Approximate counting and NP search

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Outline

TFNP

Proofs and search problems

Approximate counting

Our result

NP search problems

An *NP search problem* is specified by

- a polynomial time relation $R(x, y)$ and a polynomial p

such that

- for every x there is a y such that $R(x, y)$ and $|y| < p(|x|)$.

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- for every x there is a y such that $R(x, y)$ and $|y| < p(|x|)$.

It represents the computational task:

- given x, find y such that $R(x, y)$ and $|y| < p(|x|)$.

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Pigeonhole principle (PPP)

We are given a parametrized family of polynomial time functions f_x such that $f_x < x$.
Input: x
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Weak pigeonhole principle (WPHP)

The setup is the same as for PPP.
Input: x
Search task: find distinct $y, y' < 2x$ such that $f_x(y) = f_x(y')$.

Examples

Finite Ramsey theorem (RAMSEY)

We are given a parametrized family of graphs G_n with vertices $[0, n)$, polynomial-time structure, and edges coloured red or blue.

Input: n

Search task: find vertices $v_1, \ldots, v_{(\log n)/2} \in [0, n)^{(\log n)/2}$ forming a monochrome clique.

Reducibility

Definition

Given search problems $R(x, y)$ and $Q(x', y')$, we say

R is *polynomial-time many-one reducible* to Q if there are polynomial time functions $f(x)$ and $g(x, y')$ such that

$$\forall x, y', \ Q(f(x), y') \to R(x, g(x, y')).$$

We write $R \leq Q$.

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If reducibility holds in both directions then R and Q are equivalent, written $R \equiv Q$.

TFNP

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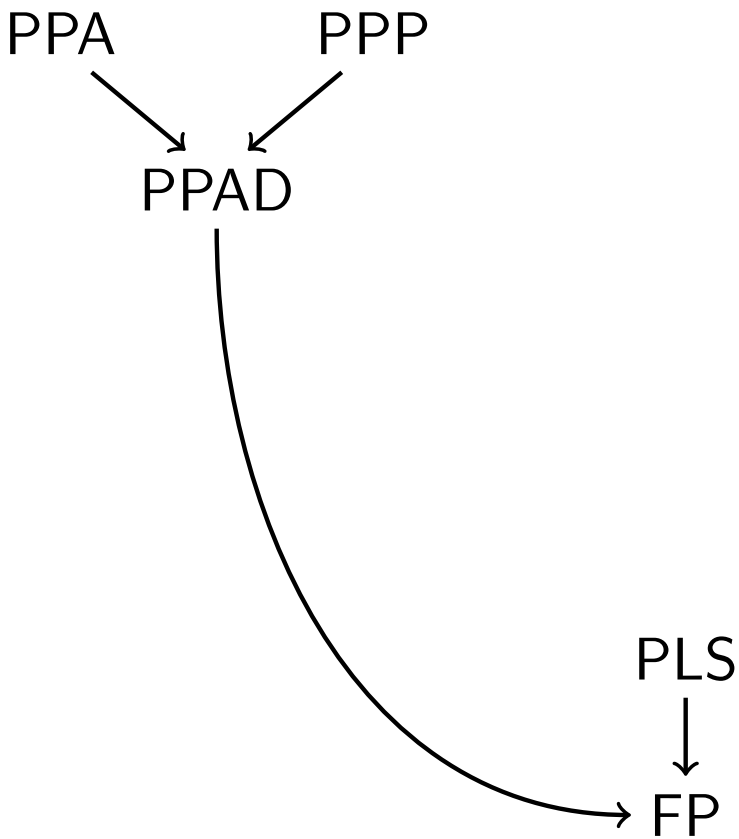
If P = NP then every problem in TFNP can be solved in P.

But in fact we think there is a rich structure inside TFNP.

[Johnson, Papadimitriou, Yannakakis 88],
[Megiddo, Papadimitriou 91], [Papadimitriou 94]

TFNP



TFNP

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Variables range over binary strings, which we identify with natural numbers.

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$$\forall \bar{x} \varphi(\bar{x})$$

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All our theories will contain BASE. It follows that our theories will prove that all many-one reductions behave as they are supposed to.

A simple bounded arithmetic

Definition

A Σ_k^b formula is one of the form

$$\exists x_1 < t_1(\bar{z})\ \forall x_2 < t_2(x_1, \bar{z})\ \ \ldots\ \ \varphi(x_1, \ldots, x_k, \bar{z})$$

where there are k quantifier alternations,

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Σ_1^b formulas express precisely the relations in NP.

Σ_k^b formulas express precisely the relations in Σ_k^p, the k-th level of the polynomial hierarchy.

A family of theories

Definition

Σ^b_k-IND is the theory consisting of BASE plus the induction scheme

$$\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$$

for all Σ^b_k formulas φ.

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Theorem [Buss 85]

Σ^b_k-IND proves:

"Every $P^{\Sigma^p_k}$ machine has a computation on every input."

For our purposes, Σ^b_k-IND is equivalent to this statement.

Search problems and theories

We can identify search problems in TFNP with true $\forall\Sigma^b_1$ sentences.

That is, true sentences of the form $\forall x \exists y < t(x)\, \varphi(x, y)$.

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The search problem is provably total in a theory T if T proves this sentence.

Write $\forall\Sigma^b_1(T)$ for the set of search problems provably total in T.

Relativized search problems

We now add an oracle tape to our machines and a symbol for an arbitrary oracle to our language.

Everything works as before, but we can now try to find oracles with respect to which we can show non-reducibility between search problems.

An old open problem

Problem

Is $\forall\Sigma^b_1(\Sigma^b_k\text{-IND})$ strictly bigger than $\forall\Sigma^b_1(\Sigma^b_j\text{-IND})$ if $k > j$?

This is open for $k \geq 2$.

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This is open for $k \geq 2$.

Put differently – do more TFNP problems arise from the axiom

every $P^{\Sigma^p_k}$ machine has a computation

than from the axiom

every $P^{\Sigma^p_j}$ machine has a computation?

The game induction principle

The *k-turn game induction principle* GI_k is an NP search problem.

It states a property of a sequence $G_0, \ldots, G_{x-1}$ of k-turn games.

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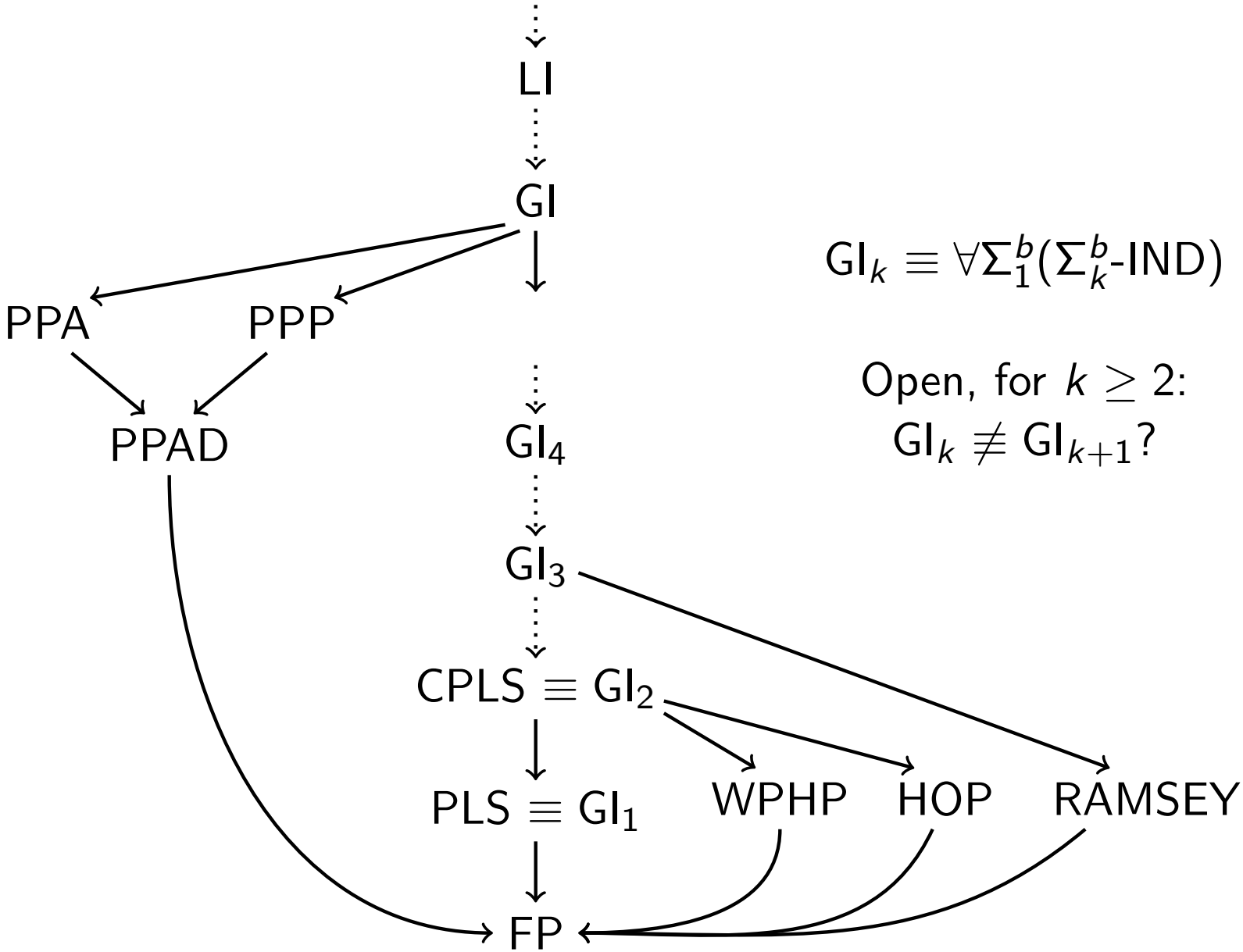
Theorem [Skelley, NT 11]

For all k, $\mathsf{GI}_k \equiv \forall\Sigma^b_1(\Sigma^b_k\text{-IND})$.

TFNP



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We define it as

$$\mathrm{APC}_2 := \Sigma^b_1\text{-IND} + \mathrm{sWPHP}(P^{NP})$$

where $\mathrm{sWPHP}(P^{NP})$ asserts:

no P^{NP} function is a surjection $x \to 2x$, for any $x > 1$.

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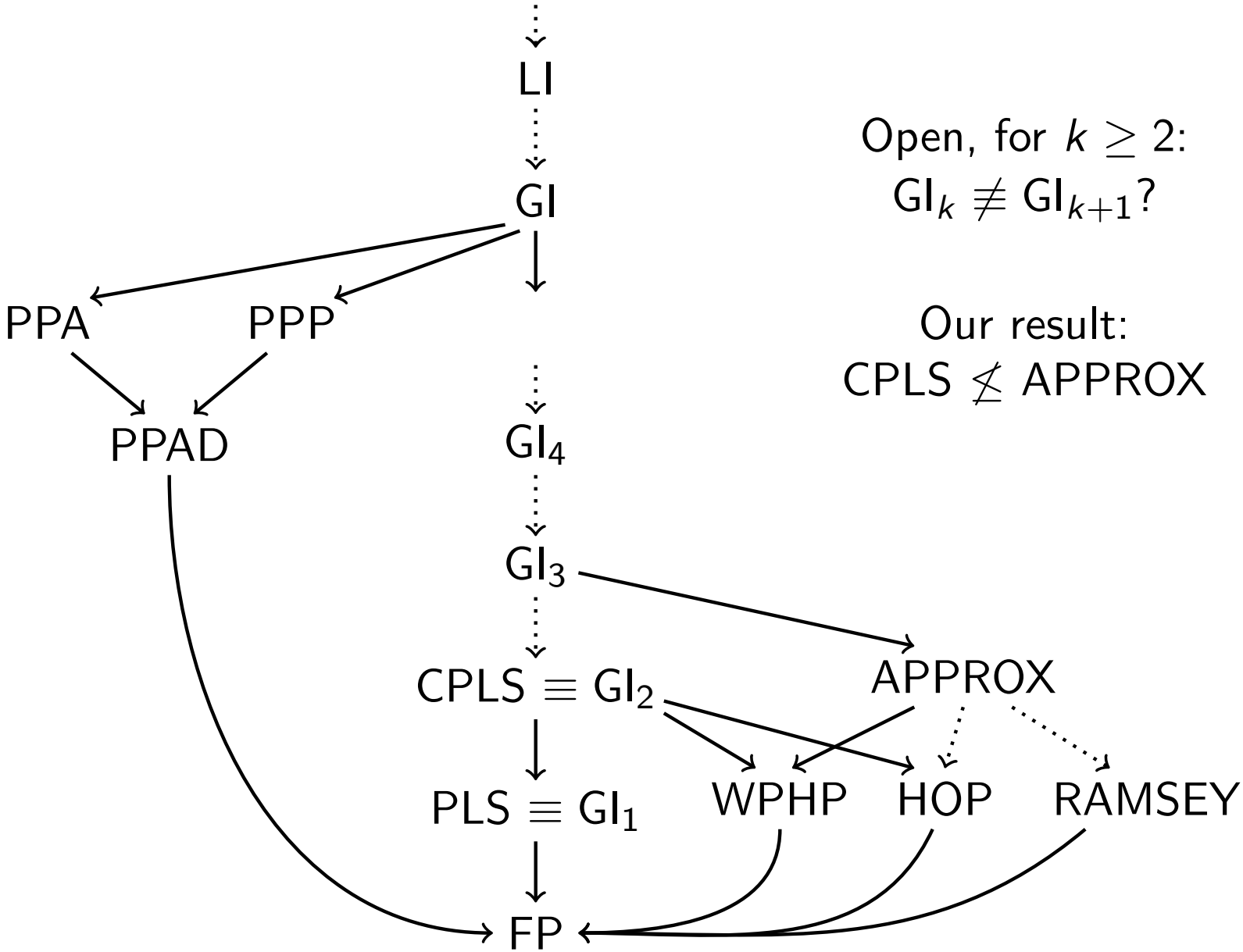
We write APPROX for $\forall\Sigma^b_1(\mathrm{APC}_2)$.

TFNP



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Choose an initial vertex $v_1 = 0$. Either at least half the edges touching v_1 are red, or at least half are blue.

Suppose more are red. Let $S(v_1, 0) = \{u \in [0, n) : (v_1, u) \text{ is red}\}$. We have $|S(v_1, 0)| \geq n/2$.

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Let v_2 be a vertex in $S(v_1, 0)$. Over vertices $u \in S(v_1, 0)$, either at least half the edges (v_2, u) are red, or at least half are blue.

Suppose more are blue.
Let $S(v_1, v_2, 0, 1) = \{u \in S(v_1, 0) : (v_2, u) \text{ is blue}\}$.
Then $|S(v_1, v_2, 0, 1)| \geq n/4$.

...

Ramsey theorem by precise counting (2)

For $v_1, \ldots, v_k \in [0, n)^k$ and $\delta_1, \ldots, \delta_k \in \{0, 1\}^k$ define

$$S(\bar{v}, \bar{\delta}) := \{u \in [0, n) : \forall i \leq k, \quad \delta_i = 0 \text{ and } (v_i, u) \text{ is red} \\ \text{or } \delta_i = 1 \text{ and } (v_i, u) \text{ is blue}\}.$$

We inductively construct $v_1, \ldots, v_k$ and $\delta_1, \ldots, \delta_k$ such that

1. For all $i < k$, $v_{i+1} \in S(v_1, \ldots, v_i, \delta_1, \ldots, \delta_i)$
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If half the bits of $\delta_1, \ldots, \delta_k$ are 0, then by item 1, $v_1, \ldots, v_k$ contains a red $k/2$-clique.

Otherwise half the bits of $\delta_1, \ldots, \delta_k$ are 1, so by item 1, $v_1, \ldots, v_k$ contains a blue $k/2$-clique.

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Suppose we can *approximately* count large, polynomial time sets.

We can write $|S| \succeq_\epsilon a$ to mean "the size of S is at least a with some small multiplicative error".

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Suppose one third are red.
Let $S(v_1, 0) = \{u \in [0, n) : (v_1, u) \text{ is red}\}$.
We have $|S(v_1, 0)| \succeq_\epsilon n/3$.

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As before, we can now find a monochrome $k/2$-clique in $v_1, \ldots, v_k$, because we can count very small sets precisely.

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As before, we can now find a monochrome $k/2$-clique in $v_1, \ldots, v_k$, because we can count very small sets precisely.

(In fact by a careful argument of this kind with smaller errors, we can still get a $(\log_2 n)/2$-clique.)

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By induction on k up to roughly $\log n$, we showed that:

There exist sequences $v_1, \ldots, v_k$ and $\delta_1, \ldots, \delta_k$ such that

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Therefore this is an induction with $\log n$ many steps, with a Σ^b_2 inductive hypothesis. This can be done in Σ^b_1-IND.

We need some form of WPHP for cardinalities to behave.

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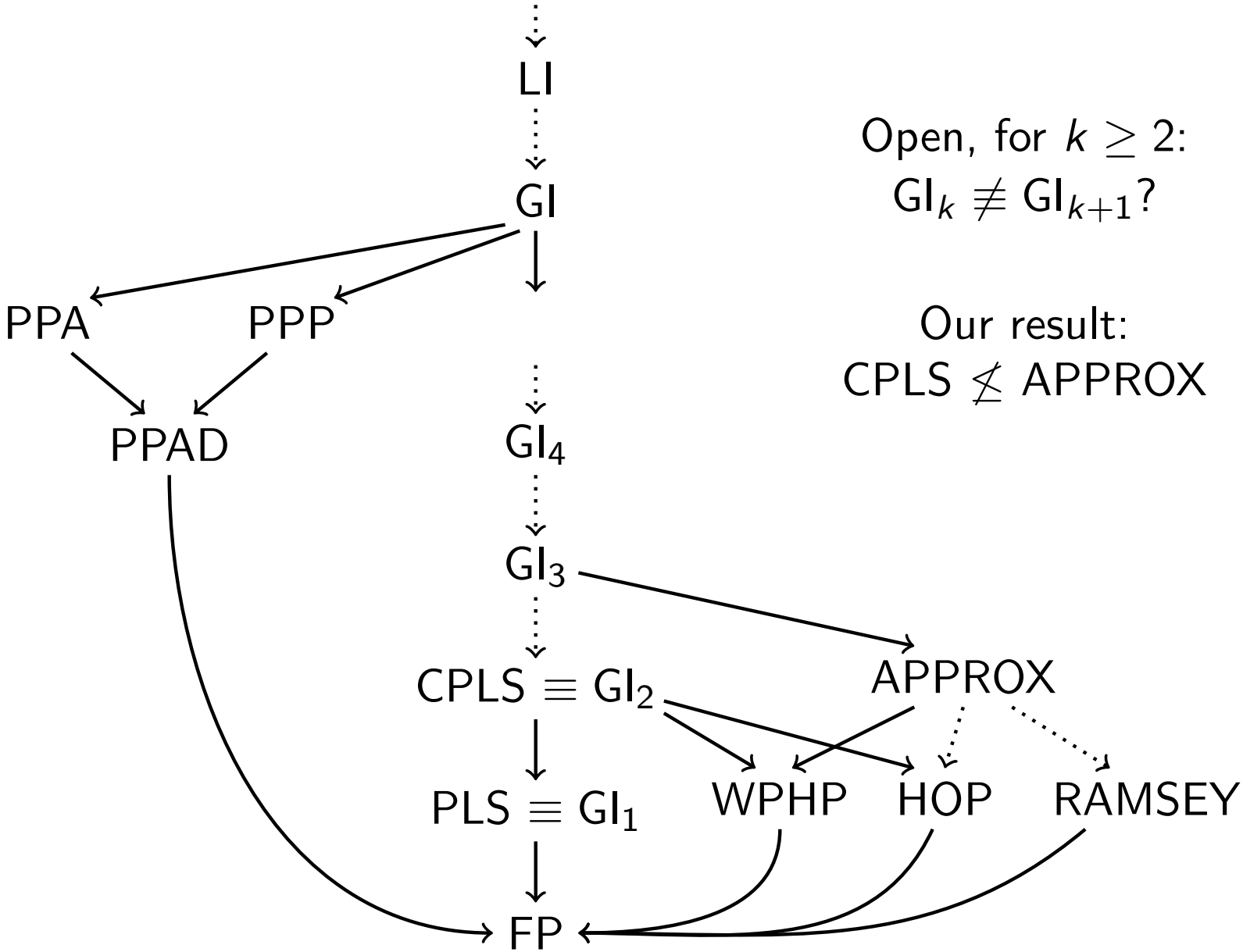

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A complexity-theoretic characterization of APPROX

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  That is: find a witness w that f_x is not a surjection $t \to 2t$ with inverse g_x.

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- Then solve Q on w to find y' such that $Q(w, y')$.

Then $R(x, y')$.

Proof that CPLS $\not\leq$ APPROX (1)

We want to find an oracle α describing an instance of CPLS which cannot be solved in this way.

Our approach is: find a partial oracle ρ such that

- ρ "forces" some witness to the retraction WPHP, but
- it is still difficult to search for a witness to CPLS in oracles extending ρ.

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Lemma [Pudlák, NT 16]

There is a distribution on partial oracles ρ, such that given any NP query $\exists v \theta(u, v)$, for a random ρ with high probability either

- there is a witness to the query in ρ, or
- there is no witness to the query in any legal $\sigma \supseteq \rho$.

In either case we say the query is *fixed* by ρ.

Proof that CPLS $\not\leq$ APPROX (3)

Lemma

Using the same distribution, for every P^{NP} machine M and input u, for a random ρ with high probability there is a computation of M on u in which every NP query made by M is fixed by ρ.

Say that such a computation is *fixed* by ρ.

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Corollary

There is a partial oracle ρ which fixes computations of M on almost all inputs u.

This guarantees that a good-enough witness to the retraction WPHP exists over ρ. The condition on legal extensions makes it difficult to find a witness to CPLS over ρ, as required. □