Periodic Solutions to Indefinite Singular Equations

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- **Solution:** $u \in AC^1\big([0, T]; \mathbb{R}_+\big)$

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Observation

If there exists a solution to (1) then

$$h_+(t) \not\equiv 0, \qquad h_-(t) \not\equiv 0, \qquad \int_0^T h(s)ds < 0$$

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$$\int_0^T h(s)ds = \int_0^T u''(s)u^\lambda(s)ds = -\lambda \int_0^T u'^2(s)u^{\lambda-1}(s)ds < 0$$

Motivation

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Trapping mechanism for a neutral atom in the vicinity of a charged wire

$$r'' = \frac{L^2}{M^2 r^3} - \frac{4\alpha q^2(t)}{M r^3}, \qquad q(t) = Q\cos(\omega t/2)$$

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J. Lei, M. Zhang, *Twist property of periodic motion of an atom near a charged wire*, Letters in Math. Physics **60** (2002), 9-17

History

$$u'' = \frac{h(t)}{u^\lambda}; \qquad u(0) = u(T), \quad u'(0) = u'(T) \tag{1}$$

$$h(t) = \begin{cases} h_+ & \text{for } \ t \in [0, a), \\ -h_- & \text{for } \ t \in [a, T], \end{cases} \qquad \lambda = 3$$

J. L. Bravo, P. J. Torres, *Periodic solutions of a singular equation with indefinite weight*, Adv. Nonlinear Stud. **10** (2010), 927-938

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- h is piecewise constant, $\lambda \geq 1$
- $h \in C^1\big([0, T]; \mathbb{R}\big)$ has only simple zeroes, $\lambda \geq 2$

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$$u'' = \frac{g_1(t)}{u^\lambda} - \frac{g_2(t)}{u^\mu} + f(t); \qquad u(0) = u(T), \quad u'(0) = u'(T)$$

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- $g_1(t) \geq 0,\ g_2(t) \geq 0$ for $t \in [0, T]$

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R. Hakl, P. J. Torres, *On periodic solutions of second-order differential equations with attractive-repulsive singularities*, J. Differ. Eq. **248** (2010), 111-126

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- Conditions in the plane $(\lambda - \mu, \overline{f})$ except $(0,0)$

Main Result

Theorem

Let $\lambda \geq 1$, $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ pairwise disjoint,

$$h(t) \geq 0 \qquad \text{for a. e. } t \in \bigcup_{k=1}^{n} [a_k, b_k],$$

$$h(t) \leq 0 \qquad \text{for a. e. } t \in [0, T] \setminus \bigcup_{k=1}^{n} [a_k, b_k].$$

Let $c_k \in (a_k, b_k)$ $(k = 1, \dots, n)$ be such that

$$\int_t^{b_k} \frac{h(s)}{(s-t)^\lambda} ds = +\infty \qquad \text{for } t \in [a_k, c_k] \quad (k = 1, \dots, n),$$

$$\int_{a_k}^{t} \frac{h(s)}{(t-s)^\lambda} ds = +\infty \qquad \text{for } t \in [c_k, b_k] \quad (k = 1, \dots, n).$$

Then (1) has a solution if and only if $\overline{h} < 0$.

Corollary

Let $\lambda \geq 1$ and let there exist pairwise disjoint intervals $[a_k, b_k] \subset [0, T]$ $(k = 1, \dots, n)$ and $\alpha > 0$ such that

$$h(t) \geq \alpha \left[(b_k - t)(t - a_k)\right]^{\lambda - 1} \qquad \text{for a. e. } \ t \in \bigcup_{k=1}^{n} [a_k, b_k],$$

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Sketch of the Proof

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Theorem

Let $\overline{h} < 0$, $0 < \delta < \varepsilon < 1$,

$$V = \big\{u \in C_T\big([0,T];\mathbb{R}\big) : \varepsilon < u(c_k)\ (k = 1,\dots,n)\big\}.$$

If, for each $\mu \in [0,1]$, there is no solution to (3) on ∂V, then (2) has at least one solution in V.

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- $u \in \overline{V} \implies u \in B(0; R_\delta)$

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Consequently,

$$u'' = h(t) g_\delta(u); \qquad u(0) = u(T), \quad u'(0) = u'(T)$$

has a solution u such that

$$u(c_k) > \varepsilon \qquad \text{for } k = 1, \dots, n.$$

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$$\int_t^{b_k} \frac{h(s)}{(s-t)^\lambda} ds = +\infty \quad \Longrightarrow \quad u(t) \geq \delta \qquad \text{for } \; t \in [a_k, c_k]$$

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Consequently,

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has a positive solution.