A feasible set theory

Neil Thapen

Institute of Mathematics
Czech Academy of Sciences

Joint work with

Arnold Beckmann, Sam Buss, Sy Friedman and Moritz Müller
Outline

Introduction

'Polynomial time' functions on sets

Theories and results

Proof of witnessing
Definable functions in arithmetic

Theorem [Parsons 70]
The provably recursive functions of $IΣ_1$ are exactly the *primitive recursive* functions.

Theorem [Buss 85]
The provably recursive functions of $S^1_2$ are exactly the *polynomial time* functions.
Definable functions in weak set theory

**Theorem [Rathjen 92]**
The provably recursive functions of \((KP^- + \Sigma_1\text{-Induction})\) are exactly the *primitive recursive set functions*.

**Theorem ?**
The provably recursive functions of \(\ldots\) are exactly the \(\ldots\) *polynomial time? \(\ldots\) set functions.
Theorem [Rathjen 92]
The provably recursive functions of \((\text{KP}^- + \Sigma_1\text{-Induction})\) are exactly the *primitive recursive set functions*.

**Theorem**
The provably recursive functions of \(\text{KP}^-\) are the *Cobham recursive set functions* (modulo adding global choice).

**Theorem**
The provably recursive functions of \(\text{KP}_{1^u}\) are exactly the *Cobham recursive set functions*. 
Definable functions in weak set theory

See also two papers by Arai:

*Predicatively computable functions on sets*, 2015

*Axiomatizing some small classes of set functions*, 2015
Primitive recursive set functions

We may take as initial functions:

- **projections:** $a_1, \ldots, a_n \mapsto a_i$
- **conditional:** $\text{cond}_{\in} (a, b, c, d) = a$ if $c \in d$, or $b$ otherwise
- **pair:** $a, b \mapsto \{a, b\}$
- **empty set:** $\emptyset$
- **union:** $a \mapsto \bigcup a$

These are closed under composition and recursion in $\in$:

$$f(x, \bar{a}) = g(x, \{f(y, \bar{a}) : y \in x\}, \bar{a})$$
Kripke-Platek set theory (KP)

- Extensionality axiom
- Pair and Union axioms
- $\Delta_0$-Separation scheme
- $\Delta_0$-Collection scheme
- Foundation scheme: for every formula $\varphi(x)$,

$$\exists x \varphi(x) \rightarrow \exists x (\varphi(x) \land \forall y \in x \neg \varphi(y)).$$

Note that foundation for $\varphi$ is equivalent to $\in$-induction for $\neg \varphi$:

$$\forall x (\forall y \in x \neg \varphi(y) \rightarrow \neg \varphi(x)) \rightarrow \forall x \neg \varphi(x).$$
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\[ \exists x \varphi(x) \rightarrow \exists x (\varphi(x) \wedge \forall y \in x \neg \varphi(y)). \]

Note that foundation for $\varphi$ is equivalent to $\in$-induction for $\neg \varphi$:

\[ \forall x (\forall y \in x \neg \varphi(y) \rightarrow \neg \varphi(x)) \rightarrow \forall x \neg \varphi(x). \]

**Theorem [Rathjen 92 again]**

If we weaken Foundation to $\Sigma_1$-Induction, the provably recursive functions of the resulting theory are exactly the primitive recursive set functions.
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‘Polynomial time’ functions on sets

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Cobham recursive set functions (CRSF)

A class of functions from (arbitrary) sets to sets

Defined by limiting the “growth rate” of functions that can be introduced by \( \in \)-recursion, as in Cobham’s definition of P
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A class of functions from (arbitrary) sets to sets

Defined by limiting the “growth rate” of functions that can be introduced by $\in$-recursion, as in Cobham’s definition of P

On finite binary strings, corresponds to polynomial time

On subsets of $\omega$, corresponds to polynomial time ITTMs

Under a natural definition of (possibly infinite) circuits, consists exactly of the set functions with “polynomial size” circuits
The *Mostowski graph* $\mathcal{G}(a)$ of a set $a$ has

- nodes $\text{tc}(\{a\})$
- edges $\{\langle x, y \rangle : x \in y \}$

$\mathcal{G}(a)$ has a single source node, 0.

$\mathcal{G}(a)$ has a single sink node, $a$. 
The *set smash* function \( \# \) is a kind of lexicographic product on Mostowski graphs.

**Definition**

Given sets \( a, b \) the smash \( a \# b \) is the set whose Mostowski graph is constructed as follows:

- Draw a disjoint copy \( G_x \) of \( \mathcal{G}(b) \) for every node \( x \in \mathcal{G}(a) \)
- For each edge \( \langle x, y \rangle \) of \( \mathcal{G}(a) \), connect the sink of \( G_x \) to the source of \( G_y \).
Important concepts - #, ⊙

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- For each edge $\langle x, y \rangle$ of $G(a)$, connect the sink of $G_x$ to the source of $G_y$.

The rank of $a\#b$ is the product of the ranks of $a$ and $b$. The same for the size of the transitive closures.
Important concepts - $\#$, $\circ$

The formal definition of $\#$ uses an auxiliary *set composition* function $a \circ b$. This is defined by drawing $G(a)$ above $G(b)$ and identifying the sink of $G(b)$ with the source of $G(a)$.
Important concepts - $\#$, $\circ$

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**Definition**

A $\#$-term is a term formed only from variables, the constant 1, and the functions $\circ$ and $\#$. 

$\#$-terms play the role of polynomial size bounds.
Important concepts - embedding

An embedding of $a$ in $b$ is an injective multifunction from $tc(a)$ to $tc(b)$ which respects the ordering given by $\in$. 
Important concepts - embedding

An *embedding* of $a$ in $b$ is an injective multifunction from $\text{tc}(a)$ to $\text{tc}(b)$ which respects the ordering given by $\in$.

**Definition**
A function $\sigma$ is an embedding of $a$ in $b$, written $\sigma : a \preceq b$, if
- For all $x \in \text{tc}(a)$, $\sigma(x)$ is a nonempty subset of $\text{tc}(b)$
- If $x \neq x'$, then $\sigma(x)$ and $\sigma(x')$ are disjoint
- If $x' \in x$, then for every $y \in \sigma(x)$ there is $y' \in \sigma(x')$ with $y' \in \text{tc}(y)$ (that is, with $y' < y$ in the ordering given by $\in$)
Important concepts - embedding

If $\sigma : a \preceq b$ then $\text{rank}(a) \leq \text{rank}(b)$ and $|\text{tc}(a)| \leq |\text{tc}(b)|$. 
Important concepts - embedding

If $\sigma : a \preccurlyeq b$ then $\text{rank}(a) \leq \text{rank}(b)$ and $|\text{tc}(a)| \leq |\text{tc}(b)|$.

For a set $e$, we write $e : a \preccurlyeq b$ if $e \subseteq \text{tc}(a) \times \text{tc}(b)$ is the graph of an embedding.

We write $a \preccurlyeq b$ for $\exists e \subseteq \text{tc}(a) \times \text{tc}(b) \ (e : a \preccurlyeq b)$. 
Important concepts - embedding

If $\sigma : a \preceq b$ then $\text{rank}(a) \leq \text{rank}(b)$ and $|tc(a)| \leq |tc(b)|$.

For a set $e$, we write $e : a \preceq b$ if $e \subseteq tc(a) \times tc(b)$ is the graph of an embedding.

We write $a \preceq b$ for $\exists e \subseteq tc(a) \times tc(b) (e : a \preceq b)$.

Later we will define a $\Sigma^1_1$ formula to be one of the form

$$\exists y \preceq t(\bar{a}) \varphi(y, \bar{a})$$

for $t$ a $\#$-term and $\varphi \in \Delta_0$.

Note that we consider quantification over members of a set as feasible (‘sharply bounded’).
Cobham recursive set functions

Initial functions:

\[ 0, 1, \text{cond}_\in, \bigcup x, \{x, y\}, x \times y, \text{tc}(x), x \odot y, x \# y \]
Cobham recursive set functions

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\[ 0, 1, \text{cond}_{\in}, \bigcup x, \{x, y\}, x \times y, \text{tc}(x), x \odot y, x\#y \]

Closed under composition, replacement

\[ f''(x, \bar{a}) = \{f(y, \bar{a}) : y \in x\} \]

and Cobham recursion –
Cobham recursive set functions

Initial functions:

0, 1, cond_{\in}, \bigcup x, \{x, y\}, x \times y, \text{tc}(x), x \odot y, x \# y

Closed under composition, replacement

\[ f''(x, \bar{a}) = \{f(y, \bar{a}) : y \in x\} \]

and Cobham recursion – informally, given \( g \) and a \#-term \( t \), we include in CRSF the function \( f \) defined by usual \( \in \)-recursion as

\[ f(x, \bar{a}) = g(x, \{f(y, \bar{a}) : y \in x\}, \bar{a}), \]

provided that \( f(x, \bar{a}) \preceq t(x, \bar{a}) \) for all \( x, \bar{a} \).
Cobham recursive set functions

Formally, we use syntactic Cobham recursion:

If \( g, \sigma \in \text{CRSF} \) and a \( t \) is a \( \# \)-term, then the function

\[
f(x) = \begin{cases} 
  g(x, f''(x)) & \text{if } \sigma : g(x, f''(x)) \preceq t(x) \\
  0 & \text{otherwise}
\end{cases}
\]

is in CRSF, where I have not written the parameters \( \bar{a} \).

(There are simpler definitions of CRSF.)
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Theories and results

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Theories - $T_0$

This is the basic theory our other theories extend. (cf. BASIC, Q)
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Language $L_0 = \{\in, 0, 1, \bigcup x, \{x, y\}, x \times y, tc(x), x \odot y, x \# y\}$
Theories - $T_0$

This is the basic theory our other theories extend. (cf. BASIC, Q)

Language $L_0 = \{ \in, 0, 1, \bigcup x, \{x, y\}, x \times y, \text{tc}(x), x \odot y, x \# y \}$

The theory $T_0$ consists of

- defining axioms for the symbols of $L_0$
- Extensionality axiom
- Set Foundation axiom $x \neq 0 \rightarrow \exists y \in x \forall u \in y (u \notin x)$
- $\Delta_0$-Separation scheme

It can prove $\Delta_0$-Induction, and many useful properties of embeddings.
A $\Sigma_1$ formula is one of the form

$$\exists y \lessdot t(\bar{a}) \, \varphi(y, \bar{a})$$

for $t$ a $\#$-term and $\varphi \in \Delta_0$. 

The theory $\text{KP}_{1}^{\lessdot}$ consists of $T_0$ plus

$\forall y \in x \exists u \, \varphi(y, u, \bar{a}) \Rightarrow \exists w \, \forall y \in x \exists u \in w \, \varphi(y, u, \bar{a})$ for $\varphi \in \Delta_0$

$\forall x \left( \forall y \in x \, \varphi(y, \bar{a}) \Rightarrow \varphi(x, \bar{a}) \right) \Rightarrow \forall x \, \varphi(x, \bar{a})$ for $\varphi \in \Sigma_1^{\lessdot}$

That is, $\text{KP}$ in an enriched language with the Foundation scheme weakened to $\Sigma_1^{\lessdot}$-Induction.
A $\Sigma_1$ formula is one of the form
\[ \exists y \approx t(\bar{a}) \varphi(y, \bar{a}) \] for $t$ a $\#$-term and $\varphi \in \Delta_0$.

The theory $\text{KP}_1$ consists of $T_0$ plus

- $\Delta_0$-Collection scheme
  \[ \forall y \in x \exists u \varphi(y, u, \bar{a}) \rightarrow \exists w \forall y \in x \exists u \in w \varphi(y, u, \bar{a}) \] for $\varphi \in \Delta_0$

- $\Sigma_1$-Induction scheme
  \[ \forall x \left( \forall y \in x \varphi(y, \bar{a}) \rightarrow \varphi(x, \bar{a}) \right) \rightarrow \forall x \varphi(x, \bar{a}) \] for $\varphi \in \Sigma_1$

That is, $\text{KP}$ in an enriched language with the Foundation scheme weakened to $\Sigma_1$-Induction.
Target theorem [definability]
Every polynomial time function is $\Sigma_1^b$-definable in $S_2^1$. 
Results

**Target theorem [definability]**
Every polynomial time function is $\Sigma^b_1$-definable in $S^1_2$.

**Theorem**
Every CRSF function is $\Sigma^c_1$-definable in $KP^c_1$. 

Proof: For $f$ obtained by Cobham recursion, 

- Write a $\Sigma^c_1$ definition of $f(x) = y$ (requires complex embeddings)
- Use $\Sigma^c_1$-induction to prove the definition is total
- Use collection to handle the induction step at infinite
Results

Target theorem [definability]
Every polynomial time function is $\Sigma^b_1$-definable in $S^1_2$.

Theorem
Every CRSF function is $\Sigma^\equiv_1$-definable in $KP^\equiv_1$.

Proof: For $f$ obtained by Cobham recursion,
- Write a $\Sigma^\equiv_1$ definition of $f(x) = y$ (requires complex embeddings)
- Use $\Sigma^\equiv_1$-induction to prove the definition is total
- Use collection to handle the induction step at infinite $x$
A problem

Target theorem [witnessing]
If $S_2^1 \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma_1^b$ then there is a polynomial time function $f$ such that $\forall x \varphi(x, f(x))$ holds.
A problem

Target theorem [witnessing]
If \( S_2^1 \vdash \forall x \exists y \varphi(x, y) \) for \( \varphi \in \Sigma^b_1 \) then there is a polynomial time function \( f \) such that \( \forall x \varphi(x, f(x)) \) holds.

The natural analogue cannot hold for KP\(^{\preceq}_1\) and CRSF. We have

\[
KP^{\preceq}_1 \vdash \forall x \exists y(x \neq 0 \rightarrow y \in x).
\]

If a function \( C \) witnesses this, then

\[
\forall x(x \neq 0 \rightarrow C(x) \in x)
\]

so \( C \) is a global choice function. No such function exists in CRSF.
Suppose there is a global choice function \( C \) on the universe (this does not follow from ZFC).

Extend CRSF to CRSF\(^C\) by adding \( C \) as an initial function.

Theorem

Suppose \( KP \equiv_1 \vdash \forall x \exists y \varphi(x, y) \) for \( \varphi \in \Sigma \equiv_1 \).

Then there is \( f \in \text{CRSF}^C \) such that \( \forall x \varphi(x, f(x)) \) holds.

Question: can we call \( C \) ‘feasible’?
Suppose there is a global choice function $C$ on the universe (this does not follow from ZFC).

Extend CRSF to $\text{CRSF}^C$ by adding $C$ as an initial function.

**Theorem**

Suppose $\text{KP}_{1}^{\mathbb{N}} \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma_{1}^{\mathbb{N}}$.

Then there is $f \in \text{CRSF}^C$ such that $\forall x \varphi(x, f(x))$ holds.

Question: can we call $C$ 'feasible'?
Second solution

Weaken the conclusion of witnessing from

$$\forall x \varphi(x, f(x)).$$
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\[ \forall x \varphi(x, f(x)). \]

Instead let \( f \) output a set containing (possibly many) solutions. That is,

\[ \forall x \exists y \in f(x) \varphi(x, y). \]

Now the formula

\[ \forall x \exists y (x \neq 0 \rightarrow y \in x) \]

is witnessed by the identity function.
Second solution

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Now the formula

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is witnessed by the identity function.

\[ \ldots \text{but we cannot prove even this kind of witnessing for } \text{KP}_{1}^{\prec}. \]
Theories - $\text{KP}_1^u$

Recall that $\text{KP}_1^u$ is the base theory $T_0$ together with the $\Delta_0$-Comprehension and $\Sigma_1^u$-Induction schemes.
Theories - $\text{KP}_1^u$

Recall that $\text{KP}_{1}^{\preceq}$ is the base theory $T_0$ together with the $\Delta_0$-Comprehension and $\Sigma_1^{\preceq}$-Induction schemes.

The theory $\text{KP}_1^u$ is like $\text{KP}_1^{\preceq}$, but weakens $\Sigma_1^{\preceq}$-Induction to the unique $\Sigma_1^{\preceq}$-Induction scheme: for each $\varphi(x, \bar{a}) \in \Sigma_1^{\preceq}$,

$$(\varphi(x, \bar{a}) \text{ has at most one witness for each } x) \implies \text{induction holds for } \varphi(x, \bar{a})$$

**Theorem**

Every CRSF function is still $\Sigma_1^{\preceq}$-definable in $\text{KP}_1^u$. 
Results

**Theorem**
Suppose $\text{KP}_1^u \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma_1^\omega$.
Then there is $f \in \text{CRSF}$ such that $\forall x \exists y \in f(x) \varphi(x, y)$ holds.

**Corollary**
The $\Sigma_1$-definable functions of $\text{KP}_1^u$ are exactly the CRSF functions.
Results

Theorem
Suppose $\text{KP}_1^u \vdash \forall x \exists y \varphi(x, y)$ for $\varphi \in \Sigma_1^{\leq}$.
Then there is $f \in \text{CRSF}$ such that $\forall x \exists y \in f(x) \varphi(x, y)$ holds.

Corollary
The $\Sigma_1$-definable functions of $\text{KP}_1^u$ are exactly the CRSF functions.

Proof of $\Rightarrow$: Suppose $F(x) = y \leftrightarrow \exists u \varphi(x, y, u)$ for $\varphi \in \Delta_0$, and $\text{KP}_1^u \vdash \forall x \exists ! y \exists u \varphi(x, y, u)$.
By witnessing, $\exists g \in \text{CRSF}$ such that $\forall x \exists y, u \in g(x) \varphi(x, y, u)$.
Then $F(x) = \bigcup \{ y \in g(x) : \exists u \in g(x) \varphi(x, y, u) \}$ is in CRSF.
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Proof of witnessing
Our proof is model-theoretic. We use an auxiliary theory, $T_{\text{crsf}}$. It is analogous to the bounded arithmetic theory $\text{PV}_1$. 
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Language $L_{\text{crsf}} = \{\text{symbol for every description of a CRSF function}\}$

The $L_{\text{crsf}}$-theory $T_{\text{crsf}}$ consists of $T_0$ plus, for each CRSF function, an axiom that the function is as described.

(E.g. if $f$ is defined by recursion from $g, \sigma, t$ then . . . )

$T_{\text{crsf}}$ is axiomatized by $\Pi_1(L_{\text{crsf}})$ sentences.
Herbrand’s theorem

$T_{\text{crsf}}$ is $\Pi_1(L_{\text{crsf}})$. It is not universal. But we can prove a version of Herbrand’s theorem:

**Lemma**

Suppose $T_{\text{crsf}} \vdash \exists y \varphi(y, \bar{x})$, where $\varphi \in \Delta_0(L_{\text{crsf}})$. Then there is a function symbol $f \in L_{\text{crsf}}$ such that

$$T_{\text{crsf}} \vdash \exists y \in f(\bar{x}) \varphi(y, \bar{x}).$$

So in $T_{\text{crsf}}$ we have the kind of witnessing we want.
Herbrand saturation

To get witnessing for $\text{KP}_1^u$, it is enough now to show that $\text{KP}_1^u$ is $\Pi_2$-conservative over $T_{\text{crsf}}$.

We adapt the method of [Avigad 2002] (after Zambella, Visser)
Herbrand saturation

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We adapt the method of [Avigad 2002] (after Zambella, Visser)

**Definition**

A structure $M$ is $\Delta_0$-Herbrand saturated if it satisfies every $\Sigma_2$-sentence with parameters from $M$ which is consistent with the $\Pi_1$-diagram of $M$. 
Herbrand saturation

To get witnessing for $\text{KP}^u_1$, it is enough now to show that $\text{KP}^u_1$ is $\Pi_2$-conservative over $T_{\text{crsf}}$.

We adapt the method of [Avigad 2002] (after Zambella, Visser)

Definition
A structure $M$ is $\Delta_0$-Herbrand saturated if it satisfies every $\Sigma_2$-sentence with parameters from $M$ which is consistent with the $\Pi_1$-diagram of $M$.

Lemma

1. In a $\Delta_0$-Herbrand saturated structure, every true $\Pi_2$ sentence is 'witnessed' by a term.

2. If every $\Delta_0$-Herbrand saturated model of $T_{\text{crsf}}$ is a model of $\text{KP}^u_1$, then $\text{KP}^u_1$ is $\Pi_2$-conservative over $T_{\text{crsf}}$. 
Conservativity proof 1

Theorem

\( \text{KP}_1^u \) is \( \Pi_2 \)-conservative over \( T_{\text{crsf}} \).
Conservativity proof 1

**Theorem**

$\text{KP}^u_1$ is $\Pi_2$-conservative over $T_{\text{crsf}}$.

**Proof sketch:** Let $M$ be a $\Delta_0$-Herbrand saturated model of $T_{\text{crsf}}$. We must show that $M \models \text{KP}^u_1$. In particular, that unique $\Sigma^1_1$ induction holds in $M$. 
Conservativity proof 1

Theorem
$\text{KP}^u_1$ is $\Pi_2$-conservative over $T_{\text{crsf}}$.

Proof sketch: Let $M$ be a $\Delta_0$-Herbrand saturated model of $T_{\text{crsf}}$. We must show that $M \models \text{KP}^u_1$. In particular, that unique $\Sigma_1^{\prec}$ induction holds in $M$.

Let $\varphi(x) \equiv \exists v \triangleleft t(x) \theta(x, v)$ be a $\Sigma_1^{\prec}$ formula with $\forall x \exists \leq^1 v \theta(x, v)$.

We may assume that the embedding $v \triangleleft t(x)$ is $\Delta_0$-definable and that the embedding bound is implicit in $\theta$.

That is, we assume $\varphi(x) \equiv \exists v \theta(x, v)$. 

Suppose the assumption of induction for $\varphi$ holds:

$$\forall x(\forall y \in x \exists u \theta(y, u) \rightarrow \exists v \theta(x, v)).$$
Conservativity proof 2

Suppose the assumption of induction for \( \varphi \) holds:

\[
\forall x (\forall y \in x \exists u \ \theta(y, u) \rightarrow \exists v \ \theta(x, v)).
\]

Suppose we have a function \( g(x, W) \) such that:

- whenever \( W \) contains witnesses to \( \exists u \ \theta(y, u) \) for every \( y \in x \),
- then \( g(x, W) \) is a witness to \( \exists v \ \theta(x, v) \).

Then we can define \( f(x) \) by recursion as

\[
\begin{align*}
    f(x) &= g(x, \{ f(y) \mid y \in x \}) \\
\end{align*}
\]

and prove by \( \Delta^0_1 \) (Lcrsf)-Induction that \( \forall x \ \theta(x, f(x)) \).

Hence \( \forall x \ \varphi(x) \), and we have shown induction for \( \varphi \).
Conservativity proof 2

Suppose the assumption of induction for $\varphi$ holds:

$$\forall x (\forall y \in x \exists u \theta(y, u) \rightarrow \exists v \theta(x, v)).$$

Suppose we have a function $g(x, W)$ such that:
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Then we can define $f(x)$ by recursion as

$$f(x) = g(x, \{f(y) : y \in x\})$$

and prove by $\Delta_0(L_{crsf})$-Induction that $\forall x \theta(x, f(x)).$

Hence $\forall x \varphi(x)$, and we have shown induction for $\varphi$. 
Conservativity proof 3

How do we get such a $g$? We have

$$\forall x (\forall y \in x \exists u \ \theta(y, u) \rightarrow \exists v \ \theta(x, v)).$$

Hence

$$\forall x \forall W (\forall y \in x \exists u \in W \ \theta(y, u) \rightarrow \exists v \ \theta(x, v)).$$

This is $\Pi_2$. 
Conservativity proof 3

How do we get such a \( g \)? We have

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\forall x (\forall y \in x \exists u \ \theta(y, u) \to \exists v \ \theta(x, v)).
\]

Hence

\[
\forall x \forall W (\forall y \in x \exists u \in W \ \theta(y, u) \to \exists v \ \theta(x, v)).
\]

This is \( \Pi_2 \). By \( \Delta_0 \)-Herbrand saturation, it is 'witnessed' in \( M \).

That is, there is a function \( h(x, W) \) such that:

- whenever \( W \) contains witnesses to \( \exists u \ \theta(y, u) \) for every \( y \in x \),
- then \( h(x, W) \) contains a witness to \( \exists v \ \theta(x, v) \).
Conservativity proof 3

How do we get such a $g$? We have

$$\forall x (\forall y \in x \exists u \theta(y, u) \rightarrow \exists v \theta(x, v)).$$

Hence

$$\forall x \forall W (\forall y \in x \exists u \in W \theta(y, u) \rightarrow \exists v \theta(x, v)).$$

This is $\Pi_2$. By $\Delta_0$-Herbrand saturation, it is 'witnessed' in $M$.

That is, there is a function $h(x, W)$ such that:

- whenever $W$ contains witnesses to $\exists u \theta(y, u)$ for every $y \in x$,
- then $h(x, W)$ contains a witness to $\exists v \theta(x, v)$.

Since such witnesses are unique, we can define

$$g(x, W) = \bigcup \{v \in h(x, W) : \theta(x, v)\}.$$
Open problems / speculation

1. Prove witnessing for KP₁ without choice.

2. At least prove witnessing using only local choice.
   E.g. if KP₁ ⊨ ∀x∃y ϕ(x, y), does this imply that there is a
   CRSF function f(x, r) such that ∀x∃y ∈ f(x, r) ϕ(x, y)
   whenever r is a well-ordering of tc(x)?

3. How simple a theory can we use instead of KP₁?
   E.g. take KP in the original language {∈}, add an axiom for
   transitive closure, and weaken Foundation to induction only
   for formulas ∃y ⊆ z θ(x, y) for θ ∈ Δ₀.

4. Infinitary propositional proof complexity

5. Arithmetic without predecessor
(Expected) connections between $\mathsf{KP}_1^\mathcal{O}$ and $S^1_2$

We can interpret $S^1_2$ in $\mathsf{KP}_1^\mathcal{O}$ as follows:

Let $L = \{\text{ordinals } \alpha \text{ such that no ordinal } \beta \leq \alpha \text{ is a limit}\}$. Let $M = \{x : x \subseteq \alpha \text{ for some } \alpha \in L\}$. Then the elements of $M$, considered as binary strings of length $\alpha$, form a model of $S^1_2$. 
(Expected) connections between $\text{KP}_1$ and $S^1_2$

We can interpret $S^1_2$ in $\text{KP}_1$ as follows:

Let $L = \{\text{ordinals } \alpha \text{ such that no ordinal } \beta \leq \alpha \text{ is a limit}\}$. Let $M = \{x : x \subseteq \alpha \text{ for some } \alpha \in L\}$. Then the elements of $M$, considered as binary strings of length $\alpha$, form a model of $S^1_2$.

We can interpret $\text{KP}_1$ in $S^1_2$ as follows:

Let $M = \{\text{strings coding Mostowski graphs}\}$. Then the functions and relations in $L_0$ are polynomial time under this encoding of sets as graphs, and with them $M$ is a model of $\text{KP}_1$. 