Coincidence of Pettis and McShane integrability

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Blahopřání k životnímu jubileu
profesora Jaroslava Kurzweila
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We say that f is Pettis integrable on [0,1] if for every $x^* \in X^*$ the composition $x^* \circ f$ is Lebesgue inegrable and for every measurable set $E \in [0,1]$ there is $x_E \in X$ such that $x^*(x_E) = \int_E x^*\big(f(t)\big)\mathrm{d}\lambda(t)$ for every $x^* \in X^*$.

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We say that f is Bochner integrable on [0,1] if f is measurable and the function $[0,1] \ni t \longmapsto \|f(t)\| \in \mathbb{R}$ is Lebesgue integrable.

We say that f is McShane integrable on [0,1] if there exists $x \in X$ such that for every $\varepsilon > 0$ there are $\eta \in (0,1)$ and a gauge function δ assigning to every $t \in [0,1]$ an open subset of $t \in \delta(t) \subset [0,1]$ such that:

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for every $r \in \mathbb{N}$, for every points $t_1, \ldots, t_r \in [0,1]$, and for every pairwise disjoint measurable subsets $E_1, \ldots, E_r$ of $[0,1]$, such that $\lambda\big(E_1 \cup \cdots \cup E_r\big) > \eta$, and $\delta(t_j) \supset E_j,\ j = 1, \ldots, r$, we have $\big\| \sum_{j=1}^r \lambda(E_j) f(t_j) - x \big\| < \varepsilon$;

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We say that f is Kurzweil-Henstock integrable on [0,1] if, in the definition of McShane integrability, we add that $E_j \ni t_j,\ j = 1, \ldots, r$.

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A compact space K is Eberlein (uniform Eberlein) if and only if the corresponding Banach space $C(K)$ is weakly compactly generated (Hilbert generated)

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Theorem 1

(Leiderman-Sokolov, Argyros-Farmaki) *Let* Δ *be an uncountable set and consider a family* $\mathcal{F} \subset \Delta^{\leq\omega}$ *such that* $K := \{1_A : A \in \mathcal{F}\}$ *is a compact space. Then* K *is Eberlein (uniform Eberlein) if and only if*

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$$\forall\, A \in \mathcal{F} \;\; \forall\, n \in \mathbb{N} \;\; \#(A \cap \Delta_n) < \omega \; (< n).$$

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Proposition 2

If $\mathcal{F} \subset [0,1]^{\leq\omega}$ is such that the corresponding $K_{\mathcal{F}}$ is a uniform Eberlein compact space, then $\mathcal{F}$ is `not` *MC-filling.*

Proposition 3

If $\mathcal{F} \subset [0,1]^{\leq\omega}$ *is such that the corresponding* $K_{\mathcal{F}}$ *is a compact space, then* $\mathcal{F}$ *is* `not` *MC-filling if and only if*

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Avilés-Plebanek-Rodriguez method of inflating Eberlein not uniformly Eberlein compact spaces

Let Γ be a given uncountable set with $\#\Gamma \leq \mathbf{c}$. By a Γ-partition of the interval $[0,1]$ we mean the equality $[0,1] = \bigcup_{\gamma\in\Gamma} Z_\gamma$ where the sets Z_γ's are pairwise disjoint and moreover $\lambda^*(Z_\gamma) = 1$ for every $\gamma \in \Gamma$;

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From now on, fix one Γ-partition $[0,1] = \bigcup_{\gamma\in\Gamma} Z_\gamma$. Define $\varphi : [0,1] \longrightarrow \Gamma$ by $\varphi\restriction_{Z_\gamma} \equiv \gamma$ for every $\gamma \in \Gamma$.

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Consider a weakly compact set $K \subset c_0(\Gamma)$ *and let* $H \subset c_0([0,1])$ *be an adequate inflation of it. Then*

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(i) The set H *is weakly compact (equivalently,* H *is norm-bounded and pointwise compact).*

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Consider a weakly compact set $K \subset c_0(\Gamma)$ and let $H \subset c_0([0,1])$ be an adequate inflation of it. Then

(i) The set H is weakly compact (equivalently, H is norm-bounded and pointwise compact).

(ii) H, in the pointwise topology, is a uniform Eberlein compact set if and only if so is K.

Theorem 4

(Main) *Let* Γ *be an uncountable set with* $\#\Gamma \leq \mathbf{c}$. *Let* $K \subset c_0(\Gamma)$ *be a weakly compact set such that it is* `not` *uniform Eberlein.*

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Then there exists a scalarly null (hence Pettis integrable) function $f : [0,1] \longrightarrow C(H)$ *which is* `not` *McShane integrable.*

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Question. Is it possible to take $H := K$ in the theorem above?

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fabian@math.cas.cz