Dichotomy, the Closed Range Theorem and Optimal Control

Pavel Brunovský
(joint work with Mária Holecyová)

Comenius University Bratislava, Slovakia

Praha 13. 5. 2016
Basic discrete time optimal control problem

Given $T \leq \infty$, $f$, $F$, $U$, maximize

$$\sum_{t=0}^{T} f(t, x(t), u(t))$$

$x(t)$ determined by recurrent dynamics

$$x(t + 1) = F(t, x(t), u(t))$$

$$x(0) = x_0$$

$$u(t) \in U \text{ for all } t$$

$u \in \mathbb{R}^m$ - control, $x \in \mathbb{R}^n$ - (state) response, $f$, $F$ $C^1$ differentiable.
Pontryagin maximum principle

Let $(\hat{u}, \hat{x})$-optimal control/response pair. Then, $\exists$ constant $\psi^0 \geq 0$ and solution $\psi$ (not both of them 0) of adjoint equation

$$\psi(t) = \psi^0 D_x f^*(t, \hat{x}(t), \hat{u}(t)) \psi^0 + A^*(t)\psi(t + 1)$$

with $A(t) = D_x F(t, \hat{x}(t), \hat{u}(t))$ such that

$$\psi^0 f(t, \hat{x}(t), \hat{u}(t)) + \psi(t + 1) F(t, \hat{x}(t), \hat{u}(t)) = \max_{u \in U}(\psi^0 f(t, \hat{x}(t), u) + \psi(t + 1) F(t, \hat{x}(t), u)).$$

Notes:

- Condition homogeneous in $\psi^0, \psi$ ("soft" MP) so if $\psi(0) \neq 0$ one can choose $\psi^0 = 1$ ("hard" MP)

- For discrete time problems PMP holds only under extra convexity assumptions

- "dynamic" view: $u(.)$ as optimization parameter, $x(.)$ as its function
Let \((\hat{u}, \hat{x})\)-optimal control/response pair. Then, \(\exists\) solution of adjoint equation

\[
\psi(t) = \psi^0 D_x f^*(t, \hat{x}(t), \hat{u}(t)) + A^*(t) \psi(t + 1)
\]

such that

\[
\psi^0 D_u f(t, \hat{x}(t), \hat{u}(t)) + \psi^*(t + 1) B(t) = 0
\]

with \(A(t) = D_x F(t, \hat{x}(t), \hat{u}(t))\) and \(B(t) = D_u F(t, \hat{x}(t), \hat{u}(t))\) ("maximum condition")

In case \(T < \infty\), \(\psi^0 = 1\): standard first order Lagrange extremum condition, the adjoint variables being the multipliers.
(x(.), u(.)) - optimization parameters, initial condition and recurrence law - constraints:

Denote $x = \{x(t)\}_{t \in \mathbb{R}^{n(T+1)}}$, $u = \{u(t)\}_{t \in \mathbb{R}^{m(T+1)}}$, $A = \{A(t)\}_{t : \mathbb{R}^{n(T+1)} \to \mathbb{R}^{n(T+1)}}$, $B = \{B(t)\}_{t : \mathbb{R}^{m(T+1)} \to \mathbb{R}^{n(T+1)}}$, $\sigma x(t) = x(t + 1)$, $F(x, u)(t) = F(x(t), u(t))$, (Nemytskii operator)

$$J(x, u) = \sum_{t=0}^{T} f(t, x(t), u(t)).$$

Then, OC problem reads:

Maximize $J(x, u)$ subject to the constraints

$$x(0) = x_0$$

$$\sigma(x) - F(x, u) = 0.$$
Universal way to obtain 1-st order extremum condition

Assume \((\hat{x}, \hat{u})\) optimal and \(DF(\hat{x}, \hat{u})\) has maximal rank. Then, along every path in the constraint set emanating from the tested point and tangent to a vector \(y, v\) the minimized function should not decrease \(\implies\)

\[
DJ(\hat{x}, \hat{u})(y, v) = 0 \text{ if } \mathcal{L}(y, v) = 0. \tag{1}
\]

with \(\mathcal{L}(y, v) = (\pi_0 y, (\sigma - A)y - Bv), \pi_t y = y(t)\).

For \(T < \infty\) one has

Elementary algebra for \(T < \infty\)

(1) holds if and only if \(DJ(\hat{x}, \hat{u}) \in \mathcal{R}(\mathcal{L}^*)\), i.e. \(\exists \psi \in \mathbb{R}^{n(T+1)}\) such that

\[
D_x J(\hat{x}, \hat{u}) = \psi_0 \pi_0 + \psi^*(\sigma - A) \tag{2}
\]

\[
D_u J(\hat{x}, \hat{u}) = \psi^* B. \tag{3}
\]

Termwise, (2) is the adjoint equation, (3) the maximum condition.
If $DF(\hat{x}, \hat{u})$ fails to have maximal rank (*iff* not surjective), the soft maximum condition

$$\psi^0 D_x J(\hat{x}, \hat{u}) = \psi_0 \pi_0 + \psi^* (\sigma - A)$$

$$\psi^0 D_u J(\hat{x}, \hat{u}) = \psi^* B$$
Continuous time: Pontrjagin et al. 1961: invertible dynamics, soft PMP

Very few references (2-3) for discrete time:
Blot - Chebbi 2000: assume invertible dynamics
Blot - Hayek 2008: assume contractive dynamics ($\|A(t)\| < 1$)
plus convexity $\rightarrow$ true PMP
OC problem with $T = \infty$ and $f(t, x, u) = \delta^t f(x, u)$ (motivation: economics): $x \in X = l^n_\infty$, $u \in U = l^m_\infty$.

To extend the necessary condition of optimality in the PMP type one has to prove

1. $J : X \times U \to \mathbb{R}$ is differentiable, its derivative is obtained by termwise differentiation
2. $\mathcal{L} : X \times U \to X$ has closed range
3. $\mathcal{N}(\mathcal{L})$ has closed component
4. the singular component of $\psi \in (l^n_\infty)^* = l^n_1 \oplus l^n_s$ vanishes on local variations.

Proofs of 1 straightforward, 4 was observed earlier by Blot-Hayek, focus on 2,3.
Closed range and exponential dichotomy

Theorem
\[ L \text{ has closed range provided } \]
\[ x(t + 1) = A(t)x(t) \]

exhibits exponential dichotomy (\( \iff \) is hyperbolic).
Exponential dichotomy

\[ x(t + 1) = A(t)x(t) \]  \hspace{1cm} (4)

exhibits exponential dichotomy: \( X = \mathbb{R}^n = X^{-}(t) \oplus X^{+}(t) \), (4) reads

\[ x^{-}(t + 1) = A^{-}(t)x^{-}(t) \]
\[ x^{+}(t + 1) = A^{+}(t)x^{+}(t) \]

with \( x^{\pm}(t) \in X^{\pm}(t) \);
\( A^{+}(t) \) are nonsingular and there are constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that
\[ |\Phi^{-}(t, s + 1)| \leq C\lambda^{t-s} \text{ for } t > s, \]
\[ |\Phi^{+}(t, s + 1)| \leq C\lambda^{s-t} \text{ for } t < s, \]

\[ \Phi^{-}(t, s + 1) = A^{-}(t-1)\ldots A^{-}(s) \text{ for } t \geq s \]
\[ \Phi^{+}(t, s + 1) = (A^{+})^{-1}(t)\ldots(A^{+})^{-1}(s-1) \text{ for } t < s. \]

Recall \( l^\infty = X \), denote \( X^{pm} = \{X^{pm}(t)\}_t \)
The range $I$

$z \in \mathcal{R}(\mathcal{L})$ iff $\exists (x, u)$ bounded such that $x(0) = 0$ and

\begin{align*}
z(0) &= x(0) \\
z(t) &= x(t + 1) - A(t)x(t) - B(t)u(t) \text{ for } t \geq 0
\end{align*}

(5)

(6)

or, equivalently,

\[ z^{\pm}(t) = x^{\pm}(t + 1) - A^{\pm}(t)x^{\pm}(t) - B(t)^{\pm}u(t) \text{ for } t \geq 0 \]

Well known (since Perron?): For given $z, u$, (5), (6) has unique bounded solution with $x^-(0) = z^-(0)$ given by

\[ x^-(t) = \Phi^-(t, 0)z^-(0) + \sum_{s=0}^{t-1} \Phi^-(t, s + 1)[z^-(s) - B^-(s)u(s)] \]

\[ x^+(t) = \sum_{t}^{\infty} \Phi^+(t, s + 1)[z^+(s) - B^+(s)u(s)]; \]

in order to satisfy also $x^+(0) = z^+(0)$ we need

\[ z^+(0) = \sum_{t=0}^{\infty} \Phi^+(0, s + 1)[z^+(s) - B^+(s)u(s)]. \]
The range II

\[ z^+(0) - \sum_{0}^{\infty} \Phi^+(0, s + 1)z^+(s) = \sum_{0}^{\infty} B^+(s)u(s). \] (7)

Denote

\[ P = \left\{ \sum_{0}^{\infty} \Phi^+(0, s + 1)B^+(s)u(s) : u \in l^m_{\infty} \right\}. \]

As subset of \( \mathbb{R}^n \), \( P \) has finite basis \( \{\xi_1, \ldots, \xi_d\} \) \( d \leq n \); choose controls \( u_j \) such that

\[ \sum_{0}^{\infty} \Phi^+(0, s + 1)B^+(s)u_j(s) = \xi_j. \]

Then, (7) can be satisfied if and only if

\[ z^+(0) - \sum_{0}^{\infty} \Phi^+(0, s + 1)z^+(s) \in P, \]
or, there exist $\alpha_1, \ldots, \alpha_d$ such that

$$z^+(0) - \sum_{0}^{\infty} \Phi^+(0, s + 1)z^+(s) = \sum_{j} \alpha_j \xi_j$$

$$= \sum_{j} \alpha_j \sum_{0}^{\infty} \Phi^+(0, s + 1)B^+(s)u_j(s); \quad (8)$$

Denote $\mathbf{Z}^+ \subset \mathbf{X}^+$ the linear space of such $z^+(.)$. $Z^+$ has finite codimension and, therefore, is closed.
\[ z = \mathcal{L}(x, u) \text{ iff } z^+ \in \mathbb{Z}^+ \text{ and } \]

\[ x^-(t) = \Phi(t, 0)z^-(0) + \sum_{s=0}^{t-1} \Phi^-(t, s + 1)[z^-(s) - B^-(s)u(s)] \]

\[ x^+(t) = \sum_{t}^{\infty} \Phi^+(t, s + 1)[z^+(s) - B^+(s)u(s)]; \quad (10) \]

where \( u(t) = \sum_j \alpha_j u_j(t) \) with \( \alpha_j \) determined by (??)

\[ \implies \mathcal{R}(\mathcal{L}) = X^- \oplus \mathbb{Z}^+. \text{ Moreover, the map given by (9), (10) maps } \mathcal{R}(\mathcal{L}) \text{ isomorphically onto a closed complement of } \mathcal{N}(\mathcal{L}). \]
Theorem

Let the system

\[ x(t + 1) = A(t)x(t), \]

admit exponential dichotomy. Then, the optimal control/response pair satisfies the (in general soft) termwise maximum condition. In \( F(x, u) \) is linear in both \( x \) and \( u \) or \( Z^+ = X^+ \) then the hard condition holds.
Is dichotomy needed?

Example

\[
\dim x = 1, \ A = 1, \ B = 0:
\]

\[
x(t + 1) = x(t)
\]

\[
R(L) = \{z \text{ bounded} : z(t) = x(t + 1) - x(t), x \text{ bounded}\}
\]

For \(\epsilon > 0\) small let

\[
z_\epsilon(0) = 0, z_\epsilon(t) = t^{-(1+\epsilon)}.
\]

For \(\epsilon \to 0\), \(z_\epsilon \to z_0\) in \(l_\infty\). For \(t > 1\) one has

\[
x_\epsilon(t) = \sum_{s=2}^{t-1} (s - 1)^{-(1+\epsilon)} \text{ so } x_\epsilon \in l_\infty \text{ for } \epsilon > 0 \text{ but } x_0 \notin l_\infty.
\]
• Dichotomy assumption satisfied if e.g. $A(t) \to A, B(t) \to B$ for $t \to \infty$ and $A, B$ hyperbolic, or if periodic and the period map hyperbolic

• Adjoint variable $\psi(.) \in l_1 \implies "transversality"$ condition

• Other spaces of variations simpler because of absence of singular component of the dual

• Analogy for continuous systems: hard optimality condition

• Dichotomy condition appears in early papers by J. Kurzweil on the "analytic construction of regulators"

• Refinements possible