GODEs: overview and trends

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“Stories” behind a joint work with Everaldo Bonotto, Matheus Bortolan, Rodolfo Collegari and Jaqueline Mesquita

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1988: IME and Non Absolute Integration

S. Schwabik, C. Hönig, M. Federson, G. Monteiro

2008

2006

2008: students with Prof. Schwabik

2009: students with Prof. Kurzweil

2009: photo by Jaqueline Mesquita who is missing...

2011: ... Milan Tvrdý, Plácido Táboas...

2012: Eduard, Patricia, Rodolfo, Maria Carolina, Fabio, Jaqueline, Rafael, Manoel.

2016: ... with Jana B. Vampolová, Irena Rachunková,

Milan Tvrdý, Joseph Diblik...

The Kurzweil integral

Example:

Consider $F : [0,1] \to \mathbb{R}$ given by

$$F(t) = \begin{cases} t^2 \sin \dfrac{1}{t^2}, & 0 < t \leq 1, \\ 0, & t = 0. \end{cases}$$

Assertion:

$$\exists\ F'(t),\ \forall t \in [0,1].$$

Let $f = F'$. Then

- f **IS NOT Lebesgue integrable.**
- f **IS Kurzweil-Henstock integrable (= Perron integrable)**

Tagged Divisions

A **tagged division** of $[a, b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, with

$$a = s_0 \leq s_1 \leq \ldots \leq s_k = b \quad \text{and} \quad \tau_i \in [s_{i-1}, s_i],$$

for $i = 1, 2, \ldots, |D|$.

Gauges

Given a function $\delta : [a, b] \to (0, +\infty)$ (called **gauge** of $[a, b]$), a tagged-division $D = (\tau_i, [s_{i-1}, s_i])$ is **δ-fine**, whenever

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)),$$

for $i = 1, 2, \ldots, |D|$.

The Kurzweil Integral

A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is **Kurzweil integrable** over $[a, b]$, if $\exists!\ I \in X$ such that $\forall\ \varepsilon > 0$, $\exists$ a gauge δ of $[a, b]$ such that $\forall$ δ-fine tagged-division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$,

$$\left\| \sum_i \left[U(\tau_i, s_i) - U(\tau_i, s_{i-1})\right] - I \right\| < \varepsilon.$$

In this case, $I = \int_a^b DU(\tau, t)$.

Cousin Lemma

Given a gauge δ of $[a, b]$, there is a δ-fine tagged-division of $[a, b]$.

The Perron-Stieltjes integral

Let X be a Banach space and let $F\colon [a,b] \to L(X)$ and $g\colon [a,b] \to X$ be s.t.

$$U(\tau, t) = F(t)g(\tau).$$

Then the integral

$$\int_a^b DU(\tau, t) = \int_a^b D[F(t)g(\tau)]$$

which is defined by means of sums of the form

$$\sum [F(t_i) - F(t_{i-1})]g(\tau_i)$$

can be rewritten as

$$\int_a^b d[F(s)]g(s).$$

Generalized ODEs

Let X be a Banach space, $\mathcal{O} \subset X$ be open $[\alpha, \beta] \subset [a, +\infty)$ and $\Omega = \mathcal{O} \times [\alpha, \beta]$.

Definition

A function $x : [\alpha, \beta] \to X$ is **a solution** on $[\alpha, \beta]$ of the GODE

$$\frac{dx}{d\tau} = DF(x, t),$$

whenever $(x(t), t) \in \Omega \ \forall \ t \in [\alpha, \beta]$ and

$$x(v) = x(\gamma) + \int_{\gamma}^{v} DF(x(\tau), t), \qquad \gamma, \ v \in [\alpha, \beta].$$

Example

Let $r\colon [0,1] \to \mathbb{R}$ be a continuous function which is nowhere differentiable in $[0,1]$ and $G(x,t) = r(t)$. Then

$$\int_{s_1}^{s_2} DG(x(\tau),t) = \int_{s_1}^{s_2} Dr(t) = r(s_2) - r(s_1).$$

Moreover, $x\colon [0,1] \to \mathbb{R}$ defined by

$$x(s) = r(s), \quad s \in [0,1]$$

is a solution of the GODE

$$\frac{dx}{d\tau} = DG(x,t) = Dr(t).$$

Impulsive measure FDEs as Measure FDEs

Consider the impulsive measure functional differential equation

$$\begin{cases} x(v) - x(u) = \int_u^v f(x_s, s)\, dg(s), \ u, v \in J_k, \ k \in \{0, \dots, m\}, \\ \Delta^+ x(t_k) := x(t_k^+) - x(t_k) = I_k(x(t_k)), \ \ k \in \{1, \dots, m\}, \\ x_{t_0} = \phi, \end{cases}$$

where

- $\sigma > 0$, g is a left-continuous function;
- $t_1, \dots, t_m$ are impulse moments, $t_0 \leq t_1 < \cdots < t_m < t_0 + \sigma$;
- $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k \in \{1, \dots, m-1\}$, and $J_m = (t_m, t_0 + \sigma]$;
- $I_k : \mathbb{R}^n \to \mathbb{R}^n$.

Remark:

The integral

$$\int_u^v f(x_s, s)\, dg(s), \quad u, v \in J_k,$$

does not change if we replace g by a function $\tilde{g}$ such that $g - \tilde{g}$ is a constant function on J_k (this follows easily from the definition of the Kurzweil-Henstock-Stieltjes integral).

Suppose

- g is left continuous and continuous at $t_1, \dots, t_m$.

Then

- $t \mapsto \int_{t_0}^{t} f(x_s, s)\, dg(s)$ is continuous

and our problem

$$\begin{cases} x(t) = x(t_0) + \int_{t_0}^{t} f(x_s, s)\, dg(s) + \sum_{k=1}^{m} I_k(x(t_k)) H_{t_k}(t), \\ x_{t_0} = \phi, \end{cases}$$

is s.t. $\Delta^+ x(t_k) = I_k(x(t_k)),\ \forall\ k \in \{1, \dots, m\}$.

Lemma - Federson, Mesquita, Slavik

Let $f : [a, b] \to \mathbb{R}$, $g \in G^{-}([a, b], \mathbb{R})$ be continuous at $t_1, \ldots, t_m$, where $a \leq t_1 < t_2 < \cdots < t_m \leq b$. Let $\tilde{f}, \tilde{g} : [a, b] \to \mathbb{R}$ be s.t.

- $\tilde{f}(t) = f(t), \forall\ t \in [a, b] \setminus \{t_1, \ldots, t_m\}$;
- $\tilde{g} - g$ is constant in $[a, t_1]$, $(t_1, t_2], \ldots, (t_{m-1}, t_m]$, $(t_m, b]$.

Then

- $\exists \displaystyle\int_a^b \tilde{f}\, d\tilde{g} \iff \exists \displaystyle\int_a^b f\, dg$;
- $\displaystyle\int_a^b \tilde{f}\, d\tilde{g} = \int_a^b f\, dg + \sum_{\substack{k\in\{1,\ldots,m\},\\ t_k<b}} \tilde{f}(t_k)\Delta^{+}\tilde{g}(t_k)$.

Theorem - Federson, Mesquita, Slavik

Let $t_0 \leq t_1 < \cdots < t_m < t_0+\sigma$, $B \subset \mathbb{R}^n$, $I_1, \ldots, I_m : B \to \mathbb{R}^n$, $P = G([-r,0], B)$, $f : P \times [t_0, t_0+\sigma] \to \mathbb{R}^n$, $g \in G^-([t_0, t_0+\sigma], \mathbb{R})$ be continuous at $t_1, \ldots, t_m$. Define

$$\tilde{f}(y,t) = \begin{cases} f(y,t), & t \in [t_0, t_0+\sigma] \backslash \{t_1, \ldots, t_m\}, \\ I_k(y(0)), & t = t_k \text{ para algum } k \in \{1, \ldots, m\}. \end{cases}$$

Let $c_1, \ldots, c_m \in \mathbb{R}$ and define $\tilde{g} : [t_0, t_0+\sigma] \to \mathbb{R}$ by

$$\tilde{g}(t) = \begin{cases} g(t), & t \in [t_0, t_1], \\ g(t) + c_k, & t \in (t_k, t_{k+1}] \text{ para algum } k \in \{1, \ldots, m-1\}, \\ g(t) + c_m, & t \in (t_m, t_0+\sigma]. \end{cases}$$

Theorem - continuation

Then $x \in G([t_0 - r, t_0 + \sigma], B)$ is a solution of

$$\begin{cases} x(t) & = & x(t_0) + \displaystyle\int_{t_0}^{t} f(x_s, s)\, dg(s) + \sum_{k=1}^{m} I_k(x(t_k)) H_{t_k}(t), \\ x_{t_0} & = & \phi, \end{cases}$$

iff x is a solution of

$$\begin{cases} x(t) = x(t_0) + \displaystyle\int_{t_0}^{t} \tilde{f}(x_s, s)\, d\tilde{g}(s), \\ x_{t_0} = \phi. \end{cases}$$

Measure neutral FDEs

as GODEs

Consider measure neutral functional differential equations (we write measure NFDEs, for short) of the form

$$\begin{aligned} y(t) &= y(0) + \int_0^t f(y_s, s)dg(s) \\ &\quad + \int_{-r}^0 d[\mu(t,\theta)]y(t+\theta) - \int_{-r}^0 d[\mu(0,\theta)]\varphi(\theta). \end{aligned}$$

where

- $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$, $\phi \in G([-r, 0], \mathbb{R}^n)$, $r > 0$;
- $\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$, μ is left continuous on $\theta \in (-r, 0)$,
- μ is BV in $\theta \in [-r, 0]$, with $\operatorname{var}_{[s,0]} \mu \to 0$ as $s \to 0$;
- $\mu(t,\theta) = 0$, $\theta \geq 0$, $\mu(t,\theta) = \mu(t,-r)$, $\theta \leq -r$.

Theorem - Federson, Frasson, Mesquita, Tacuri

Consider $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, *with* $c \geq 1$, $\phi \in P_c = \{x_t;\ x \in B_c,\ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ *nondecreasing and (H1), (H2), (H3), (H4), (H5) fulfilled. Let* $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *be as defined and* $y \in P_c$ *be a solution of the measure NFDE in* $[t_0, t_0 + \sigma]$*. Define, for* $t \in [t_0 - r, t_0 + \sigma]$,

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then $x : [t_0, t_0 + \sigma] \to B_c$ *is a solution of the GODE* $\dfrac{dx}{d\tau} = DG(x, t)$.

Theorem - Federson, Frasson, Mesquita, Tacuri

Let $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, *with* $c \geq 1$, $\phi \in P_c = \{z_t; z \in B_c,\ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ *nondecreasing and (H1), (H2), (H3), (H4), (H5) fulfilled. Let* $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ *as defined and* $x : [t_0, t_0 + \sigma] \to B_c$ *be a solution of the GODE* $\frac{dx}{d\tau} = DG(x, t)$, *with initial condition* $x(t_0)(\vartheta) = \phi(\vartheta)$ *for* $\vartheta \in [t_0 - r, t_0]$, *and* $x(t_0)(\vartheta) = x(t_0)(t_0)$ *for* $\vartheta \in [t_0, t_0 + \sigma]$. *Then* $y \in B_c$ *given by*

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0, \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

is a solution of the measure NFDE in $t \in [t_0 - r, t_0 + \sigma]$.

The generalized Feynman integral

Definition

Let $I \subset \mathbb{R}$ be an interval of the following type

$$(-\infty, v), \qquad [u, v) \qquad or \qquad [u, +\infty).$$

We say that the interval I is associated to τ if

$$\tau = -\infty, \qquad \tau = u \text{ or } v \qquad or \qquad \tau = +\infty,$$

respectively.

Definition

A partition of $\mathbb{R}$ is any finite collection of disjoints intervals I s.t. $\cup I = \mathbb{R}$.

Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Definition

Let $\delta : \overline{\mathbb{R}} \to (0, \infty)$ be a positive function, $I \subset \mathbb{R}$ be an interval associated to $\tau \in \overline{\mathbb{R}}$. The pair (τ, I) is δ-fine, whenever

$$v < -\frac{1}{\delta(\tau)}, \quad v - u < \delta(\tau) \qquad or \qquad u > \frac{1}{\delta(\tau)},$$

respectively. The function δ is called gauge.

Let $N = \{t_1, \ldots, t_n\}$ be a finite set, with $\mathbb{R}_{t_j} = \mathbb{R}$ and $\overline{\mathbb{R}}_{t_j} = \overline{\mathbb{R}}$. Then we write

$$\prod\{\overline{\mathbb{R}}_{t_j} : t_j \in N\} = \overline{\mathbb{R}}^N.$$

An element of $\overline{\mathbb{R}}^N$ is denoted by

$$\tau = (\tau(t_1), \tau(t_2), \ldots, \tau(t_n)) = (\tau_1, \tau_2, \ldots, \tau_n).$$

For each $t_j \in N$, let $I_j = I(t_j)$ be an interval associated to τ_j. Then $I = I_1 \times \ldots \times I_n$ is an interval of $\prod\{\mathbb{R}_{t_j} : t_j \in N\} = \mathbb{R}^N$ and the pair (τ, I) is associated in $\mathbb{R}^N$, if each pair (τ_j, I_j) is associated in $\mathbb{R}$, $1 \leq j \leq n$. This means that τ is a vertex of I in $\overline{\mathbb{R}}^N$.

Definition

Given a function $\widetilde{\delta} : \overline{\mathbb{R}}^N \to (0, \infty)$, an associated pair (τ, I) of the domain $\mathbb{R}^N$ is $\widetilde{\delta}$-fine, if each pair (τ_j, I_j) fulfills the conditions

$$v_j < -\frac{1}{\widetilde{\delta}(\tau)}, \quad v_j - u_j < \widetilde{\delta}(\tau) \qquad or \qquad u_j > \frac{1}{\widetilde{\delta}(\tau)},$$

depending on the corresponding interval I_j.

Definition

A finite collection $\mathcal{E} = \{(\tau_j, I_j)\}$ of associated pairs (τ_j, I_j), where each pair (τ_j, I_j) is associated in $\mathbb{R}^N$, is a tagged-division of $\mathbb{R}^N$, if the intervals I_j are disjoints s.t. $\cup I_j = \mathbb{R}^N$. The division is δ-fine, if each pair (τ_j, I_j), $1 \leq j \leq n$, is δ-fine.

Let B be an infinite set and $\mathcal{F}(B)$ the family of finite subsets of B. Consider the product space

$$\prod_{t\in B} \mathbb{R}_t = \mathbb{R}^B, \quad \text{where} \quad \mathbb{R}_t = \mathbb{R},\ t \in B.$$

Then $\mathbb{R}^B$ represents the set of functions from B to $\mathbb{R}$.

Denote by $\tau = \tau_B$ an element of $\overline{\mathbb{R}}^B$ and consider the set

$$N = N_B = \{t_1, \ldots, t_n\} \in \mathcal{F}(B).$$

An element $(\tau_1, \ldots, \tau_n) = (\tau(t_1), \ldots, \tau(t_n))$ of $\overline{\mathbb{R}}^N$ is denoted by $\tau(N) = \tau(N_B)$.

Consider the projection $P_N : \mathbb{R}^B \to \mathbb{R}^N$ given by

$$P_N(\tau) = (\tau(t_1), \ldots, \tau(t_n)).$$

Similarly, consider the projection $\overline{P}_N : \overline{\mathbb{R}}^B \to \overline{\mathbb{R}}^N$.

For each interval $I_1 \times \ldots \times I_n$ of $\mathbb{R}^N$, there is a cell

$$I[N] := P_N^{-1}(I_1 \times \ldots \times I_n) \subset \mathbb{R}^B.$$

Instead of $I_1 \times \ldots \times I_n$, we write $I(N)$ so that

$$I[N] = I(N) \times \mathbb{R}^{B \setminus N}.$$

Similarly, $\overline{P}_N(\tau_B) = \tau(N) \in \overline{\mathbb{R}}^N$, for $\tau = \tau_B \in \overline{\mathbb{R}}^B$.

Definition

Given $\tau \in \overline{\mathbb{R}}^B$ and a cell $I[N] \subset \mathbb{R}^B$, the point-cell pair $(\tau, I[N])$ is associated in $\mathbb{R}^B$, if the point-interval pair $(\tau(N), I(N))$ is associated in $\mathbb{R}^N$.

Definition

A finite collection $\mathcal{E} = \{(\tau^j, I^j[N]) : \tau^j \in \overline{\mathbb{R}}^B, N \in \mathcal{F}(B)\}$ of associated point-cell pairs is a tagged-division of $\mathbb{R}^B$, if the pairs $(\tau^j, I^j[N])$ are associated in $\mathbb{R}^B$ and the cells $I^j[N]$ are disjoints with union equal to $\mathbb{R}^B$. We denote this tagged-division by $\mathcal{E} = \{(\tau, I[N])\}$.

Consider applications

- $L_B : \overline{\mathbb{R}}^B \to \mathcal{F}(B), \quad L_B(\tau) \in \mathcal{F}(B);$
- $\delta_B : \overline{\mathbb{R}}^B \times \mathcal{F}(B) \to (0, \infty), \quad 0 < \delta_B(\tau, N) < \infty.$

Let $\gamma_B := (L_B, \delta_B)$. We call γ_B a gauge.

Definition

An associated pair $(\tau, I[N])$ is γ_B-fine, whenever

- $N \supseteqq L_B(\tau)$;
- $(\tau(N), I(N))$ is δ_B-fine in $\mathbb{R}^N$.

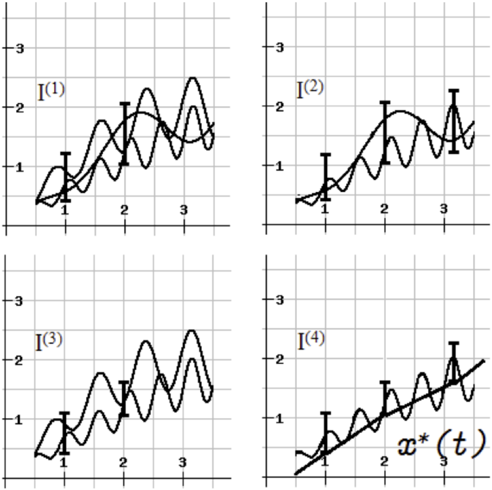
Definition

A tagged-division $\mathcal{E} = \{(\tau, I[N]) : \tau \in \overline{\mathbb{R}}^B, N \in \mathcal{F}(B)\}$ of $\mathbb{R}^B$ is γ_B-fine, if each pair $(\tau, I[N])$ is γ_B-fine. We denote $\mathcal{E}$ by $\mathcal{E}_{\gamma B}$.

Lemma (P. Muldowney)

Given an infinite set B and a gauge γ_B, there exists a γ_B-fine tagged-division of $\mathbb{R}^B$.

Given a γ_B-fine tagged-division $\mathcal{E} = \{(\tau, I[N])\}$, the set of restricted dimensions N for each cell $I[N]$ in $\mathcal{E}_{\gamma B}$ includes some minimal set of dimensions, given by a value of L_B. Thus, if we make $\delta_B(\tau)$ successively smaller and $L_B(\tau)$ successively larger, then the cells in the corresponding γ_B-fine tagged-division will "*shrink*".



Shrinking of cells in $\mathbb{R}^B$.

Let $\mathbf{I}(\mathbb{R}^B) = \{I[N]; N \in \mathcal{F}(B)\}$ be the collection of all cells in $\mathbb{R}^B$ and let $(\tau, I) \in \mathbb{R}^B \times \mathbf{I}(\mathbb{R}^B)$.

Definition

A function $U(\tau, I)$ is generalized Feynman integrable over $\mathbb{R}^B$, with integral $\alpha = \int_{\mathbb{R}^B} U(\tau, I)$ (or simply $\alpha = \int_{\mathbb{R}^B} U$), if $\forall\ \epsilon > 0$, there is a gauge γ_B s.t., $\forall\ \gamma_B$-fine tagged-division $\mathcal{E}_{\gamma B}$ of $\mathbb{R}^B$, we have

$$\left| \sum_{(\tau, I[N]) \in \mathcal{E}_{\gamma B}} U(\tau, I[N]) - \alpha \right| < \epsilon.$$

Example:

Let $C = C((a,b],\mathbb{R})$ and define $f(\tau)$, for $\tau \in \mathbb{R}^{(a,b]}$, by:

$$f(\tau) = \begin{cases} \exp\left(\int_a^b \tau(t)dt\right), & \tau \in C, \\ 0, & \tau \in \mathbb{R}^{(a,b]} \setminus C. \end{cases}$$

Let μ be defined on cells I of $\mathbb{R}^{(a,b]}$, with $I = I[N] = I_1 \times \ldots \times I_n \times \mathbb{R}^{(a,b]\setminus N}$. Take $|I_j| := v_j - u_j$ when $I_j = [u_j, v_j)$, and $|I_j| := 0$ otherwise. Then

$$\mu(I) = |I[N]| := \prod_{j=1}^{n} |I_j|.$$

We could ask if $U(\tau, I) = f(\tau)\mu(I)$ is integrable over $\mathbb{R}^{(a,b]}$.

Theorem - P. Muldowney

Every distribution function is generalized Feynman integrable.

Example: Consider

$$|I[N]| := \begin{cases} \prod_{j=1}^{n}(v_j - u_j), & I_j = [u_j, v_j),\ j = 1, 2, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}$$

The Fresnel infinite-dimensional integrand, given by

$$G(I[N]) := \left(\sqrt{\frac{-i}{2\pi}}\right)^n \prod_{j=1}^{n} \int_{I_j} e^{\frac{i}{2}\tau_j^2} d\tau_j,$$

is generalized Feynman integrable over $\mathbb{R}^B$ and $\int_{\mathbb{R}^B} G = 1$.

Thanks for your attention!