Periodic boundary value problem for second-order ordinary differential equations: resonance like case

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$$\boxed{u'' = f(t,u); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

$f\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ – Carathéodory function

Lower and upper functions

$$\sigma_1''(t) \geq f(t, \sigma_1(t)) \quad \text{pro} \quad t \in [0,\omega], \quad \sigma_1(0) = \sigma_1(\omega), \quad \sigma_1'(0) \geq \sigma_1'(\omega)$$
$$\sigma_2''(t) \leq f(t, \sigma_2(t)) \quad \text{pro} \quad t \in [0,\omega], \quad \sigma_2(0) = \sigma_2(\omega), \quad \sigma_2'(0) \leq \sigma_2'(\omega)$$

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$$\begin{array}{llll} \sigma_1''(t) \geq f(t,\sigma_1(t)) \quad \text{pro} & t \in [0,\omega], & \sigma_1(0) = \sigma_1(\omega), & \sigma_1'(0) \geq \sigma_1'(\omega) \\ \sigma_2''(t) \leq f(t,\sigma_2(t)) \quad \text{pro} & t \in [0,\omega], & \sigma_2(0) = \sigma_2(\omega), & \sigma_2'(0) \leq \sigma_2'(\omega) \end{array}$$

Definition.

- $p \in \mathcal{D}(\omega)$: $u'' = p(t)u$; $u(a) = 0$, $u(b) = 0$ only $\equiv 0$ solution $\forall\, b - a < \omega$
- $\mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)$ and $\overline{\mathcal{D}(\omega)} = \mathcal{D}(\omega)$
- $p \in \mathcal{V}^-(\omega)$ (resp. $p \in \mathcal{V}^+(\omega)$): $\left.\begin{array}{c} u''(t) \geq p(t)u(t) \\ u(0) = u(\omega), u'(0) = u'(\omega) \end{array}\right\} \Rightarrow \begin{array}{c} u(t) \leq 0 \\ (\text{resp. } u(t) \geq 0) \end{array}$
- $p \in \mathcal{V}_0(\omega)$: $\begin{array}{c} u'' = p(t)u \\ u(0) = u(\omega),\ u'(0) = u'(\omega) \end{array}$ possesses $u > 0$ solution
- $p(t) \overset{\text{def}}{=} p = Const.$ $\quad \mathcal{V}^-(\omega) =]0,+\infty[\,,\ \mathcal{V}_0(\omega) = \{0\},\ \mathcal{V}^+(\omega) = [-\frac{\pi^2}{\omega^2}\,, 0[$
- $\operatorname{Int}\mathcal{D}(\omega) = \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \operatorname{Int}\mathcal{V}^+(\omega)$
  $p \in \operatorname{Int}\mathcal{D}(\omega)$: $u'' = p(t)u$; $u(a) = 0$, $u(b) = 0$ only $\equiv 0$ solution $\forall\, b - a \leq \omega$

$$\boxed{u'' = f(t,u); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

Theorem 1. *Let* $\exists \sigma_1, \sigma_2$ *(arbitrarily ordered),*

$$f(t,x)\operatorname{sgn} x \geq p(t)|x| - q(t,|x|) \quad \textit{pro} \quad t \in [0,\omega],\ x \in \mathbb{R},$$

$$p \in \operatorname{Int}\mathcal{D}(\omega),$$

q is a sublinear function (i. e., $\lim_{r\to+\infty} \frac{1}{r}\int_0^\omega |q(s,r)|\mathrm{d}s = 0$*). Then the problem* (1) *has a solution u and* $\exists t_u$*:*

$$\alpha(t_u) \leq u(t_u) \leq \beta(t_u),$$

where $\alpha(t) \stackrel{\text{def}}{=} \min\{\sigma_1(t), \sigma_2(t)\}$, $\beta(t) \stackrel{\text{def}}{=} \max\{\sigma_1(t), \sigma_2(t)\}$

Remark. C. De Coster, M. Tarallo, *Foliations, associated reductions and lower and upper solutions*, Calc. Var. Partial Differential Equations **15** (2002), No. 1, 25–44.

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda}\operatorname{sgn} u + q(t,u); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{2}$$

- $p \in \mathcal{V}_0(\omega)$
- $h \not\equiv 0,\ \lambda \in]0,1[\,,\ q$ – Carathéodory, and $\lim_{r\to+\infty} \frac{1}{r^{\lambda}} \int_0^{\omega} |q(s,r)| \mathrm{d}s = 0$

resonance like case

$$u_0 :\ u'' = p(t)u;\ u(0) = u(\omega),\ u'(0) = u'(\omega),\ u_0 > 0,\ \text{and } \|u_0\| = 1$$

Theorem 2. *Let $\int_0^{\omega} h(s)u_0^{\lambda+1}(s)\mathrm{d}s \neq 0$. Then the problem (2) is solvable.*

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + q(t,u); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \qquad (2)$$

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Theorem 2. *Let* $\int_0^\omega h(s)u_0^{\lambda+1}(s)\mathrm{d}s \neq 0$. *Then the problem* (2) *is solvable.*

Theorem 3. *Let* $\lambda \in [\frac{1}{2}, 1[\,,\ \int_0^\omega h(s)u_0^{\lambda+1}(s)\mathrm{d}s = 0$, *and*

$$\limsup_{r\to+\infty} \frac{1}{r^{2\lambda-1}} \int_0^\omega u_0(s)|q(s,r)|\mathrm{d}s < \eta^*,$$

$$\eta^* \stackrel{\text{def}}{=} \lambda \min\left\{ \int_0^\omega \frac{1}{u_0^2(s)} \left(\int_x^s h(\xi)u_0^{\lambda+1}(\xi)\mathrm{d}\xi \right)^2 \mathrm{d}s : x \in [0,\omega] \right\}.$$

Then the problem (2) *is solvable.*

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{3}$$

- $p \in \mathcal{V}_0(\omega)$, $\lambda \in \,]0,1[\,$, $h \not\equiv 0$

Corollary 4. *Let either* $\lambda \in \,]\frac{1}{2}, 1[$ *or*

$$\lambda \in \,]0, \tfrac{1}{2}] \quad \textit{and} \quad \int_0^\omega h(s) u_0^{\lambda+1}(s)\,\mathrm{d}s \neq 0$$

Then the problem (3) *is solvable (for any* q*).*

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Corollary 4. *Let either* $\lambda \in]\frac{1}{2}, 1[$ *or*

$$\lambda \in]0, \tfrac{1}{2}] \quad \textit{and} \quad \int_0^{\omega} h(s) u_0^{\lambda+1}(s)\mathrm{d}s \neq 0$$

Then the problem (3) *is solvable (for any* q*).*

What happens if

$$\lambda \in]0, \tfrac{1}{2}] \quad \text{and} \quad \int_0^{\omega} h(s) u_0^{\lambda+1}(s)\mathrm{d}s = 0\,?$$

- $\lambda = \frac{1}{2}$

$$\boxed{u'' = p(t)u + h(t)\sqrt{|u|}\,\mathrm{sgn}\,u + q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{4}$$

- $p \in \mathcal{V}_0(\omega)$, $\int_0^\omega h(s)u_0^{\frac{3}{2}}(s)\mathrm{d}s = 0$

Corollary 5. *Let*

$$\int_0^\omega u_0(s)[q(s)]_+\mathrm{d}s < c^*, \qquad \int_0^\omega u_0(s)[q(s)]_-\mathrm{d}s < c^*,$$

where

$$c^* \stackrel{\mathrm{def}}{=} \frac{1}{2} \min\left\{\int_0^\omega \frac{1}{u_0^2(s)} \left(\int_x^s h(\xi)u_0^{\frac{3}{2}}(\xi)\mathrm{d}\xi\right)^2 \mathrm{d}s : x \in [0,\omega]\right\}.$$

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Then the problem (4) *is solvable.*

q is “small enough”

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{5}$$

- $p \in \mathcal{V}_0(\omega)$
- $\lambda \in]0, \frac{1}{2}]$, $\int_0^\omega h(s)u_0^{\lambda+1}(s)\mathrm{d}s = 0$

Theorem 6. *Problem* (5) *is solvable* $\forall\mu \in \mathbb{R}$ *if and only if*

$$\int_0^\omega u_0(s)q(s)\mathrm{d}s = 0.$$

Remark. Let $\int_0^\omega u_0(s)q(s)\mathrm{d}s \neq 0$. Then $\exists 0 < \mu^*(< +\infty)$ such that the problem (5) has no solution $\forall|\mu| > \mu^*$.

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \qquad (5)$$

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What happens for "small enough" μ ?

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu q(t); \quad u(0) = u(\omega), \ u'(0) = u'(\omega)} \tag{5}$$

- $p \in \mathcal{V}_0(\omega)$
- $\lambda \in]0, \frac{1}{2}]$, $\int_0^\omega h(s) u_0^{\lambda+1}(s) \mathrm{d}s = 0$

Theorem 7. $\forall q \ \exists \mu_q > 0$ *such that the problem* (5) *is solvable provided* $\mu \in] - \mu_q, \mu_q[$.

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu q(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{5}$$

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For $\lambda = \frac{1}{2}$ it is possible to estimate μ_q.

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Theorem 7. *$\forall q\ \exists \mu_q > 0$ such that the problem (5) is solvable provided $\mu \in]-\mu_q, \mu_q[$.*

For $\lambda = \frac{1}{2}$ it is possible to estimate μ_q.

Theorem 8. *Let $q(t) \geq 0$ and $q \not\equiv 0$. Then $\exists 0 < \mu^* (< +\infty)$ such that the problem (5) is solvable for $\mu \in]-\mu^*, \mu^*[$ and has no solution for $|\mu| > \mu^*$.*