012 airfoil, $Re = 100,000$

Streamlines and pressure contours, $t = 2.6s$, $Re = 100,000$
Unsteady flow past NACA 0012 airfoil, $Re = 100,000$

Streamlines and pressure contours, $t = 3.6s$, $Re = 100,000$
Unsteady flow past NACA 0012 airfoil, $Re = 100,000$

Streamlines and pressure contours, $t = 6s$, $Re = 100,000$
Conclusion

- **semiGLS** – modification of GLS technique of stabilization
- markably higher Reynolds numbers in solved problems reached
- evaluation of the distortion of solution affected by the stabilization
- application of *a posteriori error estimates*

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Our publications on the topic

Pavel Burda, Jaroslav Novotný, and Bedřich Sousedík.
A posteriori error estimates applied to flow in a channel with corners.

Pavel Burda, Jaroslav Novotný, and Jakub Šístek.
On a modification of GLS stabilized FEM for solving incompressible viscous flows.

Pavel Burda, Jaroslav Novotný, and Jakub Šístek.
Numerical solution of flow problems by stabilized finite element method and verification of its accuracy using a posteriori error estimates.

Pavel Burda, Jaroslav Novotný, and Jakub Šístek.
Accuracy of SemiGLS stabilization of FEM for solving Navier–Stokes equations and a posteriori error estimates.