Half-linear Euler type differential equation with periodic coefficients

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Contents

1. Introduction
2. Oscillation theory
3. Half-linear Euler equation
4. Modified Riccati equation
5. Periodic perturbations
Half-linear differential equations

Differential equation with the one-dimensional $p$-Laplacian

\[(HL) \quad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x, \quad p > 1,\]

$r, c$ continuous functions, $r(t) > 0$,

Special case $p = 2$

\[(SL) \quad (r(t)x')' + c(t)x = 0.\]

linear Sturm-Liouville differential equation
Why 1/2-linear equations

Motivation

- Partial differential equations with $N$-dimensional $p$-Laplacian

$$\Delta_p u + c(x)\Phi(u) = 0, \quad \Delta_p u = \text{div} \left( ||\nabla u||^{p-2}\nabla u \right)$$

with spherically symmetric potential $c$, i.e., $c(x) = c(\|x\|)$, can be reduced to (HL).

If $\|x\| = t$ then

$$\Delta_p u(x) = t^{1-N} (t^{N-1}\Phi(u'(t)))'$$

- Non-Newtonian fluid theory, models in glaceology,...
Equation (HL) is the Euler-Lagrange equation of the $p$-degree functional

$$\mathcal{F}(y; a, b) = \int_a^b \left[ r(t)|y'(t)|^p - c(t)|y(t)|^p \right] dt$$

Extension of the results for the linear equation (special case $p = 2$ in (HL))

$$\text{(SL)} \quad (r(t)x')' + c(t)x = 0.$$  

to (HL).

Differences between linear and half-linear

The solution space is **only homogeneous, but not additive**  $\implies$ half-linear equations.

No half-linear analogue of the linear transformation identity

\[
\begin{align*}
h \left[(rx')' + cx\right] &\xrightarrow{x=hy} \quad (Ry')' + Cy, \\
R &= rh^2, \quad C = h \left[(rh')' + ch\right]
\end{align*}
\]
No reduction of order formula (D’Alembert formula): $p = 2$ and $x(t) \neq 0$ is a solution of (SL) $\Rightarrow$

$$\tilde{x}(t) = x(t) \int_{s}^{t} \frac{ds}{r(s)x^2(s)}$$

is also a solution of (SL).

No problems with the existence, uniqueness and continuability of solutions, no singular solutions, in contrast to the Emden-Fowler differential equation (which has not homogeneity property!)

$$x'' + c(t)|x|^{p-2}x = 0,$$

where no uniqueness and no continuability is guaranteed (singular solution of the first and second kind).
Half-linear trigonometric functions.

Denote by \( S = S(t) \) the solution of

\[
(\Phi(x'))' + (p - 1)\Phi(x) = 0, \quad x(0) = 0, \quad x'(0) = 1.
\]

Multiplying by \( S' \) and using the initial condition

\[
|S(t)|^p + |S'(t)|^p = 1 \implies S' = \frac{p}{\sqrt{1 - S^p}}
\]

for \( t > 0 \) small. Further denote

\[
\pi_p := 2 \int_0^1 (1 - s^p)^{-\frac{1}{p}} ds = \frac{2}{p} B(1/p, 1/q) = \frac{2\pi}{p \sin \frac{\pi}{p}}
\]

and let

\[
\sin_p t := "2\pi_p" \text{ periodic odd continuation of } S(t),
\]

\[
\cos_p t := (\sin_p t)'.
\]
Half-linear Prüfer transformation

- Half-linear Prüfer transformation

\[
x(t) = \rho(t) \sin_\rho \varphi(t), \quad r^{q-1}(t) x'(t) = \rho(t) \cos_\rho \varphi(t),
\]

\[
\varphi' = r^{1-q}(t) |\cos_\rho \varphi|^p + \frac{c(t)}{\rho - 1} |\sin_\rho \varphi|^p,
\]

\[
\rho' = \Phi(\sin_\rho \varphi(t)) \cos_\rho \varphi(t) \left[ r^{1-q}(t) - \frac{c(t)}{\rho - 1} \right] \rho.
\]

- The right-hand side of the equation for \( \varphi \) is Lipschitzian with respect to \( \varphi \) (and does not contain \( \rho \)) \( \implies \) existence, uniqueness, and continuability for \( \varphi, \rho \) \( \implies \) existence, uniqueness and continuability for (HL).
Oscillation theory

- Linear Sturmian separation and comparison theory extends (almost) verbatim to (HL).

- Half-linear Riccati equation for \( w = r \Phi(x'/x) \):

  \[
  (RE) \quad w' + c(t) + (p - 1)r^{1-q}(t)|w|^q = 0, \quad \frac{1}{p} + \frac{1}{q} = 1
  \]

- Associated energy functional

  \[
  (F) \quad \mathcal{F}(y) = \int_a^b [r(t)|y'|^p - c(t)|y|^p] \, dt.
  \]
Roundabout theorem

The following statements are equivalent:

- (HL) is **disconjugate** on \([a, b]\), i.e., every nontrivial solution has at most one zero in \([a, b]\).

- There exists a solution of the Riccati equation

  \[
  w' + c(t) + (p - 1)r^{1-q}(t)|w|^q = 0
  \]

  defined on the whole interval \([a, b]\).

- The energy functional

  \[
  \mathcal{F}(y; a, b) = \int_a^b \left[ r(t)|y'(t)|^p - c(t)|y(t)|^p \right] dt
  \]

  is positive for every nontrivial \(y \in W^{1,p}(a, b)\) with \(y(a) = 0 = y(b)\).
The half-linear Euler equation

$$\left( \Phi(x') \right)' + \frac{\gamma}{t^p} \Phi(x) = 0$$

is oscillatory if $\gamma > \gamma_p := \left( \frac{p-1}{p} \right)^p$ and nonoscillatory in the opposite case.

If $\gamma = \gamma_p$, then $x(t) = t^{\frac{p-1}{p}}$ is a solution.

The potential $t^{-p}$ is a border line between oscillation and nonoscillation, Kneser type (non)oscillation criteria for the equation

$$\left( \Phi(x') \right)' + c(t)\Phi(x) = 0$$

$$\liminf_{t \to \infty} t^p c(t) > \gamma_p, \quad \limsup_{t \to \infty} t^p c(t) < \gamma_p.$$
The equation

\[(HL) \quad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0\]

with positive \(c\) is conditionally oscillatory if there exists \(\lambda_0 > 0\), the so-called oscillation constant, such that the equation

\[(r(t)\Phi(x'))' + \lambda c(t)\Phi(x) = 0\]

is oscillatory for \(\lambda > \lambda_0\) and nonoscillatory for \(\lambda < \lambda_0\).

If \(\int_{\infty}^{\infty} r^{1-q}(t) \, dt = \infty \quad (\frac{1}{p} + \frac{1}{q} = 1)\), then the equation

\[(r(t)\Phi(x'))' + \frac{1}{r^{q-1}(t)\left(\int_{t}^{\infty} r^{1-q}(s) \, ds\right)^p}\Phi(x) = 0\]

is conditionally oscillatory with the oscillation constant \(\lambda_0 = \gamma_p\).
**Limiting case**

What happens when

$$\lim_{t \to \infty} t^p c(t) = \gamma_p := \left( \frac{p - 1}{p} \right)^p,$$

i.e., we have

$$c(t) = \frac{\gamma_p}{t^p} + d(t) \implies \left( \Phi(x') \right)' + \left[ \frac{\gamma_p}{t^p} + d(t) \right] \Phi(x) = 0$$

with a “small” function $d$. This motivates the investigation of various perturbations of the “critical” half-linear Euler differential equation

(EE) \hspace{1cm} \left( \Phi(x') \right)' + \frac{\gamma_p}{t^p} \Phi(x) = 0.
Transformation approach in the linear case

\[(*) \quad x'' + \left[ \frac{1}{4t^2} + d(t) \right] x = 0 \quad \bigg| \quad x = \sqrt{ty} \bigg| \quad (ty')' + td(t)y = 0.\]

The change of independent variable \( s = \log t, \quad y(s) = x(t), \) in the last equation gives

\[
\frac{d^2}{ds^2} y(s) + t^2 d(t) y(s) = 0
\]

and from Euler equation we know that the limiting case is

\[
t^2 d(t) = \frac{1}{4s^2} \quad \implies \quad d(t) = \frac{1}{4t^2 \log^2 t}.
\]

If

\[
\lim_{t \to \infty} \inf t^2 \log^2 t \ d(t) > \frac{1}{4}
\]

then \((*)\) is oscillatory, if \(\lim_{t \to \infty} \sup t^2 \log^2 t \ d(t) < \frac{1}{4}\), then \((*)\) is nonoscillatory.
Modified Riccati equation

In the linear case, the transformation \( x = h(t)y \) transforms the equation

\[
(r(t)x')' + c(t)x = 0
\]

into the equation

\[
(R(t)y')' + C(t)y = 0
\]

with \( R = rh^2, \quad C = h[(rh')' + ch] \).

In terms of the associated Riccati equations

\[
w = \frac{rx'}{x} = \frac{r(h'y + hy')}{hy} = \frac{rh'}{h} + \frac{1}{h^2} \frac{rh^2 y'}{y},
\]

hence

\[
v = h^2(w - w_h), \quad v = \frac{rh^2 y'}{y}, \quad w_h = \frac{rh'}{h}.
\]
This motivates the transformation of the Riccati equation associated with (HL)

\[ w' + c(t) + (p - 1)r^{1-q}(t)|w|^q = 0, \quad \frac{1}{p} + \frac{1}{q} = 1 \]

of the form

\[ v(t) = h^p(t)(w(t) - w_h(t)), \quad w_h = \frac{r\Phi(h')}{h}. \]

By a direct computation \(v\) satisfies the Modified Riccati equation:

\[ (MRE) \quad v' + h^p(t)C(t) + (p - 1)r^{1-q}(t)h^{-q}(t)H(t, v) = 0, \]

\[ H(t, v) := |v + G(t)|^q - qv\Phi^{-1}(G(t)) - |G(t)|^q, \]

\[ G(t) := r(t)h(t)\Phi(h'(t)), \quad C = h[(r\Phi(h'))' + c\Phi(h)] \]
Special cases

- $p = 2 \implies$

$$H(t, v) = (v + G)^2 - 2Gv - G^2 = v^2,$$

hence modified RE is the equation corresponding to the transformed equation

$$(rh^2 y')' + h[(rh')' + ch]y = 0.$$

- If $r(t) = 1$, $h(t) = t^\frac{p-1}{p}$, then $G(t) \equiv \left(\frac{p-1}{p}\right)^{p-1} =: \Gamma_p$ and

$$H(v, G) = |v + \Gamma_p|^q - v + \gamma_p$$

and

$$C(t) = t^{p-1} \left( c(t) - \frac{\gamma_p}{t^p} \right).$$
Quadratization of the function $H$: Suppose that $h'(t) \neq 0$,

\[ H(t, v) = |G|^q \left\{ \left| \frac{v}{G} + 1 \right|^q - q \frac{v}{G} - 1 \right\} \]

\[ \sim \frac{q(q-1)}{2} |G|^q \left( \frac{v}{G} \right)^2 = \frac{q(q-1)}{2} |G|^{q-2} v^2 \quad \text{as} \quad v \to 0. \]

We have

\[ (p - 1) r^{1-q} h^{-q} H(t, v) \sim \frac{q}{2} \frac{v^2}{R}, \quad R := rh^2 |h'|^{p-2}. \]

hence we obtain the Approximate Riccati equation

\[ (ARE) \quad v' + C(t) + \frac{q}{2} \frac{v^2}{R(t)} = 0. \]

(ARE) is the Riccati equation associated with the linear Sturm-Liouville equation

\[ (R(t)y')' + \frac{q}{2} C(t)y = 0. \]
Application of MRE

Elbert and Schneider, 2000. The half-linear Riemann-Weber equation

\[(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right) \Phi(x) = 0, \quad \gamma_p = \left(\frac{p - 1}{p}\right)^p,\]

is oscillatory iff

\[\mu > \mu_p := \frac{1}{2} \left(\frac{p - 1}{p}\right)^{p-1}.\]

Modified Riccati equation is (with \(h(t) = t^{\frac{p-1}{p}}\))

\[v' + \frac{\mu}{t \log^2 t} + (p - 1)t^{-1} [\nu + \Gamma_p|^q - \nu + \gamma_p] = 0\]

and the Approximate Riccati equation is

\[z' + \frac{\mu}{t \log^2 t} + \frac{1}{2t\Gamma_p} z^2 = 0\]
The change of independent variable $s = \log t$ gives

$$z' + \frac{\mu}{s^2} + \frac{1}{2\Gamma_p} z^2 = 0.$$ 

The associated second order differential equation is

$$(2\Gamma_p y')' + \frac{\mu}{s^2} y = 0$$

which the same as

$$y'' + \frac{\mu}{2\Gamma_p s^2} y = 0$$

and the last equation is nonoscillatory iff

$$\frac{\mu}{2\Gamma_p} \leq \frac{1}{4} \quad \iff \quad \mu \leq \frac{1}{2} \Gamma_p = \frac{1}{2} \left( \frac{p-1}{p} \right)^{p-1} =: \mu_p.$$
More general perturbation of the critical Euler equation

\[
(\Phi(x'))' + \frac{1}{t^p} \left[ \gamma_p + \frac{\mu_p}{\log^2 t} + \frac{\mu}{\log^2 t \log^2(\log t)} \right] \Phi(x) = 0
\]

is using the same method nonoscillatory iff \( \mu \leq \mu_p \).

We consider a more general perturbation of critical Euler equation (EE):

\[
\begin{align*}
(EP) \quad & \left[ \left( 1 + \sum_{j=1}^{n} \frac{\alpha_j}{\text{Log}_j^2(t)} \right) \Phi(x') \right]' + \left[ \frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\beta_j}{t^p \text{Log}_j^2(t)} \right] \Phi(x) = 0
\end{align*}
\]

with

\[
\text{Log}_k(t) = \prod_{j=1}^{k} \log_k(t), \quad \log_k(t) = \log_{k-1}(\log t), \quad \log_1(t) = \log t.
\]
(Non)oscillation of (EP)

Recall that
\[ \gamma_p = \left( \frac{p - 1}{p} \right)^p, \quad \mu_p = \frac{1}{2} \left( \frac{p - 1}{p} \right)^p. \]

O. D. H. Funková (2012):
- If \( \beta_1 + \gamma_p \alpha_1 < \mu_p \) then (EP) is nonoscillatory and if \( \beta_1 + \gamma_p \alpha_1 > \mu_p \) then it is oscillatory.
- Let \( \beta_1 + \gamma_p \alpha = \mu_p \), if \( \beta_2 + \gamma_p \alpha_2 < \mu_p \) then (EP) is nonoscillatory and if \( \beta_2 + \gamma_p \alpha_2 > \mu_p \) then it is oscillatory.
  
  \[ \vdots \]

- Let
  \[ \beta_k + \gamma_p \alpha_k = \mu_p, \quad k = 1, \ldots, n - 1. \]

  Then (EP) is nonoscillatory if and only if
  \[ \beta_n + \gamma_p \alpha_n \leq \mu_p. \]
We consider the equation

\[(HLP)\]  
\[ (r(t)\Phi(x'))' + \frac{\lambda c(t)}{t^p} \Phi(x) = 0 \]

with positive \(\alpha\)-periodic functions \(r, c\)

- If \(r(t) \equiv 1, c(t) \equiv 1\), then (HLP) reduces to the half-linear Euler equation and its oscillation constant is \(\lambda_0 = \gamma_p = \left(\frac{p-1}{p}\right)^p\).

**Theorem** (P. Hasil, 2009). Let

\[ r = \frac{1}{\alpha} \int_0^\alpha ds \frac{1}{r^{q-1}(s)}, \quad c = \frac{1}{\alpha} \int_0^\alpha c(s) \, ds. \]

Then (HLP) is oscillatory if

\[ \lambda > \gamma_{r,c} := \frac{\gamma_p}{(\bar{r})^{p-1} \bar{c}}. \]

and nonoscillatory if \(\lambda < \gamma_{r,c}\).
The limiting case $\lambda = \gamma r, c$

Consider the equation

\[(\text{RW}) \quad (r(t)\Phi(x'))' + \left[ \frac{\gamma r, c c(t)}{t^p} + \frac{\mu d(t)}{t^p \log^2 t} \right] \Phi(x) = 0.\]

**Theorem** (O. D., P. Hasil, 2011). Let $\bar{r}, \bar{c}$ be as before and

$$
\bar{d} = \frac{1}{\alpha} \int_{0}^{\alpha} d(s) \, ds.
$$

Then (RW) is oscillatory if

$$
\mu > \mu_{r,d} := \frac{\mu p}{(\bar{r})^{p-1} \bar{d}}
$$

and nonoscillatory if $\mu < \mu_{r,d}$. In particular, equation (HLP) is nonoscillatory also in the limiting case $\lambda = \gamma r, c$. 

Half-linear Euler type differential equation with periodic coefficients
Consider the equation

\[
(EPP) \quad \left[ \left( r(t) + \sum_{j=1}^{n} \frac{\alpha_j(t)}{\log^2_j(t)} \right)^{1-p} \Phi(x') \right]' + \left[ \frac{c(t)}{t^p} + \sum_{j=1}^{n} \frac{\beta_j(t)}{t^p \log^2_j(t)} \right] \Phi(x) = 0.
\]

with \( T \) periodic functions \( \alpha_j, \beta_j, j = 1, \ldots, n \). Denote by \( \bar{r}, \bar{c}, \bar{\alpha}_j, \bar{\beta}_j \) \( j = 1, \ldots, n \), their mean values over the period \( T \), i.e.,

\[
\bar{r} = \frac{1}{T} \int_0^T r(t) \, dt, \quad \bar{c} = \frac{1}{T} \int_0^T c(t) \, dt,
\]

\[
\bar{\alpha}_j = \frac{1}{T} \int_0^T \alpha_j(t) \, dt, \quad \bar{\beta}_j = \frac{1}{T} \int_0^T \beta_j(t) \, dt.
\]
**Theorem.** (O. D. H. Funková, 2013). Let $\bar{c} r^{p-1} = \gamma_p$. If there exists $k \in \{1, \ldots, n\}$ such that

$$\bar{\beta}_j r^{p-1} + (p - 1) \gamma_p \bar{\alpha}_j r^{-1} = \mu_p, \quad j = 1, \ldots, k - 1,$$

and $\bar{\beta}_k r^{p-1} + (p - 1) \gamma_p \bar{\alpha}_k r^{-1} \neq \mu_p$, then (EPP) is oscillatory if

$$\bar{\beta}_k r^{p-1} + (p - 1) \gamma_p \bar{\alpha}_k r^{-1} > \mu_p$$

and nonoscillatory if

$$\bar{\beta}_k r^{p-1} + (p - 1) \gamma_p \bar{\alpha}_k r^{-1} < \mu_p.$$


