A feasible set theory

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Outline

Definable functions in arithmetic

Theorem [Parsons 70]

The provably recursive functions of $I\Sigma_1$ are exactly the *primitive recursive* functions.

Theorem [Buss 85]

The provably recursive functions of S^1_2 are exactly the *polynomial time* functions.

Definable functions in weak set theory

Theorem [Rathjen 92]

The provably recursive functions of ($\mathsf{KP}^- + \Sigma_1$-Induction) are exactly the *primitive recursive set functions*.

Theorem ?

The provably recursive functions of ... are exactly the ... *polynomial time?* ... set functions.

Definable functions in weak set theory

Theorem [Rathjen 92]

The provably recursive functions of ($\mathrm{KP}^- + \Sigma_1$-Induction) are exactly the *primitive recursive set functions*.

Theorem

The provably recursive functions of $\mathrm{KP}_1^{\preccurlyeq}$ are the *Cobham recursive set functions* (modulo adding global choice).

Theorem

The provably recursive functions of KP_1^{u} are exactly the *Cobham recursive set functions*.

Definable functions in weak set theory

See also two papers by Arai:

Predicatively computable functions on sets, 2015

Axiomatizing some small classes of set functions, 2015

Primitive recursive set functions

We may take as initial functions:

- projections: $a_1, \ldots, a_n \mapsto a_i$
- conditional: $\text{cond}_\in(a, b, c, d) = a$ if $c \in d$, or b otherwise
- pair: $a, b \mapsto \{a, b\}$
- empty set: $\emptyset$
- union: $a \mapsto \bigcup a$

These are closed under composition and recursion in $\in$:

$$f(x, \bar{a}) = g(x, \{f(y, \bar{a}) : y \in x\}, \bar{a})$$

Kripke-Platek set theory (KP)

- Extensionality axiom
- Pair and Union axioms
- Δ_0-Separation scheme
- Δ_0-Collection scheme
- Foundation scheme: for every formula $\varphi(x)$,

$$\exists x \varphi(x) \to \exists x \big(\varphi(x) \wedge \forall y {\in} x \, \neg \varphi(y) \big).$$

Note that foundation for φ is equivalent to $\in$-induction for $\neg\varphi$:

$$\forall x \big(\forall y {\in} x \, \neg \varphi(y) \to \neg \varphi(x) \big) \to \forall x \neg \varphi(x).$$

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Theorem [Rathjen 92 again]

If we weaken Foundation to Σ_1-Induction, the provably recursive functions of the resulting theory are exactly the primitive recursive set functions.

Introduction

'Polynomial time' functions on sets

Theories and results

Proof of witnessing

Cobham recursive set functions (CRSF)

A class of functions from (arbitrary) sets to sets

Defined by limiting the "growth rate" of functions that can be introduced by $\in$-recursion, as in Cobham's definition of P

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On finite binary strings, corresponds to polynomial time

On subsets of ω, corresponds to polynomial time ITTMs

Under a natural definition of (possibly infinite) circuits, consists exactly of the set functions with "polynomial size" circuits

Important concepts - Mostowski graph

The *Mostowski graph* $\mathcal{G}(a)$ of a set a has

- nodes $\text{tc}(\{a\})$
- edges $\{\langle x, y \rangle : x \in y\}$

$\mathcal{G}(a)$ has a single source node, 0.

$\mathcal{G}(a)$ has a single sink node, a.

Important concepts - $\#$, $\odot$

The *set smash* function $\#$ is a kind of lexicographic product on Mostowski graphs.

Definition

Given sets a, b the smash $a\#b$ is the set whose Mostowski graph is constructed as follows:

- Draw a disjoint copy G_x of $\mathcal{G}(b)$ for every node $x \in \mathcal{G}(a)$
- For each edge $\langle x, y \rangle$ of $\mathcal{G}(a)$, connect the sink of G_x to the source of G_y.

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The rank of $a\#b$ is the product of the ranks of a and b.
The same for the size of the transitive closures.

Important concepts - $\#$, $\odot$

The formal definition of $\#$ uses an auxiliary *set composition* function $a \odot b$. This is defined by drawing $\mathcal{G}(a)$ above $\mathcal{G}(b)$ and identifying the sink of $\mathcal{G}(b)$ with the source of $\mathcal{G}(a)$.

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Definition

A $\#$-*term* is a term formed only from variables, the constant 1, and the functions $\odot$ and $\#$.

$\#$-terms play the role of polynomial size bounds.

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Definition

A function σ is an embedding of a in b, written $\sigma : a \preccurlyeq b$, if

- For all $x \in \mathrm{tc}(a)$, $\sigma(x)$ is a nonempty subset of $\mathrm{tc}(b)$
- If $x \neq x'$, then $\sigma(x)$ and $\sigma(x')$ are disjoint
- If $x' \in x$, then for every $y \in \sigma(x)$ there is $y' \in \sigma(x')$ with $y' \in \mathrm{tc}(y)$ (that is, with $y' < y$ in the ordering given by $\in$)

Important concepts - embedding

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For a set e, we write $e : a \preccurlyeq b$ if $e \subseteq \text{tc}(a) \times \text{tc}(b)$ is the graph of an embedding.

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Later we will define a $\Sigma_1^{\preccurlyeq}$ formula to be one of the form

$$\exists y \preccurlyeq t(\bar{a})\, \varphi(y, \bar{a})$$

for t a $\#$-term and $\varphi \in \Delta_0$.

Note that we consider quantification over *members* of a set as feasible ('sharply bounded').

Cobham recursive set functions

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$$0,\ 1,\ \mathrm{cond}_\in,\ \bigcup x,\ \{x,y\},\ x \times y,\ \mathrm{tc}(x),\ x \odot y,\ x \# y$$

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$$f``(x, \bar{a}) = \{f(y, \bar{a}) : y \in x\}$$

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Closed under composition, replacement

$$f``(x, \bar{a}) = \{f(y, \bar{a}) : y \in x\}$$

and *Cobham recursion* – informally, given g and a $\#$-term t, we include in CRSF the function f defined by usual $\in$-recursion as

$$f(x, \bar{a}) = g(x, \{f(y, \bar{a}) : y \in x\}, \bar{a}),$$

provided that $f(x, \bar{a}) \preccurlyeq t(x, \bar{a})$ for all $x, \bar{a}$.

Cobham recursive set functions

Formally, we use *syntactic Cobham recursion*:

If $g, \sigma \in$ CRSF and a t is a $\#$-term, then the function

$$f(x) = \begin{cases} g(x, f"(x)) & \text{if } \sigma : g(x, f"(x)) \preccurlyeq t(x) \\ 0 & \text{otherwise} \end{cases}$$

is in CRSF, where I have not written the parameters $\bar{a}$.

(There are simpler definitions of CRSF.)

Theories - T_0

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The theory T_0 consists of

- defining axioms for the symbols of L_0
- Extensionality axiom
- Set Foundation axiom $\quad x \neq 0 \to \exists y{\in}x\, \forall u{\in}y\, (u \notin x)$
- Δ_0-Separation scheme

It can prove Δ_0-Induction, and many useful properties of embeddings.

Theories - $\mathrm{KP}_1^{\preccurlyeq}$

A $\Sigma_1^{\preccurlyeq}$ formula is one of the form

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The theory $\mathrm{KP}_1^{\preccurlyeq}$ consists of T_0 plus

- Δ_0-Collection scheme

$$\forall y{\in}x\, \exists u\, \varphi(y, u, \bar{a}) \to \exists w\, \forall y{\in}x\, \exists u{\in}w\, \varphi(y, u, \bar{a}) \qquad \text{for } \varphi \in \Delta_0$$

- $\Sigma_1^{\preccurlyeq}$-Induction scheme

$$\forall x\, \big(\forall y{\in}x\, \varphi(y, \bar{a}) \to \varphi(x, \bar{a})\big) \to \forall x \varphi(x, \bar{a}) \qquad \text{for } \varphi \in \Sigma_1^{\preccurlyeq}$$

That is, KP in an enriched language with the Foundation scheme weakened to $\Sigma_1^{\preccurlyeq}$-Induction.

Results

Target theorem [definability]

Every polynomial time function is Σ_1^b-definable in S_2^1.

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Theorem

Every CRSF function is $\Sigma_1^{\preccurlyeq}$-definable in $\mathsf{KP}_1^{\preccurlyeq}$.

Results

Target theorem [definability]

Every polynomial time function is Σ_1^b-definable in S_2^1.

Theorem

Every CRSF function is $\Sigma_1^{\preccurlyeq}$-definable in $\mathrm{KP}_1^{\preccurlyeq}$.

Proof: For f obtained by Cobham recursion,

- Write a $\Sigma_1^{\preccurlyeq}$ definition of $f(x) = y$ (requires complex embeddings)
- Use $\Sigma_1^{\preccurlyeq}$-induction to prove the definition is total
- Use collection to handle the induction step at infinite x

A problem

Target theorem [witnessing]

If $S^1_2 \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma^b_1$ then there is a polynomial time function f such that $\forall x \varphi(x, f(x))$ holds.

A problem

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If $S^1_2 \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma^b_1$ then there is a polynomial time function f such that $\forall x \varphi(x, f(x))$ holds.

The natural analogue cannot hold for $\mathrm{KP}^{\preccurlyeq}_1$ and CRSF. We have

$$\mathrm{KP}^{\preccurlyeq}_1 \vdash \forall x \exists y (x \neq 0 \to y \in x).$$

If a function C witnesses this, then

$$\forall x (x \neq 0 \to C(x) \in x)$$

so C is a global choice function. No such function exists in CRSF.

First solution

Suppose there is a global choice function C on the universe (this does not follow from ZFC).

Extend CRSF to CRSF^C by adding C as an initial function.

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Theorem
Suppose $\mathrm{KP}_1^{\preccurlyeq} \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma_1^{\preccurlyeq}$.
Then there is $f \in \mathrm{CRSF}^C$ such that $\forall x \varphi(x, f(x))$ holds.

Question: can we call C 'feasible'?

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Weaken the conclusion of witnessing from

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Instead let f output a set containing (possibly many) solutions. That is,

$$\forall x \exists y \in f(x)\, \varphi(x, y).$$

Now the formula

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... but we cannot prove even this kind of witnessing for $\mathsf{KP}_1^{\preccurlyeq}$.

Theories - KP_1^u

Recall that $\mathsf{KP}_1^{\preccurlyeq}$ is the base theory T_0 together with the Δ_0-Comprehension and $\Sigma_1^{\preccurlyeq}$-Induction schemes.

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Recall that $\mathrm{KP}_1^{\preccurlyeq}$ is the base theory T_0 together with the Δ_0-Comprehension and $\Sigma_1^{\preccurlyeq}$-Induction schemes.

The theory KP_1^u is like $\mathrm{KP}_1^{\preccurlyeq}$, but weakens $\Sigma_1^{\preccurlyeq}$-Induction to the *unique* $\Sigma_1^{\preccurlyeq}$*-Induction scheme*: for each $\varphi(x, \bar{a}) \in \Sigma_1^{\preccurlyeq}$,

$$(\varphi(x, \bar{a}) \text{ has at most one witness for each } x) \rightarrow \text{induction holds for } \varphi(x, \bar{a})$$

Theorem
Every CRSF function is still $\Sigma_1^{\preccurlyeq}$-definable in KP_1^u.

Results

Theorem

Suppose $\mathrm{KP}_1^u \vdash \forall x \exists y \varphi(x,y)$ for $\varphi \in \Sigma_1^{\preccurlyeq}$.
Then there is $f \in \mathrm{CRSF}$ such that $\forall x \exists y{\in}f(x)\, \varphi(x,y)$ holds.

Corollary

The Σ_1-definable functions of KP_1^u are exactly the CRSF functions.

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Theorem

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Corollary

The Σ_1-definable functions of KP_1^u are exactly the CRSF functions.

Proof of $\Rightarrow$: Suppose $F(x) = y \leftrightarrow \exists u\ \varphi(x,y,u)$ for $\varphi \in \Delta_0$, and $\mathrm{KP}_1^u \vdash \forall x\ \exists! y\ \exists u\ \varphi(x,y,u)$.

By witnessing, $\exists g \in \mathrm{CRSF}$ such that $\forall x\ \exists y, u \in g(x)\ \varphi(x,y,u)$.

Then $F(x) = \bigcup \{ y \in g(x) : \exists u {\in} g(x)\, \varphi(x,y,u)\}$ is in CRSF.

Last theory - T_{crsf}

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It is analogous to the bounded arithmetic theory PV_1.

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Language $L_{\mathrm{crsf}} = \{$symbol for every description of a CRSF function$\}$

The L_{crsf}-theory T_{crsf} consists of T_0 plus, for each CRSF function, an axiom that the function is as described.

(E.g. if f is defined by recursion from g, σ, t then ...)

T_{crsf} is axiomatized by $\Pi_1(L_{\mathrm{crsf}})$ sentences.

Herbrand's theorem

T_{crsf} is $\Pi_1(L_{\mathsf{crsf}})$. It is not universal. But we can prove a version of Herbrand's theorem:

Lemma

Suppose $T_{\mathsf{crsf}} \vdash \exists y \varphi(y, \bar{x})$, where $\varphi \in \Delta_0(L_{\mathsf{crsf}})$.
Then there is a function symbol $f \in L_{\mathsf{crsf}}$ such that

$$T_{\mathsf{crsf}} \vdash \exists y{\in}f(\bar{x})\, \varphi(y, \bar{x}).$$

So in T_{crsf} we have the kind of witnessing we want.

Herbrand saturation

To get witnessing for KP_1^u, it is enough now to show that KP_1^u is Π_2-conservative over T_{crsf}.

We adapt the method of [Avigad 2002] (after Zambella, Visser)

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Definition

A structure M is Δ_0-*Herbrand saturated* if it satisfies every Σ_2-sentence with parameters from M which is consistent with the Π_1-diagram of M.

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A structure M is Δ_0-*Herbrand saturated* if it satisfies every Σ_2-sentence with parameters from M which is consistent with the Π_1-diagram of M.

Lemma

1. In a Δ_0-Herbrand saturated structure, every true Π_2 sentence is 'witnessed' by a term.
2. If every Δ_0-Herbrand saturated model of T_{crsf} is a model of KP_1^u, then KP_1^u is Π_2-conservative over T_{crsf}.

Conservativity proof 1

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Proof sketch: Let M be a Δ_0-Herbrand saturated model of T_{crsf}. We must show that $M \vDash \mathsf{KP}_1^u$. In particular, that unique $\Sigma_1^{\preccurlyeq}$ induction holds in M.

Conservativity proof 1

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Proof sketch: Let M be a Δ_0-Herbrand saturated model of T_{crsf}. We must show that $M \vDash \mathrm{KP}_1^u$. In particular, that unique $\Sigma_1^{\preccurlyeq}$ induction holds in M.

Let $\varphi(x) \equiv \exists v \preccurlyeq t(x) \theta(x, v)$ be a $\Sigma_1^{\preccurlyeq}$ formula with $\forall x \exists^{\leq 1} v \theta(x, v)$.

We may assume that the embedding $v \preccurlyeq t(x)$ is Δ_0-definable and that the embedding bound is implicit in θ.

That is, we assume $\varphi(x) \equiv \exists v \theta(x, v)$.

Conservativity proof 2

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Suppose we have a function $g(x, W)$ such that:

whenever W contains witnesses to $\exists u\ \theta(y,u)$ for every $y \in x$,

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Then we can define $f(x)$ by recursion as

$$f(x) = g(x, \{f(y) : y \in x\})$$

and prove by $\Delta_0(L_{\mathrm{crsf}})$-Induction that $\forall x \theta(x, f(x))$.
Hence $\forall x \varphi(x)$, and we have shown induction for φ.

Conservativity proof 3

How do we get such a g? We have

$$\forall x \big(\forall y{\in}x\, \exists u\; \theta(y,u) \rightarrow \exists v\; \theta(x,v)\big).$$

Hence

$$\forall x \forall W \big(\forall y{\in}x\, \exists u{\in}W\, \theta(y,u) \rightarrow \exists v\; \theta(x,v)\big).$$

This is Π_2.

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This is Π_2. By Δ_0-Herbrand saturation, it is 'witnessed' in M.

That is, there is a function $h(x, W)$ such that:

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Since such witnesses are unique, we can define

$$g(x, W) = \bigcup \{ v \in h(x, W) : \theta(x, v) \}.$$

□

Open problems / speculation

1. Prove witnessing for $\mathsf{KP}_1^{\preccurlyeq}$ without choice.
2. At least prove witnessing using only local choice.

   E.g. if $\mathsf{KP}_1^{\preccurlyeq} \vdash \forall x \exists y \varphi(x, y)$, does this imply that there is a CRSF function $f(x, r)$ such that $\forall x \exists y \in f(x, r)\, \varphi(x, y)$ whenever r is a well-ordering of $\mathrm{tc}(x)$?
3. How simple a theory can we use instead of $\mathsf{KP}_1^{\preccurlyeq}$?

   E.g. take KP in the original language $\{\in\}$, add an axiom for transitive closure, and weaken Foundation to induction only for formulas $\exists y \subseteq z\ \theta(x, y)$ for $\theta \in \Delta_0$.
4. Infinitary propositional proof complexity
5. Arithmetic without predecessor

(Expected) connections between $\mathsf{KP}_1^{\precsim}$ and S_2^1

We can interpret S_2^1 in $\mathsf{KP}_1^{\precsim}$ as follows:

Let $L = \{$ordinals α such that no ordinal $\beta \leq \alpha$ is a limit$\}$.
Let $M = \{x : x \subseteq \alpha$ for some $\alpha \in L\}$.
Then the elements of M, considered as binary strings of length α, form a model of S_2^1.

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Then the elements of M, considered as binary strings of length α, form a model of S_2^1.

We can interpret $\mathsf{KP}_1^{\preccurlyeq}$ in S_2^1 as follows:

Let $M = \{$strings coding Mostowski graphs$\}$. Then the functions and relations in L_0 are polynomial time under this encoding of sets as graphs, and with them M is a model of $\mathsf{KP}_1^{\preccurlyeq}$.