Sturm-Liouville problem with weights via the Hardy inequality

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Sturm-Liouville problem with weights:

$$\begin{cases} \left(\rho(t)\,|u'(t)|^{p-2}u'(t)\right)' + \lambda\,\sigma(t)\,|u(t)|^{p-2}u(t) = 0, \quad t \in (a,b) \\ \lim\limits_{t\to a_+} \rho(t)\,|u'(t)|^{p-2}u'(t) = \lim\limits_{t\to b_-} u(t) = 0 \end{cases}$$

$p > 1$, $-\infty \le a < b \le +\infty$, ρ, σ continuous, positive weights

$\forall x \in (a,b)$: $\sigma \in L^1(a,x)$ $\quad \rho^{1-p'} \in L^1(x,b)$

$$\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$$

in general : $\sigma, \rho^{1-p'} \notin L^1(a,b)$!

SL-property:

All eigenvalues form increasing sequence

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots \to +\infty .$$

To each eigenvalue λ_n .... unique normalized eigenfunction with $n-1$ zeros in (a,b).

Between two consecutive zeros of u_n there is exactly one zero of u_{n+1}.

Multiplying eq. by u and integrating by parts :

$$\int_a^b \rho(t)\,|u'(t)|^p\,dt = \lambda \int_a^b \sigma(t)\,|u(t)|^p\,dt$$

Hardy inequality :

$$\int_a^b \sigma(t)\,|u(t)|^p\,dt \leq C \int_a^b \rho(t)\,|u'(t)|^p\,dt$$

$$\Downarrow$$

$$\lambda \geq \frac{1}{C}$$

Lower bound for all possible eigenvalues.

eigenvalues ... minimax for Rayleigh quotient:

$$R(u) := \frac{\int_a^b \rho(t)\,|u'(t)|^p\,dt}{\int_a^b \sigma(t)\,|u(t)|^p\,dt}$$

natural spaces:

$$W_b^{1,p}(\rho) := \Big\{ u \text{ abs. cont. on every compact in } (a,b),\ u(b)=0,\ \|u\|_{1,p;\rho} = \Big(\int_a^b \rho(t)\,|u'(t)|^p dt\Big)^{\frac{1}{p}} < \infty \Big\}$$

$$L^p(\sigma) := \Big\{ u \text{ measurable } (a,b),\ \|u\|_{p;\sigma} = \Big(\int_a^b \sigma(t)\,|u(t)|^p dt\Big)^{\frac{1}{p}} < \infty \Big\}$$

Hardy $\int_a^b \sigma(t)\,|u(t)|^p\,dt \le C \int_a^b \rho(t)\,|u'(t)|^p\,dt$

$$\Downarrow$$

$$W_b^{1,p}(\rho) \hookrightarrow L^p(\sigma)$$

$$\Updownarrow$$

$$\sup_{t\in(a,b)} \underbrace{\left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1}}_{\substack{\text{Muckenhoupt function} \\ M(t)}} < \infty$$

$$W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma)$$

$$\Updownarrow$$

$$\lim_{\substack{t \to a_+ \\ t \to b_-}} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} = 0 .$$

$$M(t) \to 0$$
$$(t \to a_+, b_-)$$

Theorem 1 (P.D. + Kufner + Kalier) : (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii)

(i) SL-property holds.

(ii) $W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma)$.

(iii) $\lim\limits_{\substack{t \to a_+ \\ t \to b_-}} M(t) = 0$.

Theorem 2 (P.D. + Kufner + Kuliev)

$\exists\ \varepsilon \in (0, p-1)\ ,\ c > 0\ :\ \forall t \in (a,b)$

$$M(t) \le c \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau \right)^{\varepsilon}$$

$$\Downarrow$$

$\exists\ c_1, c_2 > 0\ ,\ \exists\ \bar{b} \in (a,b)\ :\ \forall t \in (\bar{b}, b)$

$$c_1 \int_t^b \rho^{1-p'}(\tau)\, d\tau \le |u(t)| \le c_2 \int_t^b \rho^{1-p'}(\tau)\, d\tau$$

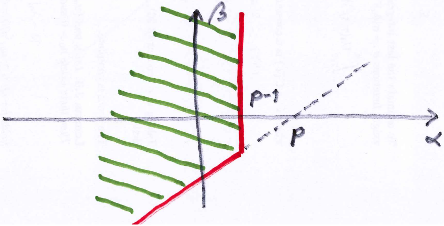
eigenfunction

Application # 1 : Radial problem on the ball

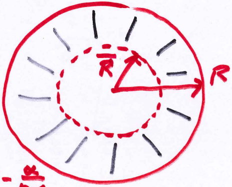
$$\begin{cases} -\operatorname{div}\left((R-|x|)^{\alpha} |\nabla u|^{p-2} \nabla u \right) = \lambda (R-|x|)^{\beta} |u|^{p-2} u & B_R(0) \\ u = 0 & \text{on } \partial B_R(0) \end{cases}$$

Assumptions of Theorem 2 hold if $1 < p < N$ and

$(\beta < -1$ and $\alpha - \beta < p)$ or $(\beta \geq -1$ and $\alpha < p-1)$



$\exists\, \bar R \in (0,R) \quad \forall x \in B_R(0) \setminus B_{\bar R}(0)$



$$c_1 (R-|x|)^{1-\frac{\alpha}{p-1}} \le |u(x)| \le c_2 (R-|x|)^{1-\frac{\alpha}{p-1}}$$

1. $\alpha = 0$ .... $u(x) \approx (R-|x|)\,,\ x \to \partial B_R(0)$
   (Vázquez m.p. $\frac{\partial u}{\partial \nu} < 0$)

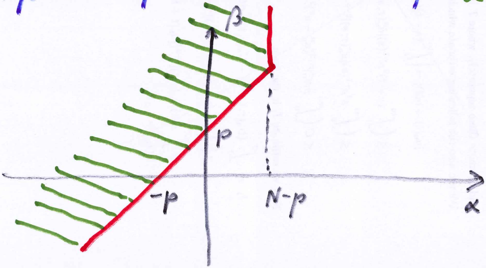
2. $\alpha \in (0,p-1)$ .... $u(x) \approx (R-|x|)^{1-\frac{\alpha}{p-1}}\,,\ x \to \partial B_R(0)$, $1-\frac{\alpha}{p-1} \in (0,1)$
   $\left(\frac{\partial u}{\partial \nu} = -\infty \right)$

3. $\alpha < 0$ .... $u(x) \approx (R-|x|)^{1-\frac{\alpha}{p-1}}\,,\ x \to \partial B_R(0)$, $1-\frac{\alpha}{p-1} > 1$
   $\left(\frac{\partial u}{\partial \nu} = 0 \right)$

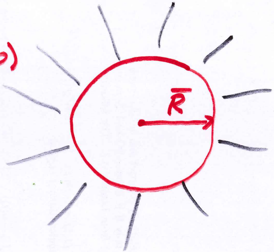
Application #2 : Radial problem in $\mathbb{R}^N$

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{(1+|x|)^{\alpha}}|\nabla u|^{p-2}\nabla u\right) = \lambda \dfrac{1}{(1+|x|)^{\beta}}|u|^{p-2}u & \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

Assumptions of Theorem 2 hold if $\alpha + p < \min\{N, \beta\}$



$\exists \bar{R}>0 \ \exists c_1, c_2>0 \ \forall x \in \mathbb{R}^N \setminus B_{\bar{R}}(0)$



$$\frac{c_1}{|x|^{\frac{N-(\alpha+p)}{p-1}}} \le |u(x)| \le \frac{c_2}{|x|^{\frac{N-(\alpha+p)}{p-1}}}$$

In particular, if $\alpha = 0$ $\quad (p < \min\{N, \beta\})$

$$u(x) \approx \frac{1}{|x|^{\frac{N-p}{p-1}}}$$

for $p=2$: $\quad u(x) \approx \frac{1}{|x|^{N-2}}$
$(N>2)$

Thank you!