Dichotomy, the Closed Range Theorem and Optimal Control

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Basic discrete time optimal control problem

Given $T \leq \infty, f, F, U$, maximize

$$\sum_{t=0}^{T} f(t, x(t), u(t))$$

x(t) determined by recurrent dynamics

$$\begin{aligned} x(t+1) &= F(t, x(t), u(t)) \\ x(0) &= x_0 \\ u(t) &\in U \text{ for all } t \end{aligned}$$

$u \in \mathbb{R}^m$ - control, $x \in \mathbb{R}^n$ - (state) response, f, F C^1 differentiable.

Pontryagin maximum principle

Let $(\hat{u}, \hat{x})$-optimal control/response pair. Then, $\exists$ constant $\psi^0 \geq 0$ and solution ψ (not both of them 0) of adjoint equation

$$\psi(t) = \psi^0 D_x f^*(t, \hat{x}(t), \hat{u}(t))\psi^0 + A^*(t)\psi(t+1)$$

with $A(t) = D_x F(t, \hat{x}(t), \hat{u}(t))$ such that

$$\begin{aligned} \psi^0 f(t, \hat{x}(t), \hat{u}(t)) \quad &+ \quad \psi(t+1)^* F(t, \hat{x}(t), \hat{u}(t)) \\ &= \quad \max_{u \in U}(\psi^0 f(t, \hat{x}(t), u) + \psi(t+1)^* F(t, \hat{x}(t), u)). \end{aligned}$$

Notes:

- Condition homogeneous in ψ^0, ψ ("soft" MP) so if $\psi(0) \neq 0$ one can choose $\psi^0 = 1$ ("hard" MP)
- For discrete time problems PMP holds only under extra convexity assumptions
- "dynamic" view: $u(.)$ as optimization parameter, $x(.)$ as its function

The unconstrained case $U = \mathbb{R}^m$

Let $(\hat{u}, \hat{x})$-optimal control/response pair. Then, $\exists$ solution of adjoint equation

$$\psi(t) = \psi^0 D_x f^*(t, \hat{x}(t), \hat{u}(t)) + A^*(t)\psi(t+1)$$

such that

$$\psi^0 D_u f(t, \hat{x}(t), \hat{u}(t)) + \psi^*(t+1)B(t) = 0$$

with $A(t) = D_x F(t, \hat{x}(t), \hat{u}(t))$ and $B(t) = D_u F(t, \hat{x}(t), \hat{u}(t))$ ("maximum condition")

In case $T < \infty$, $\psi^0 = 1$: standard first order Lagrange extremum condition, the adjoint variables being the multipliers.

"Functional" formulation (static view)

$(x(.), u(.))$ - optimization parameters, initial conditon and recurrence law - constraints:

Denote $\mathbf{x} = \{x(t)\}_t \in \mathbb{R}^{n(T+1)}$, $\mathbf{u} = \{u(t)\}_t \in \mathbb{R}^{m(T+1)}, \mathbf{A} = \{A(t)\}_t : \mathbb{R}^{n(T+1)} \to \mathbb{R}^{n(T+1)}$, $\mathbf{B} = \{B(t)\}_t : \mathbb{R}^{m(T+1)} \to \mathbb{R}^{n(T+1)}, \sigma\mathbf{x}(t) = \mathbf{x}(t+1), \mathbf{F}(\mathbf{x}, \mathbf{u})(t) = F(x(t), u(t))$, (Nemytskii operator)

$$J(\mathbf{x}, \mathbf{u}) = \sum_{t=0}^{T} f(t, x(t), u(t)).$$

Then, OC problem reads:

Maximize $J(\mathbf{x}, \mathbf{u})$ subject to the constraints

$$\begin{aligned} x(0) &= x_0 \\ \sigma(\mathbf{x}) &- \mathbf{F}(\mathbf{x}, \mathbf{u}) = 0. \end{aligned}$$

Assume $(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ optimal and $DF(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ has maximal rank. Then, along every path in the constraint set emanating from the tested point and tangent to a vector $\mathbf{y}, \mathbf{v}$ the minimized function should not decrease $\implies$

$$DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}})(\mathbf{y}, \mathbf{v}) = 0 \text{ if } \mathcal{L}(\mathbf{y}, \mathbf{v}) = 0. \tag{1}$$

with $\mathcal{L}(\mathbf{y}, \mathbf{v}) = (\pi_0 \mathbf{y}, (\sigma - \mathbf{A})\mathbf{y} - \mathbf{B}\mathbf{v}),\ \pi_t \mathbf{y} = y(t)$.

For $T < \infty$ one has

Elementary algebra for $T < \infty$

(1) holds if and only if $DJ(\hat{\mathbf{x}}, \hat{\mathbf{u}}) \in \mathcal{R}(\mathcal{L}^*)$, i. e. $\exists \psi \in \mathbb{R}^{n(T+1)}$ such that

$$\begin{aligned} D_{\mathbf{x}} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) &= \psi_0 \pi_0 + \psi^* (\sigma - \mathbf{A}) && (2) \\ D_{\mathbf{u}} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) &= \psi^* \mathbf{B}. && (3) \end{aligned}$$

Termwise, (2) is the adjoint equation, (3) the maximum condition.

If $DF(\hat{\mathbf{x}}, \hat{\mathbf{u}})$ fails to have maximal rank (*iff* not surjective), the soft maximum condition

$$\begin{aligned} \psi^0 D_{\mathbf{x}} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) &= \psi_0 \pi_0 + \psi^* (\sigma - \mathbf{A}) \\ \psi^0 D_{\mathbf{u}} J(\hat{\mathbf{x}}, \hat{\mathbf{u}}) &= \psi^* \mathbf{B} \end{aligned}$$

Continuous time: Pontrjagin et al. 1961: invertible dynamics, soft PMP

Very few references (2-3) for discrete time:
Blot - Chebbi 2000: assume invertible dynamics
Blot - Hayek 2008: assume contractive dynamics ($\|A(t)\| < 1$)
plus convexity $\rightarrow$ true PMP

The discounted infinite horizon problem

OC problem with $T = \infty$ and $f(t, x, u) = \delta^t f(x, u)$ (motivation: economics): $\mathbf{x} \in \mathbf{X} = l^n_\infty, \; \mathbf{u} \in \mathbf{U} = l^m_\infty$.

To extend the necessary condition of optimality in the PMP type one has to prove

1. $J : X \times U \to \mathbb{R}$ is differentiable, its derivative is obtained by termwise differentiation
2. $\mathcal{L} : X \times U \to X$ has closed range
3. $\mathcal{N}(\mathcal{L})$ has closed component
4. the singular component of $\psi \in (l^n_\infty)^* = l^n_1 \oplus l^n_s$ vanishes on local variations.

Proofs of 1 straightforward, 4 was observed earlier by Blot-Hayek, focus on 2,3.

Theorem

$\mathcal{L}$ has closed range provided

$$x(t+1) = A(t)x(t)$$

exhibits exponential dichotomy ($\iff$ is hyperbolic).

$$x(t+1) = A(t)x(t) \tag{4}$$

exhibits exponential dichotomy: $X = \mathbb{R}^n = X^-(t) \oplus X^+(t)$, (4) reads

$$\begin{aligned} x^-(t+1) &= A^-(t)x^-(t) \\ x^+(t+1) &= A^+(t)x^+(t) \end{aligned}$$

with $x^\pm(t) \in X^\pm(t)$;
$A^+(t)$ are nonsingular and there are constants $C > 0$ and $0 < \lambda < 1$ such that $|\Phi^-(t, s+1)| \leq C\lambda^{t-s}$ for $t > s$, $|\Phi^+(t, s+1)| \leq C\lambda^{s-t}$ for $t < s$,

$$\begin{aligned} \Phi^-(t, s+1) &= A^-(t-1)\dots A^-(s) \text{ for } t \geq s \\ \Phi^+(t, s+1) &= (A^+)^{-1}(t)\dots(A^+)^{-1}(s-1) \text{ for } t < s. \end{aligned}$$

Recall $l_\infty^n = \mathbf{X}$, denote $\mathbf{X}^{pm} = \{X^{pm}(t)\}_t$

$\mathbf{z} \in \mathcal{R}(\mathcal{L})$ iff $\exists(\mathbf{x}, \mathbf{u})$ bounded such that $x(0) = 0$ and

$$\begin{aligned} z(0) &= x(0) && (5) \\ z(t) &= x(t+1) - A(t)x(t) - B(t)u(t) \text{ for } t \geq 0 && (6) \end{aligned}$$

or, equivalently,

$$z^{\pm}(t) = x^{\pm}(t+1) - A^{\pm}(t)x^{\pm}(t) - B(t)^{\pm}u(t) \text{ for } t \geq 0$$

Well known (since Perron?): For given $\mathbf{z}, \mathbf{u}$, (5),(6) has unique bounded solution with $x^{-}(0) = z^{-}(0)$ given by

$$\begin{aligned} x^{-}(t) &= \Phi^{-}(t,0)z^{-}(0) + \sum_{0}^{t-1} \Phi^{-}(t,s+1)[z^{-}(s) - B^{-}(s)u(s)] \\ x^{+}(t) &= \sum_{t}^{\infty} \Phi^{+}(t,s+1)[z^{+}(s) - B^{+}(s)u(s)]; \end{aligned}$$

in order to satisfy also $x^{+}(0) = z^{+}(0)$ we need

$$z^{+}(0) = \sum^{\infty} \Phi^{+}(0,s+1)[z^{+}(s) - B^{+}(s)u(s)].$$

The range II

$$z^+(0) - \sum_0^\infty \Phi^+(0, s+1) z^+(s) = \sum_0^\infty B^+(s) u(s). \tag{7}$$

Denote

$$P = \{\sum_0^\infty \Phi^+(0, s+1) B^+(s) u(s) : \mathbf{u} \in l_\infty^m\}.$$

As subset of $\mathbb{R}^n$, P has finite basis $\{\xi_1, \dots \xi_d\}$ $d \leq n$; choose controls $\mathbf{u_j}$ such that

$$\sum_0^\infty \Phi^+(0, s+1) B^+(s) u_j(s) = \xi_j.$$

Then, (7) can be satisfied if and only if

$$z^+(0) - \sum_0^\infty \Phi^+(0, s+1) z^+(s) \in P,$$

or, there exist $\alpha_1, \ldots, \alpha_d$ such that

$$
\begin{aligned}
z^+(0) &- \sum_0^\infty \Phi^+(0, s+1) z^+(s) = \sum_j \alpha_j \xi_j \\
&= \sum_j \alpha_j \sum_0^\infty \Phi^+(0, s+1) B^+(s) u_j(s); \qquad (8)
\end{aligned}
$$

Denote $\mathbf{Z}^+ \subset \mathbf{X}^+$ the linear space of such $z^+(.)$. Z^+ has finite codimension and, therefore, is closed.

The range IV

$\mathbf{z} = \mathcal{L}(\mathbf{x}, \mathbf{u})$ iff $\mathbf{z}^+ \in \mathbf{Z}^+$ and

$$x^-(t) = \Phi(t,0)z^-(0) + \sum_0^{t-1} \Phi^-(t,s+1)[z^-(s) - B^-(s)u(s)] \quad (9)$$

$$x^+(t) = \sum_t^{\infty} \Phi^+(t,s+1)[z^+(s) - B^+(s)u(s)]; \quad (10)$$

where $u(t) = \sum_j \alpha_j u_j(t)$ with α_j determined by (**??**)
$\Longrightarrow \quad \mathcal{R}(\mathcal{L}) = \mathbf{X}^- \oplus \mathbf{Z}^+$. Moreover, the map given by (9), (10) maps $\mathcal{R}(\mathcal{L})$ isomorphically onto a closed complement of $\mathcal{N}(\mathcal{L})$.

Theorem

Let the system

$$x(t+1) = A(t)x(t),$$

admit exponential dichotomy. Then, the optimal control/response pair satisfies the (in general soft) termwise maximum condition. In $F(x,u)$ is linear in both x and u or $\mathbf{Z}^+ = \mathbf{X}^+$ then the hard condition holds.

Example

dim $x = 1,\ A = 1,\ B = 0$:

$$x(t+1) = x(t)$$

$$\mathcal{R}(\mathcal{L}) = \{\mathbf{z} \text{ bounded } : z(t) = x(t+1) - x(t), \mathbf{x} \text{ bounded}\}$$

For $\epsilon > 0$ small let

$$z_\epsilon(0) = 0, z_\epsilon(t) = t^{-(1+\epsilon)}.$$

For $\epsilon \to 0$, $\mathbf{z}_\epsilon \to \mathbf{z_0}$ in l_∞. For $t > 1$ one has
$x_\epsilon(t) = \sum_{s=2}^{t-1}(s-1)^{-(1+\epsilon)}$ so $\mathbf{x}_\epsilon \in l_\infty$ for $\epsilon > 0$ but $\mathbf{x_0} \notin l_\infty$.

Concluding remarks

- Dichotomy assumption satisfied if e. g. $A(t) \to A, B(t) \to B$ for $t \to \infty$ and A, B hyperbolic, or if periodic and the period map hyperbolic
- Adjoint variable $\psi(.) \in l_1 \implies$ "transversality" condition
- Other spaces of variations simpler because of absence of singular component of the dual
- Analogy for continuous systems: hard optimality condition
- dichotomy condition appears in early papers by J. Kurzweil on the "analytic construction of regulators"
- Refinements possible