Nonlocal problems for the generalized Bagley-Torvik fractional differential equation

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Workshop on differential equations
Malá Morávka, 28. 5. 2012
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1. INTRODUCTION

In modelling the motion of a rigid plate immersing in a Newtonian fluid, Torvik and Bagley (1984) considered the fractional differential equation

\[ u''(t) + AD^{3/2}u(t) = au(t) + \varphi(t), \quad A, a \in \mathbb{R}, \quad A \neq 0, \quad (1) \]

subject to the initial homogeneous conditions

\[ u(0) = 0, \quad u'(0) = 0, \quad (2) \]

where

\[ D^{3/2}u(t) = \frac{1}{\Gamma(\frac{1}{2})} \frac{d^2}{dt^2} \int_0^t (t - s)^{-1/2} u(s) \, ds \]

is the Riemann-Liouville fractional derivative of order \( \frac{3}{2} \).

In the literature equation (1) is called the Bagley-Torvik equation.

Numerical solutions of the problem

\[ u''(t) + A^{cD^\alpha} u(t) = au(t) + \varphi(t), \]

\[ u(0) = y_0, \quad u'(0) = y_1, \]

\[ c^{D^\alpha} u(t) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dt^2} \int_{0}^{t} (t - s)^{1-\alpha} (u(s) - u(0) - u'(0)s) \, ds, \quad \alpha \in (1, 2), \]

(Caputo fractional derivative of order \( \alpha \))

were discussed for \( \alpha = \frac{3}{2} \) by Cenesiz, Keskin, Kurnaz (2010) and by Diethelm, Ford (2002), and by Edwards, Ford, Simpson (2002) for \( \alpha \in (1, 2) \). Applying the Adomian decomposition method, analytical solutions of the above problem were obtained by Deftardar-Gejji, Jafari (2005) for \( \alpha \in (1, 2) \).

Analytical and numerical solutions of the boundary value problem

\[ u''(t) + A^{cD^{\frac{3}{2}}} u(t) = au(t) + \varphi(t), \]

\[ u(0) = y_0, \quad y(T) = y_1. \]

were discussed by Al-Mdallal, Syam, Anwar (2010).


\[ u''(t) + A^{cD^{\frac{3}{2}}} u(t) + u(t) = 0, \quad u''(t) + AD^{\frac{3}{2}} u(t) + u(t) = 0. \]
Existence and uniqueness results for the generalized Bagley-Torvik fractional differential equation

\[ u''(t) + A^cD^\alpha u(t) = f(t, u(t), cD^\mu u(t), u'(t)), \quad A \in \mathbb{R} \setminus \{0\}, \]

subject to the boundary conditions

\[ u'(0) = 0, \quad u(T) + au'(T) = 0, \quad a \in \mathbb{R}, \]

where \( \alpha \in (1, 2), \mu \in (0, 1), f \in \text{Car}([0, T] \times \mathbb{R}^3) \) were given by S.S. (2012).

Note that

\[ cD^\alpha u(t) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dt^2} \int_0^t (t - s)^{1-\alpha} (u(s) - u(0) - u'(0)s) \, ds, \quad \alpha \in (1, 2), \]

\[ cD^\mu u(t) = \frac{1}{\Gamma(1 - \mu)} \frac{d}{dt} \int_0^t (t - s)^{-\mu} (u(s) - u(0)) \, ds, \quad \mu \in (0, 1). \]
The Riemann-Liouville fractional integral $I^\gamma v$ of $v : [0, T] \rightarrow \mathbb{R}$ of order $\gamma > 0$ is defined as

$$I^\gamma v(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1} v(s) \, ds$$

Properties of fractional integral:

- $I^\gamma : C[0, T] \rightarrow C[0, T], \quad I^\gamma : L^1[0, T] \rightarrow L^1[0, T], \quad \gamma \in (0, 1),$
- $I^\gamma : AC[0, 1] \rightarrow AC[0, 1], \quad \gamma \in (0, 1),$
- $I^\gamma : L^1[0, T] \rightarrow AC[0, T], \quad \gamma \in [1, 2),$
- $I^\beta I^\gamma v(t) = I^{\beta+\gamma} v(t)$ for $t \in [0, T]$, where $v \in L^1[0, T], \beta, \gamma > 0, \beta + \gamma \geq 1$ (semigroup property)
- $\frac{d}{dt} I^{\gamma+1} v(t) = I^\gamma v(t)$ for a.e. $t \in [0, T]$, where $v \in L^1[0, T]$ and $\gamma > 0$. 
**Lemma 1.** Let $w \in C[0, 1]$, $b \in C^1[0, T]$, $\alpha \in (1, 2)$ and let $\varphi_1 \in AC[0, 1]$ be such that $\varphi_1(0) = 0$. Suppose that

$$w(t) = b(t)l^{2-\alpha}w(t) + \varphi_1(t) \quad \text{for } t \in [0, 1].$$

Then for each $n \in \mathbb{N}$ there exists $\varphi_n \in AC[0, 1]$ such that $\varphi_n(0) = 0$ and the equality

$$w(t) = b^n(t)l^{n(2-\alpha)}w(t) + \varphi_n(t) \quad \text{for } t \in [0, 1]$$

holds.

**Corollary.** Let the assumptions of Lemma 1 hold. Then $w \in AC[0, 1]$.

**Proof.** Choose $n \in \mathbb{N}$ such that $n(2-\alpha) > 2$. Then $l^{n(2-\alpha)}w = l^1l^{n(2-\alpha)-1}w \in C^1[0, 1]$. Since $w(t) = b^n(t)l^{n(2-\alpha)}w(t) + \varphi_n(t)$ for $t \in [0, 1]$, we have $w \in AC[0, 1]$. 

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The following result is a generalization of the Gronwall lemma for integrals with singular kernels (Henry (1989)).

**LEMMA 2.** Let $0 < \gamma < 1$, $b \in L^1[0, T]$ be nonnegative and let $K$ be a positive constant. Suppose $w \in L^1[0, T]$ is nonnegative and

$$w(t) \leq b(t) + K \int_0^t (t - s)^{\gamma - 1} w(s) \, ds \quad \text{for a.e. } t \in [0, T].$$

Then

$$w(t) \leq b(t) + LK \int_0^t (t - s)^{\gamma - 1} b(s) \, ds \quad \text{for a.e. } t \in [0, T],$$

where $L = L(\gamma)$ is a positive constant.

$L = K\Gamma(\gamma)E_{\gamma\gamma}(K\Gamma(\gamma) \max\{1, T\})$,

$E_{\beta\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\beta + \gamma)}$ Mittag-Leffler function
Fractional derivatives

The Caputo fractional derivative \( cD^\beta v \) of order \( \beta > 0, \beta \notin \mathbb{N} \), of \( x : [0, T] \rightarrow \mathbb{R} \) is defined by

\[
cD^\beta x(t) = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\beta-1} \left( x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k \right) ds,
\]

where \( n = [\beta] + 1 \) and where \([\beta]\) means the integral part of \( \beta \).

The Riemann-Liouville fractional derivative \( D^\beta v \) of \( v : [0, T] \rightarrow \mathbb{R} \) of order \( \beta > 0 \) is given by

\[
D^\beta x(t) = \frac{1}{\Gamma(n - \beta)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\beta-1} x(s) ds \left( = \frac{d^n}{dt^n} I^{n-\beta} x(t) \right),
\]

where \( n = [\beta] + 1 \).
3. FORMULATION OF OUR PROBLEM

Let \( \mathcal{A} \) be the set of functionals \( \phi : C[0, 1] \to \mathbb{R} \) which are

(i) continuous,

(ii) \( \lim_{c \in \mathbb{R}, c \to \pm \infty} \phi(c) = \pm \infty \),

(We identify the set of constant functions on \([0, 1]\) with \(\mathbb{R}\))

(iii) there exists a positive constant \( L = L(\phi) \) such that

\[
\begin{align*}
  x \in C[0, 1], & \quad |x(t)| > L \text{ for } t \in [0, 1] \Rightarrow \phi(x) \neq 0. \\
(\phi \in \mathcal{A}, & \quad \phi(x) = 0 \text{ for some } x \in C[0, 1] \Rightarrow \exists \xi \in [0, 1] : |x(\xi)| \leq L)
\end{align*}
\]

EXAMPLE. Let \( p, g_j \in C(\mathbb{R}), \) \( p \) be bounded, \( \lim_{v \to \pm \infty} g_j(v) = \pm \infty, j = 0, 1, \ldots, n, \) and let \( 0 \leq t_1 < t_2 < \cdots < t_n \leq 1. \) Then the functionals

\[
\begin{align*}
  \phi_1(x) = g_0 \left( \max_{t \in [0,1]} x(t) \right), & \quad \phi_2(x) = g_0 \left( \min_{t \in [0,1]} x(t) \right), \\
  \phi_3(x) = p(\|x\|) + \int_0^1 g_0(x(t)) \, dt, & \quad \phi_4(x) = \sum_{j=1}^n g_j(x(t_j))
\end{align*}
\]

belong to the set \( \mathcal{A} \).
LEMMA 3. Let $\phi \in \mathcal{A}$. Then there exists a positive constant $L = L(\phi)$ such that the estimate $|c| < L$ holds for each $\lambda > 0$ and each solution $c \in \mathbb{R}$ of the equation

$$\lambda \phi(c) - \phi(-c) = 0.$$
We investigate the Bagley-Torvik fractional functional differential equation

\[ u''(t) + a(t) cD^\alpha u(t) = (Fu)(t) \]  

(3)

together with the nonlocal boundary conditions

\[ u'(0) = 0, \quad \phi(u) = 0, \quad (\phi \in A). \]  

(4)

Here \( \alpha \in (1, 2) \), \( cD \) is the Caputo fractional derivative, \( a \in C^1[0, 1] \) and \( F : C^1[0, 1] \to L^1[0, 1] \) is continuous.

Note that

\[ cD^\alpha u(t) = \frac{1}{\Gamma(2 - \alpha)} \frac{d^2}{dt^2} \int_0^t (t - s)^{1 - \alpha} (u(s) - u(0) - u'(0)s) \, ds, \quad \alpha \in (1, 2) \]

We say that a function \( u \in AC^1[0, 1] \) is a solution of problem (3), (4) if \( u \) satisfies (4) and (3) holds for a.e. \( t \in [0, 1] \).
We work with the following conditions on the function $a$ and the operator $F$ in (3).

**(H$_1$)** $a \in C^1[0, 1]$ and $a(t) \neq 0$ for $t \in [0, 1],$

**(H$_2$)** $F : C^1[0, 1] \to L^1[0, 1]$ is continuous and for a.e. $t \in [0, 1]$ and all $x \in C^1[0, 1]$, the estimate

$$|(Fx)(t)| \leq \varphi(t)\omega(\|x\| + \|x'\|)$$

holds, where $\varphi \in L^1[0, 1]$ and $\omega \in C[0, \infty)$ are nonnegative, $\omega$ is nondecreasing and

$$\lim_{v \to \infty} \frac{\omega(v)}{v} = 0.$$
In order to prove the solvability of problem (3), (4), we define an operator $\mathcal{F}$ acting on $[0, 1] \times C^1[0, 1] \times \mathbb{R}$ by the formula

$$\mathcal{F}(\lambda, x, c) = (\mathcal{F}_1(\lambda, x, c), \mathcal{F}_2(x, c)),$$

where

$$\mathcal{F}_1(\lambda, x, c)(t) = c + \int_0^t (Qx)(s) \, ds + \lambda \int_0^t (t - s)(Fx)(s) \, ds,$$
$$\mathcal{F}_2(x, c) = c - \phi(x),$$

and

$$(Qx)(t) = -a(t)l^{2-\alpha}x'(t) + \int_0^t a'(s)l^{2-\alpha}x'(s) \, ds. \quad (5)$$

Here the function $a$ and the operator $F$ are from equation (3) and $\phi \in \mathcal{A}$ is from the boundary conditions (4)
Properties of $Q$, $F_1$ and $F_2$

- Let $(H_1)$ hold. Then $Q : C^1[0, 1] \to C[0, 1]$ and $Q$ is completely continuous.
- Let $(H_1)$ and $(H_2)$ hold. Then $F_1 : [0, 1] \times C^1[0, 1] \times \mathbb{R} \to C^1[0, 1]$ and $F_1$ is completely continuous.
- Let $\phi \in \mathcal{A}$. Then $F_2 : C^1[0, 1] \times \mathbb{R} \to \mathbb{R}$ and $F_2$ is completely continuous.

\[
F_1(\lambda, x, c)(t) = c + \int_0^t (Qx)(s) \, ds + \lambda \int_0^t (t - s)(Fx)(s) \, ds,
\]
\[
F_2(x, c) = c - \phi(x),
\]
\[
(Qx)(t) = -a(t)t^{2-\alpha}x'(t) + \int_0^t a'(s)t^{2-\alpha}x'(s) \, ds.
\]
LEMMA 4. Let \((H_1)\) and \((H_2)\) hold. Then

(a) \(\mathcal{F} : [0,1] \times C^1[0,1] \times \mathbb{R} \to C^1[0,1] \times \mathbb{R}\) and \(\mathcal{F}\) is completely continuous,

(b) if \((x, c)\) is a fixed point of \(\mathcal{F}(1, \cdot, \cdot)\), then \(x\) is a solution of problem (3), (4) and \(c = x(0)\).

Proof.

(b) Let \((x, c)\) be a fixed point of \(\mathcal{F}(1, \cdot, \cdot)\). Then \(x \in C^1[0,1]\),

\[
x(t) = c + \int_0^t (Qx)(s) \, ds + \int_0^t (t-s)(Fx)(s) \, ds, \quad t \in [0,1],
\]

and \(\phi(x) = 0\). Differentiating (6) gives

\[
x'(t) = -a(t)I^{2-\alpha}x'(t) + \int_0^t a'(s)I^{2-\alpha}x'(s) \, ds + \int_0^t (Fx)(s) \, ds, \quad t \in [0,1].
\]

Therefore, \(x'(0) = 0\), and so \(x\) satisfies the boundary conditions (4). Since

\[
\int_0^t a'(s)I^{2-\alpha}x'(s) \, ds \in C^1[0,1] \text{ and } \int_0^t (Fx)(s) \, ds \in AC[0,1],
\]

(7) shows that

\[
x'(t) = -a(t)I^{2-\alpha}x'(t) + \psi(t), \quad t \in [0,1],
\]

where \(\psi \in AC[0,1]\) and \(\psi(0) = 0\). Hence, by Corollary, \(x' \in AC[0,1]\). It follows from the R.-L. factional integrals that \(I^{2-\alpha}x' \in AC[0,1]\).
Next we have
\[
\frac{d}{dt} \left[ \frac{1}{a(t)} \left( -x'(t) + \int_0^t a'(s) l^{2-\alpha} x'(s) \, ds + \int_0^t (Fx)(s) \, ds \right) \right] = \frac{(Fx)(t) - x''(t)}{a(t)} \in L^1[0, 1] \text{ for a.e. } t \in [0, 1].
\]

Since, by (7), the equality
\[
I^{2-\alpha} x'(t) = \frac{1}{a(t)} \left( -x'(t) + \int_0^t a'(s) l^{2-\alpha} x'(s) \, ds + \int_0^t (Fx)(s) \, ds \right)
\]
holds for \( t \in [0, 1] \), we have
\[
\frac{d}{dt} I^{2-\alpha} x'(t) = \frac{(Fx)(t) - x''(t)}{a(t)} \text{ for a.e. } t \in [0, 1].
\]

Consequently,
\[
x''(t) + a(t) \frac{d}{dt} I^{2-\alpha} x'(t) = (Fx)(t) \text{ for a.e. } t \in [0, 1].
\]
\[\underbrace{\ cD^\alpha x(t) \}_{cD^\alpha x(t)} \]
Since $I^{2-\alpha}x'(t) = I^{3-\alpha}x''(t) = I^1 I^{2-\alpha}x''(t)$, we have $\frac{d}{dt}I^{2-\alpha}x'(t) = I^{2-\alpha}x''(t)$ a.e. on $[0,1]$. Since $x'' \in L^1[0,1]$, it follows that $^cD^\alpha x(t) = I^{2-\alpha}x''(t)$ for a.e. $t \in [0,1]$. Hence $\frac{d}{dt}I^{2-\alpha}x'(t) = ^cD^\alpha x(t)$ a.e. on $[0,1]$, and therefore $x$ is a solution of (3). As a result $x$ is a solution of problem (3), (4), and (6) gives $c = x(0)$. 
LEMMA 5. Let $(H_1)$ and $(H_2)$ hold. Then there exists a positive constant $S$ such that for each $\lambda \in [0, 1]$ and each fixed point $(x, c)$ of the operator $\mathcal{F}(\lambda, \cdot, \cdot)$ the estimate

$$\|x\| < S, \quad \|x'\| < S, \quad |c| < S$$

holds.
5. EXISTENCE RESULTS

We need the following result (Deimling (1985)).

**LEMMA 6.** Let $X$ be a Banach space and let $\Omega \subset X$ be open bounded and symmetric with respect to $0 \in \Omega$. Let $\mathcal{F} : \overline{\Omega} \to X$ be a compact operator and $\mathcal{G} = I - \mathcal{F}$, where $I$ is the identical operator on $X$. If $x \neq \mathcal{F}x$ for $x \in \partial \Omega$ and $\mathcal{G}(-x) \neq \lambda \mathcal{G}(x)$ on $\partial \Omega$ for all $\lambda \geq 1$, then $\deg(I - \mathcal{F}, \Omega, 0) \neq 0$. 
THEOREM 1. Let \((H_1)\) and \((H_2)\) hold. Then problem (3), (4) has at least one solution.

Proof. We have to show that \(\mathcal{F}(1, \cdot, \cdot)\) has a fixed point \((x, c)\). Then \(x\) is a solution of problem (3), (4) and \(c = x(0)\). Let \(S\) be a positive constant from Lemma 5 and let \(L = L(\phi)\) be from Lemma 3 (note that \(|c| < L\) holds for each \(\lambda > 0\) and each solution \(c \in \mathbb{R}\) of \(\lambda \phi(c) - \phi(-c) = 0\)). Let \(W = \max\{S, L\}\) and

\[
\Omega = \{(x, c) \in C^1[0, 1] \times \mathbb{R} : \|x\| < W, \|x'\| < W, |c| < W\}.
\]

We prove by Lemma 6 that \(\deg\{\mathcal{I} - \mathcal{F}(0, \cdot, \cdot), \Omega, 0\} \neq 0\), where \(\mathcal{I}\) is the identical operator on \(C^1[0, 1] \times \mathbb{R}\). Note that

\[
G(x, c) = (x, c) - \mathcal{F}(0, x, c) = \left(x(t) - c - \int_0^t (Qx)(s) \, ds, \phi(x)\right).
\]

Let \((x, c)\) be a fixed point of \(\mathcal{F}(\lambda, \cdot, \cdot)\) for some \(\lambda \in [0, 1]\). Then, by Lemma 5, \((x, c) \notin \partial\Omega\), and therefore, by the homotopy property,

\[
\deg(\mathcal{I} - \mathcal{F}(1, \cdot, \cdot), \Omega, 0) = \deg(\mathcal{I} - \mathcal{F}(0, \cdot, \cdot), \Omega, 0)\). Hence \(\deg(\mathcal{I} - \mathcal{F}(1, \cdot, \cdot), \Omega, 0) \neq 0\).

The last relation implies that \(\mathcal{F}(1, \cdot, \cdot), \Omega, 0)\) has a fixed point.
EXAMPLE. Let $\varphi_1, \varphi_2 \in L^1[0, 1]$, $h \in C[0, \infty)$, $p \in C(\mathbb{R})$, $\lim_{v \to \infty} \frac{h(v)}{v} = 0$ and $\lim_{|v| \to \infty} \frac{p(v)}{v} = 0$. Define an operator $F : C^1[0, 1] \to L^1[0, 1]$ by

$$(Fx)(t) = \varphi_1(t) \left( h(\|x'\|) + \int_0^t p(x(s)) \, ds \right) + \varphi_2(t).$$

Then $F$ satisfies condition $(H_2)$. To check it we take $\varphi(t) = |\varphi_1(t)| + |\varphi_2(t)|$ and $\omega(v) = \tilde{h}(v) + \tilde{p}(v)$, where $\tilde{h}(v) = \max\{h(\nu) : 0 \leq \nu \leq v\}$, $\tilde{p}(v) = \max\{p(\nu) : |\nu| \leq v\}$, $v \in [0, \infty)$. 
The special case of (3) is the fractional differential equation

\[ u''(t) + a(t) cD^\alpha u(t) = f(t, u(t), cD^\gamma u(t), u'(t)), \tag{8} \]

where \( \alpha \in (1, 2) \), \( \gamma \in (0, 1) \) and \( f \) satisfies the condition

\[ (H_3) \quad f \in Car([0, 1] \times \mathbb{R}^3) \quad \text{and for a.e. } t \in [0, 1] \quad \text{and all } (x, y, z) \in \mathbb{R}^3 \quad \text{the estimate} \]

\[ |f(t, x, y, z)| \leq \varphi(t)\rho(|x| + |y| + |z|) \]

holds, where \( \varphi \in L^1[0, 1] \) and \( \rho \in C[0, \infty) \) are nonnegative, \( \rho \) is nondecreasing and \( \lim_{v \to \infty} \frac{\rho(v)}{v} = 0 \).
The following theorem gives an existence result for problem (8), (4).

**THEOREM 2.** Let \((H_1)\) and \((H_3)\) hold. Then problem (8), (4) has at least one solution.

**Proof.** Let \(F\) be an operator acting on \(C^1[0, 1]\) and given by

\[
(Fx)(t) = f(t, x(t), \frac{\partial D}{\partial x} x(t), x'(t)).
\]

\(F\) satisfies condition \((H_2)\) for \(\omega(v) = \rho \left( \frac{2v}{\Gamma(2-\gamma)} \right)\). The solvability of problem (8), (4) follows from Theorem 1.
6. UNIQUENESS RESULTS

Let $\mathcal{B}$ be the set all functionals $\phi : C[0, 1] \rightarrow \mathbb{R}$ which are

(i) continuous,

(ii) increasing, that is,

$$x, y \in C[0, 1] \quad x(t) < y(t) \text{ for } t \in [0, 1] \Rightarrow \phi(x) < \phi(y).$$

EXAMPLE. Let $g_j \in C(\mathbb{R})$ be increasing ($j = 0, 1, \ldots, n$), and let

$0 \leq t_0 \leq t_1 < \cdots < t_n \leq 1$. Then the functionals

$$\phi_1(x) = g_0 \left( \max_{t \in [0,1]} x(t) \right), \quad \phi_2(x) = g_0 \left( \min_{t \in [0,1]} x(t) \right),$$

$$\phi_3(x) = \int_0^1 g_0(x(t)) \, dt, \quad \phi_4(x) = \sum_{j=1}^n g_j(x(t_j))$$

belong to $\mathcal{B}$. 

We discuss equation (8), where \( f(t, x, y, z) = \varphi(t)p(t, x, y, z) \), that is, the equation

\[
u''(t) + a(t)^cD^\alpha u(t) = \varphi(t)p(t, u(t), ^cD^\gamma u(t), u'(t)), \tag{9}\]

where \( \alpha \in (1, 2), \gamma \in (0, 1) \). Together with (9) the boundary conditions

\[
u'(0) = 0, \quad \phi(u) = 0, \quad (\phi \in \mathcal{B}) \tag{10}\]

and

\[
u'(0) = 0, \quad \phi(u) = 0, \quad (\phi \in \mathcal{A} \cap \mathcal{B}) \tag{11}\]

equation are investigated.
THEOREM 3. Let

\( (S_1) \) \( a \in C^1[0, 1], \varphi \in L^1[0, 1] \) are such that \( a < 0, a' \geq 0 \) on \( [0, 1] \) and \( \varphi > 0 \) a.e. on \( [0, 1] \),

\( (S_2) \) \( p \in C([0, 1] \times \mathbb{R}^3) \) and \( p(t, x, y, z) \) is increasing in the variable \( x \) and nondecreasing in the variables \( y \) and \( z \),

\( (S_3) \) there exists \( \kappa > 0 \) such that for each \( \rho \in \mathbb{R} \) the estimate

\[ |p(t, \rho + x_1, y_1, z_1) - p(t, \rho + x_2, y_2, z_2)| \leq k_{\rho}(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|) \]

holds for \( x_j, y_j, z_j \in [-\kappa, \kappa] \), where \( k_{\rho} \in C[0, 6\kappa] \), \( k_{\rho} \) is nondecreasing and

\[ \limsup_{v \to 0^+} \frac{k_{\rho}(v)}{v} < \infty, \]

hold. Then problem (9), (10) has at most one solution.
EXAMPLE. Let $q_1 \in C^1(\mathbb{R})$, $q_2, q_3 \in C(\mathbb{R}) \cap C^1[-1, 1]$, $q_1$ be increasing and $q_2, q_3$ be nondecreasing. Let $p_j \in C([0, 1] \times \mathbb{R}^2)$ ($j = 1, 2, 3$) be positive and bounded. Then the function

$$p(t, x, y, z) = p_1(t, y, z)q_1(x) + p_2(t, x, z)q_2(y) + p_3(t, x, y)q_3(z)$$

satisfies conditions $(S_2)$ and $(S_3)$ with $\kappa = 1$. 
THEOREM 4. Let \((S_1) - (S_3)\) and \((S_4)\) for \(t \in [0, 1]\) and \((x, y, z) \in \mathbb{R}^3\) the estimate

\[ |p(t, x, y, z)| \leq h(|x| + |y| + |z|) \]

is fulfilled, where \(h \in C[0, \infty)\), \(h\) is nondecreasing and

\[ \lim_{v \to \infty} \frac{h(v)}{v} = 0. \]

hold. Then problem (9), (11) has a unique solution.

EXAMPLE. Let \(q_1 \in C^1(\mathbb{R})\), \(q_2, q_3 \in C(\mathbb{R}) \cap C^1[-1, 1]\), \(q_1\) be increasing and \(q_2, q_3\) be nondecreasing. Let \(p_j \in C([0, 1] \times \mathbb{R}^2)\) \((j = 1, 2, 3)\) be positive and bounded. Besides, \(\lim_{v \to \infty} \frac{1}{v} \max\{|q_j(-v)|, |q_j(v)|\} = 0\) for \(j = 1, 2, 3\). Then the function

\[ p(t, x, y, z) = p_1(t, y, z)q_1(x) + p_2(t, x, z)q_2(y) + p_3(t, x, y)q_3(z) \]

satisfies conditions \((S_2) - (S_4)\).


