On regulated functions and selections

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Regulated functions

Let X be a Banach space. A function $u : [0,1] \to X$ is said to be regulated if there exist the limits $u(t^+)$ and $u(s^-)$ for any point $t \in [0,1)$ and $s \in (0,1]$.

The name for this class of functions was introduced by Dieudonné.

The set of discontinuities of a regulated function is at most countable.

Not all functions with countable set of discontinuity points are regulated. A simple example is the characteristic function $\chi_{\{1,1/2,1/3,...\}} \notin G([0,1],\mathbb{R})$.

Regulated functions are bounded.

Regulated functions II

When $(X, \|\cdot\|)$ is a Banach algebra with the multiplication $*$ the space $G([0,1],X)$ is a Banach algebra too endowed with the pointwise product, i.e. $(f \cdot g)(x) = f(x) * g(x)$.

In contrast to the case of continuous functions the composition of regulated functions need not to be regulated. The simplest example is a composition $(g \circ f)$ of functions $f, g : [0,1] \to \mathbb{R}$: $f(x) = x \cdot \sin \frac{1}{x}$ and $g(x) = sgn\, x$ (both are regulated), which has no one-side limits at 0. Thus even a composition of a regulated and continuous functions need not to be regulated.

The space $G([0,1],X)$ of regulated functions

The space $G([0,1],X)$ of regulated functions on $[0,1]$ into the Banach space X is a Banach space too, endowed with the topology of uniform convergence, i.e. with the norm $\|u\|_\infty = \sup\limits_{t\in[0,1]} \|u(t)\|$.

The space $G([0,1],X)$ is **not separable**, contains. as a proper subset, the space of continuous functions $C([0,1],X)$.

It can be represented (as an isometrical isomorphic copy) as a space of continuous functions on some Hausdorff compact non-metrizable space $\mathbb{K}$ (but different than $[0,1]$ as $G([0,1],X)$ is not separable).

(cf. [Michalak]) Put

$$\mathbb{K} = \{(t,0) : 0 < t \leq 1\} \cup \{(t,1) : 0 \leq t \leq 1\} \cup \{(t,2) : 0 \leq t < 1\}$$

(called the (Alexandroff) arrow space) and equip this set with the order topology given by the lexicographical order (i.e., $(s,i) \prec (t,j)$ if either $s < t$, or $s = t$ and $i < j$).

The neighborhoods of the point (t,r) in this topology are of the form

$$\begin{aligned} V_b(t,0) &= \{(s,r) : b < s < t, r = 0,1,2\} \cup \{(t,0)\} \\ V_c(t,2) &= \{(s,r) : t < s < c, r = 0,1,2\} \cup \{(t,2)\} \\ V_d(t,1) &= \{(t,1)\}. \end{aligned}$$

Isometry

Theorem

([Michalak] for $X = \mathbb{R}$) *The Banach spaces* $G([0,1], X)$ *and* $C(\mathbb{K}, X)$ *are isometrically isomorphic in the following way: given functions* $f \in G([0,1], X)$*, and* $\kappa(f) = g \in C(\mathbb{K}, X)$*, as corresponding to each other if*

$g(t,r) = \lim_{s \to t-} f(s)$ *if* $r = 0$ *and* $t \in (0,1]$*,*

$g(t,r) = f(t)$ *if* $r = 1$ *and* $t \in [0,1]$ *and*

$g(t,r) = \lim_{s \to t+} f(s)$ *if* $r = 2$ *and* $t \in [0,1)$*.*

Compactness in $G([0,1],X)$

Definition

[Fraňková] A set $\mathcal{A} \subset G([0,1],X)$ is said to be equi-regulated if for every $\varepsilon > 0$ and every $t_0 \in [0,1]$ there exists $\delta > 0$ such that for all $x \in \mathcal{A}$:

i) for any $t_0 - \delta < s < t_0$: $\|x(s) - x(t_0^-)\| < \varepsilon$;

ii) for any $t_0 < \tau < t_0 + \delta$: $\|x(\tau) - x(t_0^+)\| < \varepsilon$.

Lemma

[Fraňková] *For a set $\mathcal{A} \subset G([0,1],\mathbb{R}^d)$ the following assertions are equivalent:*

(i) $\mathcal{A} \subset G([0,1],\mathbb{R}^d)$ is relatively compact;

(ii) $\mathcal{A}$ is equi-regulated and, for every $t \in [0,1]$,

$\mathcal{A}(t) = \{x(t), x \in \mathcal{A}\}$ *is relatively compact in $\mathbb{R}^d$.*

Compactness in $G([0,1],X)$ via isometry

Theorem

A subset $\mathcal{A} \subset G([0,1],X)$ is equi-regulated if and only if $\kappa(\mathcal{A}) \subset C(\mathbb{K},X)$ is equicontinuous in $C(\mathbb{K},X)$.

Theorem

A subset $\mathcal{A} \subset G([0,1],X)$ is relatively compact if and only if $\kappa(\mathcal{A}) \subset C(\mathbb{K},X)$ is relatively compact in $C(\mathbb{K},X)$.

Weak compactness in $G([0,1],X)$

Lemma

The dual space of $G([0,1],X)$ is isometrically isomorphic to the space $rcabv(\mathcal{B}_o(\mathbb{K}),X^*)$ of regular countably additive X^* valued Borel vector measures on $\mathbb{K}$ with bounded variation.

Dobrakov's Theorem for regulated functions:

Theorem

([MC, KC, B. Satco]) *A sequence (x_n) of regulated functions $x_n \in G([0,1],X)$ is weakly convergent to x in $G([0,1],X)$ if and only if is (norm) bounded and for any $t \in [0,1]$ a sequence $(x_n(t))$ is weakly convergent to $x(t)$ in X for each $t \in [0,1]$.*

Theorem

[Michalak] *A superposition operator $F(x) = f(\cdot, x(\cdot))$ maps $G([0,1])$ into itself if and only if the function f has the following properties:*

1. *the limit* $\lim_{[0,s)\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $(s,x) \in (0,1] \times \mathbb{R}$*,*
2. *the limit* $\lim_{(t,1]\times\mathbb{R}\ni(u,y)\to(s,x)} f(u,y)$ *exists for every* $(t,x) \in [0,1) \times \mathbb{R}$*.*

In It means, that for the composition operator (autonomous superposition operator) $F(x)(t) = f(x(t))$ maps $G([0,1])$ into itself iff f is continuous.

Definition

([MC, Metwali]) For a bounded subset $A \subset G([0,1],X)$ we define

$$\begin{aligned}\omega_\delta^G(A) &= \sup_{x\in A}\ \sup_{t\in(0,1]}\ \sup_{s\in(0,1),t-\delta<s<t} \|x(s)-x(t^-)\| \\ &+ \sup_{x\in A}\ \sup_{t\in[0,1)}\ \sup_{s\in(0,1),t<s<t+\delta} \|x(s)-x(t^+)\|\end{aligned}$$

and then a function

$$\omega^G(A) = \lim_{\delta\to 0} \omega_\delta^G(A)$$

will be called a (uniform) modulus of equi-regularity of the set A.

Similarly we define the pointwise modulus of equi-regularity at the point $t_0 \in (0,1)$ by

$$\begin{aligned} \omega^G(A, t_0) &= \lim_{\delta \to 0} \Bigg(\sup_{x \in A} \sup_{s \in (0,1), t_0 - \delta < s < t_0} \|x(s) - x(t_0^-)\| \\ &+ \sup_{x \in A} \sup_{s \in (0,1), t_0 < s < t_0 + \delta} \|x(s) - x(t_0^+)\| \Bigg). \end{aligned}$$

Lemma

For a subset A of $G([0,1], X)$ we have $\omega^G(A) = 0$ if and only if A is equi-regulated. Consequently, for any relatively compact subsets B of $G([0,1], X)$ we have $\omega^G(B) = 0$.

Lemma

Let A be a subset of $C([0,1],X)$. Then:

$$\omega^G(A) \leq 2\omega^C(A).$$

To show, that our modulus is, in some sense, uniform we get the following lemma:

Lemma

Let A be a subset of $G([0,1],X)$. Then

$$\sup_{t_0 \in (0,1)} \omega^G(A,t_0) \leq \omega^G(A) \leq 2 \cdot \sup_{t_0 \in (0,1)} \omega^G(A,t_0).$$

Definition

The following function μ_G is a regular measure of noncompactness in the space $G([0,1],X)$:

$$\mu_G(A) = \omega^G(A) + \sup_{t\in[0,1]} \mu_X(A(t)),$$

where μ_X stands for a measure of noncompactness in X.

Theorem

A bounded subset A of $G([0,1],X)$ is relatively compact if and only if $\mu_G(A) = 0$. Consequently, it is relatively compact iff is equi-regulated and $A(t)$ are relatively compact in X for $t \in [0,1]$.

Denote by β_E the Hausdorff measure of noncompactness in the space E. Then we able to estimate β_G by the above defined measures.

Theorem

For any subset A of $G([0,1],X)$ we have

$$\beta_G(A) \leq \omega^G(A) + \sup_{t\in[0,1]} \beta(A(t))$$

and

$$\beta_G(A) \leq \sup_{t\in[0,1]} [\omega^G(A,t) + \beta(A(t))].$$

The space $D([0,1],X)$

The space of X-valued functions right-continuous admit finite left-limits at every point will be denoted by $D([0,1],X)$ (càdlàg functions).

In such a case we are able to simplify the space $\mathbb{K}$ in the construction of an isomorphic copy of $D([0,1],X)$.

$$\mathbb{L} = \{(t,0) : 0 < t \leq 1\} \cup \{(t,1) : 0 \leq t < 1\}$$

Theorem

The Banach spaces $D([0,1],X)$ and $C(\mathbb{L},X)$ are isometrically isomorphic in the following way: given functions $f \in D([0,1],X)$, and $\kappa(f) = g \in C(\mathbb{L},X)$, as corresponding to each other if $g(t,r) = \lim_{s\to t-} f(s)$ if $r = 0$ and $t \in (0,1]$, $g(t,r) = f(t)$ if $r = 1$ and $t \in [0,1)$.

Regulated selections

Given a multivalued mapping $F : A \to 2^X$ we have several results related the the question of existence of regulated selections for F under various regularity conditions for F.

The selection problem, i.e. existence of a mapping $f(t) \in F(t)$ for arbitrary $t \in A$, seems to be is one of the most interesting problems in the theory of multivalued analysis.

Some special cases

Lemma

([MC, KC, B. Satco]) *Let a multivalued mapping* $F : [0,1] \to 2^X$ *has nonempty, bounded, closed values. Then there exists a selection* $f : [0,1] \to X$ *being:*

(a) *continuous provided* F *is lower semicontinuous with convex values,*

(b) *of bounded* φ*-variation if* F *has compact values and* $F \in BV_\varphi([0,1], c(X))$*,*

(c) *of bounded variation if* F *has compact values and* $F \in BV([0,1], c(X))$*,*

(d) *Riemann measurable if* F *is lower semicontinuous at almost every point,*

(e) *measurable if* F *is measurable and* X *is separable.*

Corollary

If the convex-valued multifunction $F \in BV([0,1], c(X))$ *is lower semicontinuous outside the countable set* $\{t_1, t_2, ...\}$*, then there exists a regulated selection* $f(t) \in F(t)$*.*

Theorem

([MC, KC, B. Satco]) *If the multifunction* $F : [0,1] \to 2^X$ *with convex values is lower semicontinuous outside the countable set* $\{t_1, t_2, ...\}$ *and at every point* t_k *(*$k = 1, 2, ...$*) there exist lower limits* $Li_{t \to t_k^-} F(t) = \{\widetilde{x_k}\}$*,* $Li_{t \to t_k^+} F(t) = \{\overline{x_k}\}$ *(*$\widetilde{x_k}, \overline{x_k} \in X$*) and* $lim_{t \to t_k} diam(F(t)) = 0$*, then there exists a regulated selection* $f(t) \in F(t)$*,* $t \in [0,1]$*.*

Example

Consider $F(t) = \{t \cdot \sin \frac{1}{t}\}$ for $-1 \leq t < 0$, $F(t) = \{3 + t \cdot \sin \frac{1}{t}\}$ for $1 \geq t > 0$ and $F(0) = [1, 2]$. All the values of F are compact, F is not lower semicontinuous at $t = 0$ and $F \notin BV([-1, 1], \mathbb{R})$, but $Li_{t \to t^-} F(t) = \{0\} \neq \emptyset$ and $Li_{t \to t^+} F(t) = \{3\} \neq \emptyset$. Clearly, F **has a regulated selection**, but there is neither continuous nor bounded variation selections for F. It is unique up to a value at 0. If we expect more selections, we can modify our example by putting $G(t) = [t \cdot \sin \frac{1}{t} - |t|, t \cdot \sin \frac{1}{t} + |t|]$ for $-1 \leq t < 0$ and an analogous change for the remaining points. However G has compact values (not singletons) and satisfies $\lim_{t \to 0} diam G(t) = \{0\}$. Note, that G has many regulated selections.

Theorem

([MC, KC, B. Satco]) *Let* $F : [0,1] \to cl(X)$ *be a convex-valued multifunction such that at each point* $t_0 \in [0,1]$ *the sets* $F(t_0-) = Li_{t \to t_0-} F(t)$, $F(t_0+) = Li_{t \to t_0+} F(t)$ *are nonempty and they satisfy the following conditions:*

i) $F(t_0-) \subset Li_{t \to t_0-} F(t-) \cap \; Li_{t \to t_0-} F(t+)$;

ii) $F(t_0+) \subset Li_{t \to t_0+} F(t-) \cap \; Li_{t \to t_0+} F(t+)$.

Then F *has at least one regulated selection.*

The use of Hausdorff left and right limits instead of Kuratowski inferior limit allows one to get one more selection result.

For a closed, bounded and convex-valued multifunction $F : [0,1] \to 2^X$, define the left limit $F_H(t_0\pm) = Lim_{t\to t_0\pm} F(t)$ in the sense of Pompeiu-Hausdorff distance.

Theorem

([MC, KC, B. Satco]) *Let $F : [0,1] \to 2^X$ be a closed and convex-valued multifunction such that at each point $t_0 \in [0,1]$ there exist limits $F_H(t_0-), F_H(t_0+)$ and they are nonempty. Then F has at least one regulated selection.*

Thank you!

Professor Jaroslav Kurzweil

HAPPY 90th BIRTHDAY !!!