Singular boundary value problems

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Overview of our common research with Svaťa
We have studied the functional differential equation

\[ x''(t) = f(t, x(t), (Fx)(t), x'(t), (Hx')(t)), \]

where \( f \) satisfies the local Carathéodory conditions on \( ([0, 1] \times \mathbb{R}^4) \) and \( F, H \) are continuous and bounded operators. We proved the existence results for various types of two-point or functional boundary conditions. Typical assumptions for \( f \) - sign conditions.


2003, *Singular problems: Sign-changing solutions*


We have studied the singular Dirichlet boundary value problem with a positive parameter \( \mu \)

\[
(r(x(t))x'(t))' = \mu q(t)f(t, x(t)), \quad t \in (0, T),
\]

\[
x(0) = x(T) = 0, \quad \max_{t \in [0, T]} \{x(t)\} \cdot \min_{t \in [0, T]} \{x(t)\} < 0,
\]

where \( T \in (0, \infty) \) and \( f(t, x) \) is singular at the point \( x = 0 \) of the phase variable \( x \) in the following sense

\[
\lim_{x \to 0^-} f(t, x) = -\infty, \quad \lim_{x \to 0^+} f(t, x) = \infty \quad \text{for} \ t \in [0, T].
\]
\[(r(x(t))x'(t))' = \mu q(t)f(t, x(t))\]

The basic assumptions:

(H1) \( r \in C(\mathbb{R}), r(x) \geq r_0 > 0 \) for \( x \in \mathbb{R} \),

(H2) \( q \in C[0, T], q < 0 \) on \((0, T)\),

(H3) \( f \in C([0, T] \times D) \) fulfils (3), \( f(t, \cdot) \) is nonincreasing on \( D \) for \( t \in [0, T] \), \( D = (-\infty, 0) \cup (0, \infty) \).

**Definition**

A function \( x \in C^1[0, T] \) is a solution of problem (1), (2) if \( x \) has precisely one zero \( t_0 \) in \((0, T)\), \( r(x)x' \in C^1((0, T) \setminus \{t_0\}) \), \( x \) fulfils (2) and there exists \( \mu_0 > 0 \) such that (1) is satisfied for \( \mu = \mu_0 \) and \( t \in (0, T) \setminus \{t_0\} \).
We say that $f$ has the \textit{weak singularity at $x = 0$} if $f$ satisfies (3) and
\begin{equation}
0 < f(t, x) \text{sign } x \leq g(x) \quad \text{for } (t, x) \in [0, T] \times D,
\end{equation}
where $g \in C(D)$ fulfils
\begin{equation}
\int_0^0 g(x) \, dx < \infty, \quad \int_0^0 g(x) \, dx < \infty.
\end{equation}

For example
\begin{equation}
f(t, x) = \frac{\text{sign } x}{|x|^\alpha}, \quad \alpha \in (0, 1).
\end{equation}
Assume that there exist a positive function $p \in C(D)$ such that

$$0 < p(x) \leq f(t, x) \text{sign } x \quad \text{for } (t, x) \in [0, T] \times D. \quad (6)$$

We say that $f$ has the **strong left singularity at $x = 0$** if

$$\int_{0}^{1} p(x) \, dx = \infty \quad (7)$$

and we say that $f$ has the **strong right singularity at $x = 0$** if

$$\int_{0}^{1} p(x) \, dx = \infty. \quad (8)$$

For example

$$f(t, x) = \frac{\text{sign } x}{|x|^\alpha}, \quad \alpha \geq 1.$$
1) If $f$ has the one-sided strong singularity at $x = 0$, then problem (1), (2) has no solution;

2) If $f$ has the week singularity at $x = 0$ then:
   
   (i) for each $A > 0$, there exists at least one solution having its maximum value $\leq A$ on $[0, T]$,

   (ii) for each $B < 0$ there exists at least one solution having its minimum value $\geq B$ on $[0, T]$,

   (iii) zeros of solutions on $(0, T)$ are not precisely localized.
Steps of the proof:

- The existence of positive and negative solutions of auxiliary Dirichlet problems on the intervals \([0, c]\) and \([c, T]\) for \(\mu = \mu(c),\ c \in (0, T)\).
- The construction of a sign-changing "solution" from these positive and negative solutions.
- The existence of a point \(t_0 \in (0, T)\) and a parameter \(\mu_0\) such that the sign-changing solution belongs to \(C^1[0, 1]\).
Steps of the proof:

- The existence of positive and negative solutions of auxiliary Dirichlet problems on the intervals \([0, c]\) and \([c, T]\) for \(\mu = \mu(c), c \in (0, T)\).
- The construction of a sign-changing "solution" from these positive and negative solutions.
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2003-2004, Higher order singular problems

- Two-point higher order BVPs with singularities in phase variables. *Computers and Mathematics with Applications* 46 (12) (2003), 1799-1826.
The **singular Lidstone** boundary value problem

\[ (-1)^n x^{(2n)}(t) = f(t, x(t), \ldots, x^{(2n-2)}(t)), \quad (9) \]

\[ x^{(2j)}(0) = x^{(2j)}(T) = 0, \quad 0 \leq j \leq n - 1, \quad (10) \]

where \( n \geq 1 \) and \( f \) satisfies the local Carathéodory conditions on the set \([0, T] \times \mathcal{D}\),

\[
\mathcal{D} = \begin{cases}
\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_+ & \text{if } n = 2k - 1, \\
\underbrace{\mathbb{R}_+ \times \mathbb{R}_0 \times \mathbb{R}_- \times \mathbb{R}_0 \times \cdots \times \mathbb{R}_-}_{4k-3} & \text{if } n = 2k.
\end{cases}
\]

The function \( f(t, x_0, \ldots, x_{2n-2}) \) may be singular at the points \( x_i = 0, \ 0 \leq i \leq 2n - 2 \), of the space variables \( x_0, \ldots, x_{2n-2} \).
Assumptions:

\((H_1)\) \(f \in \text{Car}([0, T] \times D)\) and there exists \(\psi \in L_1[0, T]\) such that

\[0 < \psi(t) \leq f(t, x_0, \ldots, x_{2n-2})\]

for a.e. \(t \in [0, T]\) and each \((x_0, \ldots, x_{2n-2}) \in D;\)

\((H_2)\) For a.e. \(t \in [0, T]\) and for each \((x_0, \ldots, x_{2n-2}) \in D,\)

\[f(t, x_0, \ldots, x_{2n-2}) \leq \phi(t) + \sum_{j=0}^{2n-2} q_j(t)\omega_j(|x_j|) + \sum_{j=0}^{2n-2} h_j(t)|x_j|\]

where \(\phi, h_j \in L_1[0, T], h_j \text{ are sufficiently small and} q_j \in L_\infty[0, T] \text{ are nonnegative, } \omega_j : \mathbb{R}_+ \to \mathbb{R}_+ \text{ are nonincreasing,}\)

\[
\int_0^T \omega_j(s) \, ds < \infty, \quad \omega_j(uv) \leq \Lambda \omega_j(u)\omega_j(v).
\]
Definition

A function \( x \in AC^{2n-1}[0, T] \) is said to be a solution of BVP (9), (10) if \((-1)^j x^{(2j)}(t) > 0\) for \( t \in (0, T) \) and \( 0 \leq j \leq n - 1 \), \( x \) satisfies the boundary conditions (10) and (9) holds a.e. on \( J \).

Existence theorem - Lidstone

*Let assumptions \((H_1)\) and \((H_2)\) be satisfied. Then there exists a solution of the Lidstone BVP (9), (10).*

Proofs are based on:

- a priori estimates of solutions,
- the Green’s functions,
- investigation of zeros of solutions,
- the general existence principle
Singular Lidstone

Irena Rachůnková  Singular boundary value problems
Let \( n \in \mathbb{N} \), \([0, T] \subset \mathbb{R}\) and \( D \subset \mathbb{R}^n \), \( D \neq \overline{D} \).

We study the singular BVP

\[
u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)),\]

\[u \in B,
\]

where \( f \) satisfies the local Carathéodory conditions on \([0, T] \times D\), \( f(t, x_0, \ldots, x_{n-1}) \) may be singular on the boundary \( \partial D \) at the points \( x_i = 0 \), \( 0 \leq i \leq n-1 \), of the space variables \( x_0, \ldots, x_{n-1} \) and \( B \) is a closed subset in \( C^{n-1}([0, T]) \).

We look for solutions \( u \in AC^{n-1}([0, T]) \cap B \) such that

\[\left((u(t), \ldots, u^{(n-1)}(t)) \in \overline{D} \subset \mathbb{R}^n \right) \text{ for all } t \in [0, T].\]
Consider the singular boundary value problem

\[ u^{(n)}(t) = f(t, u(t), \ldots, u^{(n-1)}(t)), \quad u \in \mathcal{B}, \quad (11) \]

and the sequence of regular boundary value problems

\[ u^{(n)}(t) = f_m(t, u(t), \ldots, u^{(n-1)}(t)), \quad u \in \mathcal{B}, \quad (12) \]

where \( f_m \) satisfies the local Carathéodory conditions on the set 
\([0, T] \times \mathbb{R}^n), \quad m \in \mathbb{N}.

Assume that for a.e. \( t \in [0, T] \) and all \( m \in \mathbb{N}, (x_0, \ldots, x_{n-1}) \in \mathcal{D}, 0 \leq i \leq n-1, \)

\[ f_m(t, x_0, \ldots, x_{n-1}) = f(t, x_0, \ldots, x_{n-1}) \text{ if } |x_i| \geq \frac{1}{m}. \]
Theorem - General existence principle

Assume that there is a bounded set \( \Omega \subset C^{n-1}([0, T]) \) such that

\[(i)\] \( \forall m \in \mathbb{N} \), the regular BVP (12) has a solution \( u_m \in \Omega \);

\[(ii)\] \( \forall \varepsilon > 0 \) \( \exists \delta > 0 \) such that for each \( m \in \mathbb{N} \) and each system of mutually disjoint intervals \( \{(\tau_j, t_j)\}_{j=1}^{\infty} \) in \([0, T] \),

\[
\sum_{j=1}^{\infty} (t_j - \tau_j) < \delta \Rightarrow \sum_{j=1}^{\infty} \int_{\tau_j}^{t_j} |f_m(t, u_m(t), \ldots, u_m^{(n-1)}(t))| \, dt < \varepsilon.
\]

Then:

\[(a)\] there exist \( u \in cl(\Omega) \) and a subsequence \( \{u_k\} \subset \{u_m\} \) such that \( \lim_{k \to \infty} \|u_k - u\|_{C^{n-1}} = 0 \),

\[(b)\] if \( \mu(\mathcal{V}) = 0 \), where \( \mathcal{V} \) is the set of all zeros of the functions \( u^{(i)} \) with \( 0 \leq i \leq n - 1 \), then \( u \) is a solution of the singular BVP (11), (12) and \( u > 0 \) on \((0, T)\).
The proof of the **General existence principle** is based on:
- topological degree arguments,
- the combination of regularization and sequential techniques,
- the Vitali convergence theorem.

**Vitali convergence theorem**

Let \( \varphi_m \subset L_1[0, T] \) for \( m \in \mathbb{N} \) and let

\[
\lim_{m \to \infty} \varphi_m(t) = \varphi(t) \quad \text{for a.e.} \ t \in [0, T].
\]

Then the following statements are equivalent:

(i) \( \varphi \in L_1[0, T] \) and \( \lim_{m \to \infty} \|\varphi_m - \varphi\|_1 = 0 \),

(ii) the sequence \( \{\varphi_m\} \) is **uniformly integrable** on \([0, T]\).
The singular \((n, p)\) boundary value problem

\[-x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),\]
\[x^{(i)}(0) = 0, \ 0 \leq i \leq n - 2, \ x^{(p)}(T) = 0,\]

(13) \hspace{1cm} (14)

where \(n \geq 2, \ 0 \leq p \leq n - 1,\) and \(f\) satisfies the local Carathéodory conditions on the set \([0, T] \times \mathcal{D},\)

\[
\mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_0^{n-2} \times \mathbb{R}.
\]

The function \(f(t, x_0, \ldots, x_{n-1})\) may be singular at the points \(x_i = 0, \ 0 \leq i \leq n - 2\) of the space variables \(x_0, \ldots, x_{n-2}.\)
The singular Sturm-Liouville boundary value problem

\[-x^n(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),\]
\[x^{(i)}(0) = 0, \quad 0 \leq i \leq n - 3,\]
\[\alpha x^{(n-2)}(0) - \beta x^{(n-1)}(0) = 0,\]
\[\gamma x^{(n-2)}(T) + \delta x^{(n-1)}(T) = 0,\]

where \( n > 2, \alpha, \gamma > 0, \beta, \delta \geq 0, \) and \( f \) satisfies the local Carathéodory conditions on the set \([0, T] \times \mathcal{D},\)

\[\mathcal{D} = \mathbb{R}_{+}^{n-1} \times \mathbb{R}_0.\]

The function \( f(t, x_0, \ldots, x_{n-1}) \) may be singular at the points \( x_i = 0, \ 0 \leq i \leq n - 1, \) of all its space variables \( x_0, \ldots, x_{n-1}.\)
Other higher order singular problems

The singular \((p, n - p)\) right focal boundary value problem

\[
(-1)^{n-p}x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),
\]

\[
x^{(i)}(0) = 0, \ 0 \leq i \leq p - 1, \ x^{(i)}(T) = 0, \ p \leq i \leq n - 1,
\]

where \(n > 2, 1 \leq p \leq n - 1\), and \(f\) satisfies the local Carathéodory conditions on the set \([0, T] \times D\),

\[
D = \begin{cases} 
\mathbb{R}_{+}^{p+1} \times \mathbb{R}_{-} \times \mathbb{R}_{+} \times \mathbb{R}_{-} \times \cdots \times \mathbb{R}_{+} & \text{if } n - p \text{ is odd} \\
\mathbb{R}_{+}^{p+1} \times \mathbb{R}_{-} \times \mathbb{R}_{+} \times \mathbb{R}_{-} \times \cdots \times \mathbb{R}_{-} & \text{if } n - p \text{ is even.}
\end{cases}
\]

The function \(f(t, x_0, \ldots, x_{n-1})\) may be singular at the points \(x_i = 0, \ 0 \leq i \leq n - 1\), of all its space variables \(x_0, \ldots, x_{n-1}\).
Other higher order singular problems

The singular \((p, n - p)\) conjugate boundary value problem.

\[
(-1)^p x^{(n)}(t) = f(t, x(t), \ldots, x^{(n-1)}(t)),
\]

\[
x^{(i)}(0) = 0, \quad x^{(j)}(T) = 0, \quad 0 \leq i \leq n - p - 1, \quad 0 \leq j \leq p - 1,
\]

where \(n > 2, p \leq n - 1,\) and \(f\) satisfies the local Carathéodory conditions on the set \([0, T] \times \mathcal{D},\)

\[
\mathcal{D} = ((0, \infty) \times \mathbb{R}_0^{n-1}), \quad \mathbb{R}_0 = \mathbb{R} \setminus \{0\}.
\]

The function \(f(t, x_0, \ldots, x_{n-1})\) may be singular at the points \(x_i = 0, \ 0 \leq i \leq n - 1,\) of all its space variables \(x_0, \ldots, x_{n-1}.\)
1. Principles of solvability of singular higher order BVPs
2. Existence results for singular higher order BVPs
3. Principles of solvability of singular second order BVPs with \(\phi\)-Laplacian
4. Singular Dirichlet BVP with \(\phi\)-Laplacian
5. Singular Robin BVP with \(\phi\)-Laplacian
6. Other types of singular second order BVPs

3.0. Initial \(\phi\)-Lapl.
3.1. Regularization of singular \(p\)-Lapl. with \(\phi\)-Laplacian (e.g., linear, p.e. pol., spec. Dir.) – scheme dichot.
3.2. Existence principle for singular problems with \(\phi\)-Lapl. – scheme dichot.
3.3. Historical and tabl. remarks

5.1. Operator representation (\(\alpha > 0\mbox{, } \beta > 0\))
\[
\begin{pmatrix}
\phi + \beta I_d & \alpha I_d \\
\alpha I_d & \phi + \beta I_d
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2
\end{pmatrix} = \begin{pmatrix}
u_0 \\
u_0 + \alpha u_1 + \beta u_2
\end{pmatrix}
\]
5.2. Stability of regular problems
5.3. Existence and multiplicity results for singular problems

4.1. Singularities in \(\phi\)-Lapl. (e.g. of solutions, if changes sign)
4.2. Singular in the \(\phi\)-Lapl. (e.g., \(w \in \mathbb{R}\))
4.3. Singular in \(\phi\)-Lapl.
4.4. Multiplicity

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Singular boundary value problems
We investigate the singular second order ordinary differential equation

\[ u'' - \frac{a}{t} u' = \lambda f(t, u, u'), \]

where \( a \in \mathbb{R} \setminus \{0\} \), \( \lambda > 0 \) and \( f(t, x, y) \) satisfies the local Carathéodory conditions on \([0, T] \times \mathbb{R}^2\).

\[ u'' - \frac{a}{t} u' = \lambda f(t, u, u'), \quad (15) \]

\[ u(0) = u(T), \quad u'(0) = u'(T). \quad (16) \]

A function \( u \) is called a solution of equation (15) if \( u \in AC^1[0, T] \) fulfils the equation for a.e. \( t \in [0, T] \).

If \( a \neq 0 \), then the solution \( u \) satisfies

\[ \lim_{t \to 0^+} u'(t) = 0. \]


- Opposite-ordered lower and upper functions.
- Auxiliary boundary conditions: \( u(0) = u(T), \ u'(T) = 0. \)
- Leray-Schauder degree method.
Existence theorem - periodic problem

Let $a > 0$. Suppose that there exist $A, B \in \mathbb{R}$, such that $A < B$ and

\[ f(t, x, y) > 0 \quad \text{for a.e. } t \in [0, T] \text{ and all } x \leq A, y \in \mathbb{R} \quad (17) \]
\[ f(t, x, y) < 0 \quad \text{for a.e. } t \in [0, T] \text{ and all } x \geq B, y \in \mathbb{R}. \quad (18) \]

Further, assume that

\[ |f(t, x, y)| \leq g(t)\omega(|y|) \quad \text{for a.e.} \ t \in [0, T], \text{ all } (x, y) \in \mathbb{R}^2, \quad (19) \]

where $g \in L_1[0, T]$, $\omega(y) \in C[0, \infty)$ are positive and $\omega$ is nondecreasing.

Then problem (15), (16) has a solution for each $\lambda \in (0, \lambda^*)$, where

\[ \lambda^* = \int_0^\infty \frac{ds}{\omega(s)} \cdot \left( \int_0^T g(t) \, dt \right)^{-1}. \]
The function

\[ f(t, x, y) = \frac{t}{3} - \frac{1}{4\sqrt{t}}(1 + y^2) \arctan x \]  

(20)

satisfies assumptions (17)-(19) on \([0, 1] \times \mathbb{R}^2\) with \(g(t) = \frac{t}{3} + \frac{\pi}{8\sqrt{t}}\) and \(\omega(s) = 1 + s^2\).

We have \(A = 0\) and \(B = \tan \left( \frac{4}{3} \right)\) and

\[ \lambda^* = \int_0^\infty \frac{ds}{\omega(s)} \cdot \left( \int_0^1 g(t) \, dt \right)^{-1} = \frac{6\pi}{2 + 3\pi}. \]

For each \(\lambda \in \left(0, \frac{6\pi}{2 + 3\pi}\right)\) there exists a solution.
Example

Let $\mu \in (0, 1)$. Consider the function

$$f(t, x, y) = h(t, x, y) - \frac{1}{\beta^2 t} \frac{x}{\sqrt{1 + x^2}} (1 + |y|^\alpha)$$ (21)

depending on the parameters $\beta \in (1, \infty)$ and $\alpha \in (0, \infty)$. Here $h \in C([0, 1] \times \mathbb{R}^2)$ and $|h(t, x, y)| \leq \mu$ on $[0, 1] \times \mathbb{R}^2$. Then $f$ satisfies assumptions (17)-(19) on $[0, 1] \times \mathbb{R}^2$ with $g(t) = \mu + \frac{1}{\beta^2 t}$, $\omega(s) = (1 + s)^\alpha$.

We have $A \leq -\frac{\mu}{\sqrt{1-\mu^2}}$ and $B \geq \frac{\mu}{\sqrt{1-\mu^2}}$ and

$$\lambda^* = \begin{cases} \infty & \text{for } \alpha \leq 1, \\ \frac{\beta - 1}{(\alpha - 1)(\beta + \mu(\beta - 1))} & \text{for } \alpha > 1. \end{cases}$$

For each $\lambda \in (0, \lambda^*)$ there exists a solution.

We consider the singular Dirichlet boundary value problem

\begin{align*}
  u''(t) + \frac{a}{t}u'(t) - \frac{a}{t^2}u(t) &= f(t, u(t), u'(t)), \quad (22) \\
  u(0) &= 0, \quad u(T) = 0, \quad (23)
\end{align*}

where \( a \in (-\infty, -1) \). For \( D = [0, \infty) \times \mathbb{R} \) we assume that \( f \) satisfies the local Carathéodory conditions on \( [0, T] \times D \).

For \( c \geq 0 \) we consider the additional condition

\[ u'(T) = -c. \quad (24) \]
Assumptions:

\((H_1)\) There exists \(\varphi \in L^1[0, T]\) such that

\[0 < \varphi(t) \leq f(t, x, y)\] for a.e. \(t \in [0, T]\) and all \((x, y) \in D\).

\((H_2)\) For a.e. \(t \in [0, T]\) and all \((x, y) \in D\) the estimate

\[f(t, x, y) \leq h(t, x, |y|),\]

is fulfilled, where \(h \in Car([0, T] \times [0, \infty)^2)\), \(h(t, x, z)\) is nondecreasing in the variables \(x, z\), and

\[
\lim_{x \to \infty} \frac{1}{x} \int_0^T h(t, x, x) \, dt = 0.
\]
We denote the set of all positive solutions of problem (22), (23), (24) by $S_c$.

**Theorem about the set $S_c$**

Let $(H_1), (H_2)$ hold. Then for each $c \geq 0$ the set $S_c$ is nonempty and compact in $C^1[0, T]$.

$$
\beta(t) = \max\{u(t) : u \in S_0\} \quad \text{for } t \in [0, T].
$$

$$
S = \bigcup_{c \geq 0} S_c.
$$

**Theorem about the set $S \setminus S_0$**

Let $(H_1), (H_2)$ hold. Then for each $t_0 \in (0, T)$ and each $A > \beta(t_0)$ there exists a positive solution $u$ of problem (22), (23) satisfying $u(t_0) = A$. 
Operators:

\[
\tilde{f}(t, x, y) = \begin{cases} 
  f(t, x, y) & \text{if } x \geq 0 \\
  f(t, 0, y) & \text{if } x < 0.
\end{cases}
\]

We define

\[
(Hx)(t) = t \int_t^T s^{-a-2} \left( \int_s^T \xi^{a+1} \tilde{f}(\xi, x(\xi), x'(\xi)) \, d\xi \right) \, ds.
\]

Further, for each \( t_0 \in (0, T) \) and \( A \geq 0 \) we define

\[
(K_{t_0} A x)(t) = \frac{t T^{-a-1} - t_0^{-a-1}}{T^{-a-1} - t_0^{-a-1}} \max\{0, A - (Hx)(t_0)\} + (Hx)(t),
\]

and for each \( c \geq 0 \) we define

\[
(L_c x)(t) = t \frac{c T^{a+1}}{a+1} (T^{-a-1} - t^{-a-1}) + (Hx)(t).
\]
A special case

\[ u''(t) + \frac{a}{t} u'(t) - \frac{a}{t^2} u(t) = f(t, u(t)). \]  

(25)

Now we will work with the following assumptions on \( f \):

\((H_1^*)\) \( f \in \text{Car}([0, T] \times [0, \infty)). \)

\((H_2^*)\) \( 0 < f(t, x) \) for a.e. \( t \in [0, T] \) and all \( x \in [0, \infty). \)

\((H_3^*)\) \( f(t, x) \) is increasing in \( x \) for a.e. \( t \in [0, T] \) and \n
\[
\lim_{x \to \infty} \frac{1}{x} \int_0^T f(t, x) \, dt = 0.
\]

- There exist \textit{minimal and maximal} solutions \( u_{c, \text{min}}, u_{c, \text{max}} \in S_c \) for each \( c \geq 0 \).

- If the interior of the set \( \{(t, x) \in \mathbb{R}^2 : 0 \leq t \leq T, \ u_{c, \text{min}}(t) \leq x \leq u_{c, \text{max}}(t)\} \) is nonempty, then this interior is \textit{covered by graphs} of other solutions of \( S_c \) for each \( c > 0 \).
Theorem

Let \((H_1^*) - (H_3^*)\) hold.

- Assume that there exists \(t_0 \in (0, T)\) such that \(u_{c,\text{min}}(t_0) < u_{c,\text{max}}(t_0)\) for some \(c > 0\). Then for each \(A \in (u_{c,\text{min}}(t_0), u_{c,\text{max}}(t_0))\) there exists \(u \in S_c\) satisfying \(u(t_0) = A\).

- The set \(S_c\) is one-point for each \(c \in [0, \infty) \setminus \Gamma\), where \(\Gamma \subset [0, \infty)\) is at most countable.
Example

Let us choose $\alpha \in [0, 1)$ and for a.e. $t \in [0, T]$ and all $x \in [0, \infty)$, define the function $f$ by

$$f(t, x) = h_1(t) + h_2(t, x)x^\alpha,$$

or

$$f(t, x) = h_1(t) + h_2(t, x)\frac{x}{\ln(x + 2)},$$

where $h_1 \in L^1[0, T]$, $h_1 > 0$ a.e. on $[0, T]$, $h_2$ is nonnegative, bounded and continuous on $[0, T] \times [0, \infty)$ and increasing in $x$. Then $(H_1^*) - (H_3^*)$ hold.
Happy birthday, Svaťa !!!