Small circuits for feasible set functions

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Joint work with

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Some previous work

- Sazonov – Bounded set theory
- Hamkins and Lewis – Infinite time Turing machines
- Beckmann, Buss, Friedman – Safe recursive set functions (SRSF)
- Arai – Predicatively computable set functions (PCSF)
The Cobham recursive set functions (CRSF) are defined by taking some basic initial functions and closing under composition and a limited recursion.
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We add a smash function \( \# \) as an initial function, to allow “polynomial increase in complexity”.
Introduction - CRSF

The Cobham recursive set functions (CRSF) are defined by taking some basic initial functions and closing under composition and a limited recursion.

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We add a smash function # as an initial function, to allow “polynomial increase in complexity”.

There is rather little set theory involved. CRSF also make sense as a way of talking about parallel computing where the inputs and outputs are directed acyclic graphs.
Arai recently introduced a class PCSF of set functions which capture polynomial time if restricted to finite binary strings.

**Theorem**

$\text{CRSF} = \text{PCSF}^+$, which is a slightly relaxed version of PCSF.
Introduction - results

**Theorem**
CRSF contains exactly the functions from sets to sets which are computed by uniform “polynomial size” families of infinite circuits.
Introduction - results

We define a weak fragment of Kripke-Platek set theory $\mathsf{KP}^\mathord{\leq}_1$, modelled on the bounded arithmetic theory $S^1_2$. It weakens Foundation to $\in$-induction for formulas of the form

$$\exists y \triangleleft t(x) \theta(x, y)$$

where $\theta \in \Delta_0$.

**Almost-Theorem**

CRSF contains exactly the $\Sigma_1$-definable functions of $\mathsf{KP}^\mathord{\leq}_1$. 

Define a function class RS by taking Jensen’s rudimentary set functions Rud, adding transitive closure, and closing under subset-bounded recursion.

(We do not add the smash function ≠)

**Theorem**

The functions in RS and CRSF are the same, modulo a certain way of coding the output.

In particular, RS and CRSF contain the same 0/1-valued functions.
Outline
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- The definition of CRSF
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- Turing machines
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- Turing machines
- Circuits
Cobham’s definition of \( \mathcal{P} \)

The smallest set of functions from binary strings to binary strings which contains as initial functions

- constant \( \epsilon \) (empty string)
- functions \( s \mapsto s0 \) and \( s \mapsto s1 \)
- projection functions \( a_1, \ldots, a_n \mapsto a_i \)
- the smash function \( a, b \mapsto a\#b = 0^{|a||b|} \)

and is closed under composition and recursion, where we introduce a new function \( f \) from functions \( g, h_0, h_1, k \) in \( \mathcal{P} \) by

\[
\begin{align*}
f(\bar{a}, \epsilon) &= g(\bar{a}) \\
f(\bar{a}, s0) &= h_0(\bar{a}, f(s), s) \\
f(\bar{a}, s1) &= h_1(\bar{a}, f(s), s)
\end{align*}
\]

provided that \( |f(\bar{a}, s)| \leq k(|a_1|, \ldots, |a_n|, |s|) \) for all \( \bar{a}, s \).
The primitive recursive set functions

Initial functions are:

- projections: \( a_1, \ldots, a_n \mapsto a_i \)
- conditional: \( \text{cond}_{\in} (a, b, c, d) = a \) if \( c \in d \), or \( b \) otherwise
- pair: \( a, b \mapsto \{a, b\} \)
- empty set: \( \emptyset \)
- union: \( a \mapsto \bigcup a \)

These are closed under composition and recursion:

\[
f(\bar{a}, b) = g(\bar{a}, b, \{f(\bar{a}, c) : c \in b\}).
\]
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- pair: \( a, b \mapsto \{a, b\} \)
- empty set: \( \emptyset \)
- union: \( a \mapsto \bigcup a \)

and we may add:

- transitive closure: \( a \mapsto \text{tc}(a) \)
- cartesian product: \( a, b \mapsto a \times b \)

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The Cobham recursive set functions

Initial functions are:

\[ \ldots \text{as before, plus} \]

- set composition: \( a, b \mapsto a \circ b \)
- set smash: \( a, b \mapsto a \# b \)
- embedded Mostowski collapse: \( a, E \mapsto M(a, E) \)

These are closed under composition and \textit{subset-bounded} recursion.

This is defined by: if \( g, h \in \text{CRSF} \) then so is

\[
f(\bar{a}, b) = g(\bar{a}, b, \{ f(\bar{a}, c) : c \in b \}) \cap h(\bar{a}, b).
\]
Mostowski graphs

The *Mostowski graph* of a set $a$ is the directed graph $\langle \text{tc}(\{a\}), E \rangle$ with nodes $\text{tc}(\{a\})$ and edges $E$ given by $\langle x, y \rangle \in E \iff x \in y$. We denote it by $G(a)$. It has a unique source 0 and sink $a$.

Examples: $G(2), G(\{1, 2\}), G(\omega)$
Definition
Given sets $a, b$ the set composition $a \circ b$ is the set whose Mostowski graph is constructed as follows:

- Draw the graph $G(a)$ above the graph $G(b)$
- Identify the source of $G(a)$ with the sink of $G(b)$. 

Equivalently, $a \circ b = \begin{cases} b & \text{if } a = 0 \\ x \circ b : x \in a & \text{if } a \neq 0 \end{cases}$
Extra initial functions – composition

Definition
Given sets $a, b$ the set composition $a \odot b$ is the set whose Mostowski graph is constructed as follows:

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- Identify the source of $G(a)$ with the sink of $G(b)$.

Equivalently, $a \odot b = \begin{cases} b & \text{if } a = 0 \\ \{x \odot b : x \in a\} & \text{if } a \neq 0. \end{cases}$
Extra initial functions – composition ⊙

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Given sets $a$, $b$ the set composition $a ∘ b$ is the set whose Mostowski graph is constructed as follows:

1. Draw the graph $G(a)$ above the graph $G(b)$
2. Identify the source of $G(a)$ with the sink of $G(b)$.

Equivalently, $a ∘ b = \begin{cases} b & \text{if } a = 0 \\ \{x ∘ b : x ∈ a\} & \text{if } a \neq 0. \end{cases}$

Notice

1. $\text{rank}(a ∘ b) = \text{rank}(b) + \text{rank}(a)$
2. $|\text{tc}(a ∘ b)| = |\text{tc}(a)| + |\text{tc}(b)|$. 
Definition
Given sets $a, b$ the set smash $a \# b$ is the set whose Mostowski graph is constructed as follows:

- Draw a disjoint copy $G_x$ of $G(b)$ for every node $x \in G(a)$
- For each edge $\langle x, y \rangle$ of $G(a)$, connect the sink of $G_x$ to the source of $G_y$. 

Equivalently, $a \# b = b \circ \{ x \# b : x \in a \}$.
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Equivalently, $a#b = b \odot \{x#b : x \in a\}$.

Notice
1. $\text{rank}(a#b) + 1 = (\text{rank}(b) + 1)(\text{rank}(a) + 1)$
2. $|\text{tc}(\{a#b\})| = |\text{tc}(\{a\})| \cdot |\text{tc}(\{b\})|$. 
Extra initial functions – embedded Mostowski collapse

...we will come back to this
Basic properties of CRSF

Definition
The CRSF relations are relations of the form $f(\bar{a}) \neq 0$ for a CRSF function $f$.

Lemma

1. CRSF is closed under separation. That is, if $\varphi(\bar{a}, c)$ is a CRSF relation then the following function is in CRSF:

\[
f(\bar{a}, b) = \{ c \in b : \varphi(\bar{a}, c) \}.
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2. The CRSF relations are closed under $\Delta_0$ quantification.

3. CRSF contains the rank function, and ordinal addition and multiplication.
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2. The CRSF relations are closed under \( \Delta_0 \) quantification.
3. CRSF contains the rank function, and ordinal addition and multiplication.
4. CRSF does not contain ordinal exponentiation.
Turing machines

Let $n, t \in \omega$. Consider a Turing machine limited to tape length $n$, in the binary language. A configuration of the machine is a binary string of length $n$ (if we code just the tape contents, and ignore the state and the head position).

We represent such a binary string as a subset of the ordinal $n$. E.g. we represent $01101 \in \{0, 1\}^5$ by $\{1, 2, 4\} \subseteq 5$.

Similarly, we can represent a complete history of a computation up to time $t$ as a subset $W \subseteq (t + 1) \times n$.

We can recover the configuration at time $i$ as $\{x \in n : \langle i, x \rangle \in W\}$. 

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We can recover the configuration at time $i$ as $\{x \in n : \langle i, x \rangle \in W\}$. 
Lemma
Let the function $f : E, n, t \mapsto W$ take as input numbers $n, t \in \omega$ and an initial configuration $E \subseteq n$ of the machine, and output the computation up to time $t$. Then $f \in CRSF$. 
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Proof
We use recursion to compute successively

- the computation up to time 0 (this is given by $E$)
- the computation up to time 1
  
  ...  

  the computation up to time $t$.
These are all subsets of $(t + 1) \times n$, so subset bounded recursion is sufficient.

It is easy to extend by one computation step, using separation with a $\Delta_0$ formula.
Turing machines

Let us continue to identify finite binary strings with subsets of finite ordinals.
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If we restrict the input and output to subsets of finite ordinals, then CRSF is exactly polynomial time.
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**Proof**
\( P \subseteq \text{CRSF} \): We use the lemma, plus ordinal arithmetic to compute the bounds on time and tape length.

\( \text{CRSF} \subseteq P \): Every initial CRSF function on sets corresponds to a simple function on their Mostowski graphs, and it is straightforward to manipulate graphs in \( P \). We can also simulate recursion in \( P \), and the subset bound stops this from blowing up the size of graphs too much.
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Infinite time Turing machines

These are defined as finite machines, except

- we replace the tape length $n$ with $\omega$
- we replace the time $t$ with an ordinal $\tau$
- we put in rules for what happens at a time which is a limit ordinal; in particular, each cell of the tape contains the lim sup of the contents at previous times.
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Polynomial time now means time $\omega^k$ for some $k < \omega$. 
Theorem
If we restrict the input and output to infinite subsets of \( \omega \), then CRSF contains exactly the functions computed by polynomial time ITTMs.

Proof
\( P \subseteq CRSF \): As before, with tweaks for the limit case.

\( CRSF \subseteq P \): This is more complicated, but the same in spirit as before. We use a result of Friedman and Welch characterizing \( P \) on ITTMs as functions uniformly definable in the \( L \) hierarchy.
Circuits in complexity theory
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A finite (binary, DeMorgan) circuit is a directed acyclic graph with
- source nodes labelled as inputs, or as constants 0 or 1
- possibly some nodes labelled as outputs
- internal nodes labelled as $\land$, $\lor$ or $\neg$ gates
  where $\neg$ gates have exactly one incoming edge.
Circuits in complexity theory

If we have

- input nodes labelled $x_0, \ldots, x_{n-1}$
- output nodes labelled $y_0, \ldots, y_{m-1}$

then the circuit computes a function from $\{0, 1\}^n$ to $\{0, 1\}^m$
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To compute a function $F$ with domain $\{0, 1\}^*$ we use a family of circuits $\langle C_n \rangle_{n \in \omega}$ where $C_n$ computes $F$ restricted to $\{0, 1\}^n$. 

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$F$ has *polynomial size circuits* if $|C_n| \leq p(n)$ for some polynomial $p$.

**Theorem**
The polynomial time functions are exactly those with uniform polynomial size circuits.
Infinite circuits

A circuit is a well-founded directed graph with

- source nodes labelled as inputs (or as constants 0 or 1)
- possibly some nodes labelled as outputs
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Here $\land$ and $\lor$ gates may have infinitely many incoming edges.
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...but it may require strong recursion to construct the computation.
Some notation

We write $<$ for the order induced by $\in$. That is, $a < b$ if and only if $a \in \text{tc}(b)$.
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We write $[b]$ for the set $\text{tc}(\{b\})$. Hence $[b] = \{a : a \leq b\}$. 
Infinite circuits

Formally, an *embedded circuit* is a triple $\langle c, E, \lambda \rangle$ where

- $c$ is any set. The nodes of the circuit are the members of $c$.
- $E \subseteq c \times c$ is the underlying graph of the circuit satisfying $\langle x, y \rangle \in E \rightarrow x < y$.
- $\lambda$ is a labelling function $c \rightarrow \{0, 1, *, \& , \lor, \neg\}$. The nodes labelled $*$ are input nodes. Each node labelled $\neg$ has exactly one $E$-predecessor.

We say that the circuit has size $c$. A computation is a correct assignment of 0/1 values to all nodes. We identify computations with subsets of $c$. We can thus construct computations in CRSF using subset-bounded recursion.
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We can thus construct computations in CRSF using subset-bounded recursion.
Infinite circuits

That is, let $C = \langle c, E, \lambda \rangle$ be a circuit and let $a$ be its set of input nodes.

Definition

Given any $A \subseteq a$, a computation of $C$ on $A$ is a subset $W \subseteq [c]$ which, informally, assigns 0/1 values to the nodes of $C$ using the usual rules of a Boolean circuit with input $A$.

That is, for all $u \in [c]$,

- if $\lambda(u) = 0$ then $u \notin W$
- if $\lambda(u) = 1$ then $u \in W$
- if $\lambda(u) = \ast$ then $u \in W \iff u \in A$
- if $\lambda(u) = \land$ then $u \in W \iff \forall v < u (\langle v, u \rangle \in E \rightarrow v \in W)$
- if $\lambda(u) = \lor$ then $u \in W \iff \exists v < u (\langle v, u \rangle \in E \land v \in W)$
- if $\lambda(u) = \neg$ then $u \in W \iff \exists v < u (\langle v, u \rangle \in E \land v \notin W)$. 
Strings

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**Definition**
For any set $a$, an *a-string* is just a subset of $a$. We think of it as assigning a 0/1 value to every member of $a$.

For example, for $n \in \omega$ we can identify the usual binary strings in $\{0,1\}^n$ with $n$-strings.
Circuits with string inputs and outputs

Definition
An embedded circuit with input size $a$ and output size $p$ is a tuple $C = \langle c, E, \lambda, a, p, \mu, \nu \rangle$ where

- $\langle c, E, \lambda \rangle$ is a circuit

and we are also given

- A partial function $\mu : [c] \rightarrow a$ mapping every input node to a member of $a$
- A function $\nu : p \rightarrow [c]$ which maps every element of $p$ to some node in $[c]$. 
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  which maps every element of $p$ to some node in $[c]$.

This computes a function which takes as input an $a$-string $A$ and outputs a $p$-string $P$.

Lemma
The function $\langle C, A \rangle \mapsto P$ is in CRSF.
Coding sets as strings

We know how to code natural numbers as finite binary strings which can then be processed by a circuit.

How do we code sets as strings?
Recall that the Mostowski graph $G(a)$ of a set $a$ is

- well-founded
- extensional
- “accessible pointed with sink $a$”
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On the other hand, any well-founded, extensional, accessible pointed graph is isomorphic to the Mostowski graph of some set, which we call its Mostowski collapse.
Initial functions - embedded Mostowski collapse

Recall $x < y$ means $x \in \text{tc}(y)$ and $[a] = \text{tc}([a])$. We write $[a]^2$ for $[a] \times [a]$.

**Definition**
A *diagram* is a pair $\langle a, E \rangle$ of sets such that

$$\langle x, y \rangle \in E \rightarrow x < y.$$
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Definition
A diagram is a pair \( \langle a, E \rangle \) of sets such that

\[
\langle x, y \rangle \in E \rightarrow x < y.
\]

A diagram represents the graph with nodes \([a]\) and edges \(E \cap [a]^2\).

This graph must be well-founded, because \(G(a)\) is.
In general it is not extensional or accessible pointed. Nevertheless, we can define its Mostowski collapse.
Initial functions - embedded Mostowski collapse

Definition
The embedded Mostowski collapse function $M(a, E)$ is defined by

$$M(a, E) = \{ M(b, E) : b < a \text{ and } \langle b, a \rangle \in E \}.$$
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Example: $M(4, \{ \langle 3, 4 \rangle, \langle 2, 3 \rangle, \langle 0, 3 \rangle \}) = \{ 1 \}.$
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Notice:

- this is definable by a simple $\in$-recursion
- $|M(a, E)| \leq |a|$
- $\text{rank}(M(a, E)) \leq \text{rank}(a)$. 
Definition
For sets $a, b$ we say that $b$ is embeddable in $a$ if

$$b = M(a, E)$$

for some $E$. 

Any set $a$ is embeddable in itself: we have $a = M(a, E)$ where $E = \{ \langle x, y \rangle \in [a]^2 : x \in y \}$.

We think of such a $b$ as having "size" $a$ and being coded by $E$.

Without loss of generality, we may assume $E$ is a $[a]^2$-string.

We can use $E$ as the input or output of a circuit, which can then compute things about the set $b$. 

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Circuits computing set functions

Definition
Let $f$ be a function from sets to sets. Let $a, p$ be sets.
Let $C$ be an embedded circuit with input size $[a]^2$ and output size $[p]^2$ such that for every $[a]^2$-string $E$,

$$M(p, C(E)) = f(M(a, E))$$

where $C(E)$ is the $[p]^2$-string output by $C$ on input $E$.
Then we say that $C$ computes $f$ on sets embeddable in $a$. 
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The definition extends naturally to functions with many inputs.
Small circuits

Recall that the size of a circuit $\langle c, E, \lambda, a, p, \mu, \nu \rangle$ is the set $c$.

**Definition**

A function $f$ from sets to sets has small circuits if there is a family $C_a$ of circuits, parametrized by a set $a$, such that for every $a$

- $C_a$ computes $f$ on sets embeddable in $a$.
- $C_a$ has size $s(a)$
- $C_a$ has output size $t(a)$

where $s$ and $t$ are smash-terms.

**Definition**

*Smash-terms* are functions built from variables, the constant 0 and the functions pair, $\times$, tc, $\circ$ and $\#$.

They play the role that polynomials usually play in complexity theory.
Small circuits for CRSF

Theorem
Every CRSF function has small circuits.
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Proof
Recall that the initial functions of CRSF are: 0, projection, pairing, union, conditional, transitive closure, cartesian product, $\odot$, $\#$, $M$. 
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**Theorem**
Every CRSF function has small circuits.

**Proof**
Recall that the initial functions of CRSF are: 0, projection, pairing, union, conditional, transitive closure, cartesian product, $\odot$, $\#$, $M$. These correspond to straightforward operations on diagrams.
Example: small circuits for union
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Given $a$ and $E$, consider the set

$$F = \left\{ \langle x, y \rangle \in [a]^2 : \begin{array}{ll}
\langle x, y \rangle \in E & \text{if } y \neq a \\
\exists z \in [a], \langle x, z \rangle \in E \land \langle z, a \rangle \in E & \text{if } y = a
\end{array} \right\}.$$

Then $M(a, F) = \bigcup M(a, E)$. 
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Then $M(a, F) = \bigcup M(a, E)$.

This formula can easily be made into a small circuit family $C_a$, where each $C_a$ takes an $[a]^2$-string $E$ as input and outputs the $[a]^2$-string $F$. 

Proof continued

Closure under composition is straightforward.
Proof continued

For closure under subset bounded recursion, suppose

\[ f(a) = g(a, \{ f(b) : b \in a \}) \cap h(a) \]

and we have small circuits for \( g \) and \( h \).
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We first construct circuits \( C_b \) for the function \( g(b, S) \cap h(b) \).

We may assume all circuits \( C_b \) have the same fixed size \( t(a) \),
where \( t \) is a smash-term. In other words, the underlying graph of
every \( C_b \) is the Mostowski graph \( G(t(a)) \) of \( t(a) \).
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To compute \( f(a) \), we simulate the recursion. We do this with a circuit, of size \( a \# t(a) \), constructed as follows:

- take the graph \( G(a) \)
- replace each node \( b \) with a copy of the circuit \( C_b \) and add suitable wires.
The circuits given by the theorem are highly uniform.

In particular, the circuit $C_a$ computing $f$ on sets embeddable in $a$ can be computed from $a$ by a CRSF function.
Uniform small circuits

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Corollary

The CRSF functions are exactly the functions from sets to sets computable by uniform families of small circuits.
Circuit lower bounds

Theorem [Hastad ’89]
There is no family of finite circuits with number of nodes polynomial in $n$ and depth polynomial in $\log(\log n)$ which calculates the parity of $n$-bit strings.
Theorem: There is no function $f$ in CRSF such that $f(a)$ is the parity of $|a|$ for every finite $a$.

Proof: Such an $f$ would have circuits $C_a$ of size some smash-term $t(a)$. Choose $n \in \omega$ of the form $2^{2^k}$ and let $a = P(P(k))$. Then $\text{rank}(a) = k + 2$, $|a| = n$ and $|\ll a \gg| \leq 2^n$. Smash terms increase rank and cardinality of transitive closure by at most a polynomial amount. Therefore $C_a$ has depth polynomial in $k = \log(\log n)$ and number of nodes polynomial in $n$. We can use $C_a$ to compute the parity of subsets of $a$, and therefore of subsets of $n$. This is a contradiction.
Corollary
There is no function $f$ in CRSF such that $f(a)$ is the parity of $|a|$ for every finite $a$.

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Proof
Such an $f$ would have circuits $C_a$ of size some smash-term $t(a)$. Choose $n \in \omega$ of the form $2^{2^k}$ and let $a = \mathcal{P}(\mathcal{P}(k))$. Then $\text{rank}(a) = k + 2$, $|a| = n$ and $|[a]| \leq 2n$. 
Circuit lower bounds

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P vs NP on sets

There is a natural analogue of NP for CRSF, namely relations of the form

$$\varphi(a) \iff \exists E \text{ a } t(a)\text{-string such that } \theta(a, E)$$

where $t$ is a smash-term and $\theta$ is a CRSF relation.
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This class is strictly bigger than CRSF. (This already follows indirectly from a result of Schindler, that $P \neq NP$ for infinite time Turing machines.)
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We can show this directly, even for hereditarily finite \( a \), by writing an NP expression for parity:

\(|a|\) is even if and only if there exists an \( a \times a\)-string \( E \) giving a partition of \( a \) into disjoint pairs.