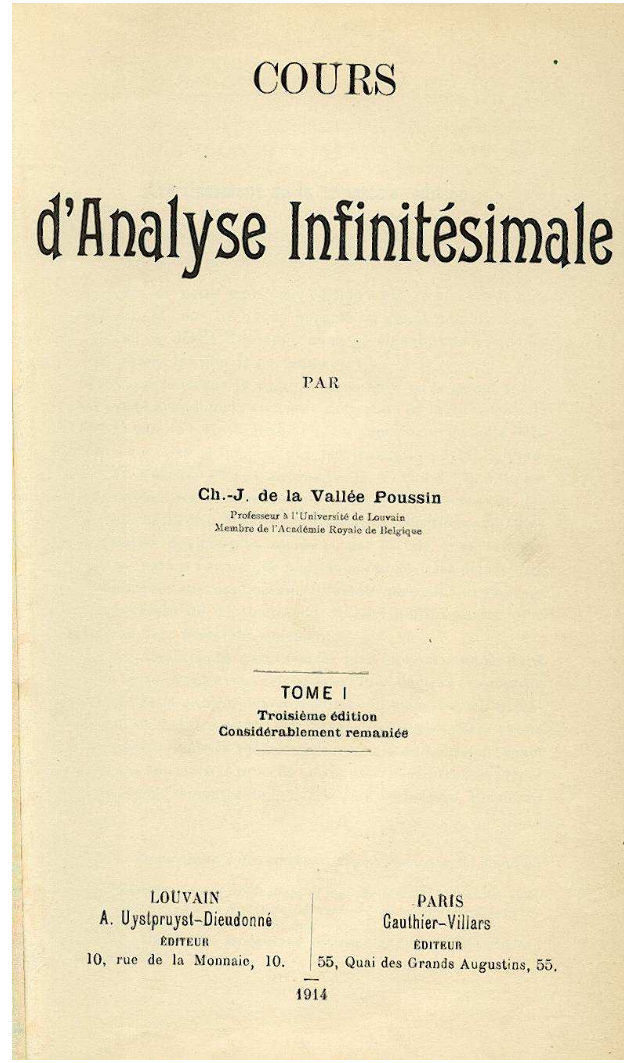
Four steps to Kurzweil's integral

Jean Mawhin

Université Catholique de Louvain

Charles-Jean de La Vallée Poussin



COURS

d'Analyse Infinitésimale

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1914

(1866-1962) 1914

Cours d'analyse infinitésimale

- 1909 : *Cours d'analyse infinitésimale*, 2nd ed., vol. 1 :
  - introduces LEBESGUE's integral

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- 1914 : *Cours d'analyse infinitésimale*, 3rd ed., vol. 1
  - **aim** : simpler proofs of the relations between integral and primitive for **Lebesgue integral**
  - *If the derivative function $f(x)$ is not bounded, passing to the primitive function requires more delicate reasonings. The fundamental theorems have been obtained by M. Lebesgue, but we will adopt to obtain them a procedure which is different from his one, that we have introduced in the previous edition of this course and that we will make a little more precise in this one. This procedure consists in constructing, as we will do it,* auxiliary functions whose derivatives satisfy everywhere some inequality conditions

de la Vallée Poussin's theorem

- $\underline{D}g(x) = \liminf_{y \to x} \frac{g(y)-g(x)}{y-x}$, $\overline{D}g(x) = \limsup_{y \to x} \frac{g(y)-g(x)}{y-x}$
- $g : [a,b] \to$ is **absolutely continuous** if $\forall\, \varepsilon > 0, \exists\, \eta > 0, \forall\, a \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \leq a_m < b_m \leq b : \sum_{j=1}^{m} (b_j - a_j) \leq \eta : \sum_{j=1}^{m} |g(b_j) - g(a_j)| \leq \varepsilon$

de la Vallée Poussin's theorem

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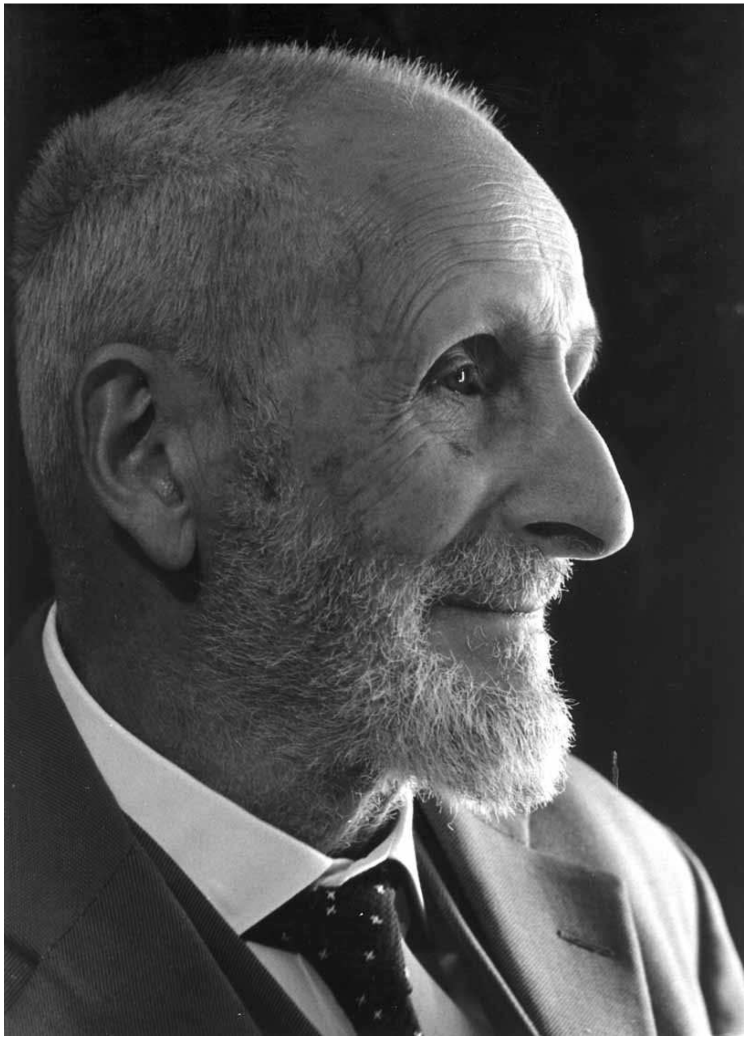
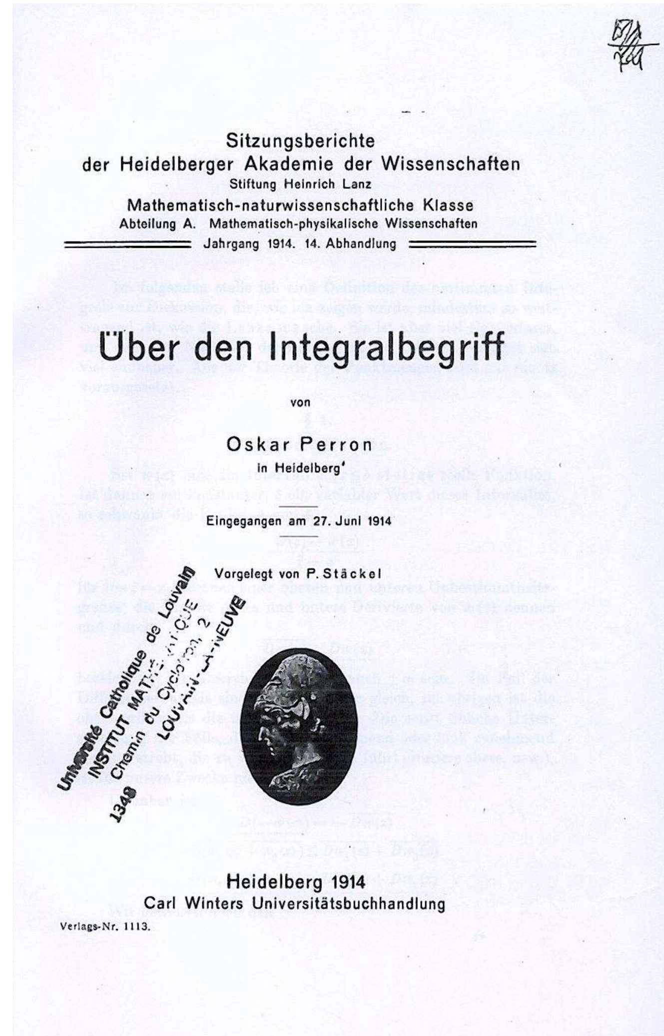
- **thm** : $\forall\, f : [a, b] \to \mathbb{R}$ L-integrable on $[a, b]$, $\forall\, \varepsilon > 0$, $\exists$ absolutely continuous functions $F_+,\ F_- : [a, b] \to \mathbb{R}$ with $F_+(a) = F_-(a) = 0$, such that, $\forall\, x \in [a, b]$,
  1. $\underline{D}F_+(x) \ge f(x)$, $\overline{D}F_-(x) \le f(x)$, when $f(x)$ is finite
  2. $F_-(x) \le (L) \int_a^x f \le F_+(x)$
  3. $F_+(x) - \varepsilon < (L) \int_a^x f < F_-(x) + \varepsilon$
- F_+ (resp. F_-) is called a **major function (fonction majorante)** (resp. **minor function (fonction minorante)**) of f on $[a, b]$

Oskar Perron



(1880-1975)



Sitzungsberichte
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Abteilung A. Mathematisch-physikalische Wissenschaften
Jahrgang 1914. 14. Abhandlung

Über den Integralbegriff

von

Oskar Perron
in Heidelberg

Eingegangen am 27. Juni 1914

Vorgelegt von P. Stäckel

Heidelberg 1914
Carl Winters Universitätsbuchhandlung

Verlags-Nr. 1113.

1914

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  - **aim** : new definition of the **integral of** $f : [a, b] \to \mathbb{R}$ **on** $[a, b]$

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  - **aim** : new definition of the **integral of** $f : [a, b] \to \mathbb{R}$ **on** $[a, b]$
- **tools** : minor and major functions
  - $F_- \in C[a, b] : F_-(a) = 0$ is a **Perron minor function (Unterfunktion)** of f on $[a, b]$ if $\forall\, x \in [a, b] : \overline{D}F_-(x) \leq f(x)$
  - $F_+ \in C[a, b] : F_+(a) = 0$ is a **Perron major function (Oberfunktion)** of f on $[a, b]$ if $\forall\, x \in [a, b] : \underline{D}F_+(x) \geq f(x)$

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- $\mathcal{P}_-(f)$ (resp. $\mathcal{P}_+(f)$) : set of Perron minor functions (resp. Perron major functions) of f on $[a, b]$
  - $\mathcal{P}_-(f)$ and $\mathcal{P}_+(f)$ are nonempty if f is e.g. bounded on $[a, b]$

Lower and upper P-integrals

- **lower P-integral of f on $[a,b]$:**

  $(P)\underline{\int_a^b} f := \sup_{F_- \in \mathcal{P}_-(f)} F_-(b)$ if $\mathcal{P}_-(f) \neq \emptyset$

  $(P)\underline{\int_a^b} f := -\infty$ if $\mathcal{P}_-(f) = \emptyset$

- **upper P-integral of f on $[a,b]$:**

  $(P)\overline{\int_a^b} f := \inf_{F_+ \in \mathcal{P}_+(f)} F_+(b)$ if $\mathcal{P}_+(f) \neq \emptyset$

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Lower and upper P-integrals

- **lower P-integral of f on $[a,b]$** :
  $(P)\underline{\int_a^b} f := \sup_{F_- \in \mathcal{P}_-(f)} F_-(b)$ if $\mathcal{P}_-(f) \neq \emptyset$
  $(P)\underline{\int_a^b} f := -\infty$ if $\mathcal{P}_-(f) = \emptyset$
- **upper P-integral of f on $[a,b]$** :
  $(P)\overline{\int_a^b} f := \inf_{F_+ \in \mathcal{P}_+(f)} F_+(b)$ if $\mathcal{P}_+(f) \neq \emptyset$
  $(P)\overline{\int_a^b} f := +\infty$ if $\mathcal{P}_+(f) = \emptyset$
- **lemma** : $(P)\underline{\int_a^b} f \leq (P)\overline{\int_a^b} f$; proof based upon
- **lemma** : if $g : [a,b] \to$ and if $\exists\, C \in : \forall x \in [a,b] : \overline{D}g(x) \leq C$,
  then $\forall\, y \neq z \in [a,b] : \frac{g(y)-g(z)}{y-z} \leq C$
  and corresponding one for $\underline{D}g$; lemma proved, by contradiction, using Weierstrass thm on achieved bounds

Perron's integral

- $f : [a, b] \to$ is **P-integrable** over $[a, b]$ if $(P)\underline{\int_a^b} f$ and $(P)\overline{\int_a^b} f$ are both finite and equal

Perron's integral

- $f:[a,b] \to$ is **P-integrable** over $[a,b]$ if $(P)\underline{\int_a^b} f$ and $(P)\overline{\int_a^b} f$ are both finite and equal
- the **P-integral** $(P)\int_a^b f$ is their common value
  - f *primitivable on* $[a,b]$ *with primitive* F $\Rightarrow f \in P[a,b]$ *and* $(P)\int_a^b f = F(b) - F(a)$
  - $f \in R[a,b] \;\Rightarrow\; f \in P[a,b]$, *with the same integral*
  - $f \in L[a,b] \;\Rightarrow\; f \in P[a,b]$, *with the same integral* :

    Indeed, although naturally under another formulation, this theorem has already been proved by M. de la Vallée Poussin in his Cours d'analyse infinitésimale. *What follows is, in its principle, a reproduction of his reasoning*

Augustus (Gus) John Ward

The Perron-Stieltjes Integral.

By

A. J. Ward in Cambridge.

Considerable attention has been given to the question of integration with respect to a function; various authors have shown how to define an integral which generalises the Stieltjes integral in the same way as Lebesgue generalised the ordinary integral. The Lebesgue-Stieltjes integral, however, is by its very nature restricted to the case in which the integrating function is of bounded variation, while the ordinary Stieltjes integral has no such limitation. An integral which includes both these (Riemann-Stieltjes and Lebesgue-Stieltjes) is therefore not without interest. Such an integral may be obtained by applying the methods of Perron [1]). It is interesting also on account of its close connection with differentiation.

In the first section of this paper I give definitions of differentiation with respect to a function. It is shown that the theorem, due to Denjoy [2]), which governs the possible values of the derivates of a function, can be extended to the case when the base-function is VBG^* on the set of points considered [3]). It seems likely that the class of functions VBG^* on a given set of points is the most general class for which it is profitable to define derivates, and it is noteworthy that the same class of functions proves to be of importance for the theory of the Perron-Stieltjes integral. The second chapter is devoted to the definition and simple properties of the integral. Finally, I consider the case of a base-function which is VBG^*, and give the analogue of the theorem which states that every function ACG^* in an interval is the Perron integral of its derivative, and conversely [4]).

---

[1]) Other authors have defined a Perron-Stieltjes integral with respect to functions of bounded variation; see R. L. Jeffery, *Trans. American Math. Soc.* 34 (1932), S. 645; J. Ridder, *Math. Zeitschr.* 40 (1935), S. 127. The definitions employed are not applicable to general base-functions. (The present work was carried out independently of the work of Ridder, and in some places conventions are used which are different from his.)

[2]) A. Denjoy, *Journal de Math.* (1915) S. 105—240, especially S. 190—192.

[3]) S. Saks, *Théorie de l'Intégrale* (Warsaw 1933), S. 158 ff. The definitions are reproduced here for completeness. See Nr. 1.

[4]) Saks, loc. cit. S. 198, 216.

at each point of E_n, except possibly the extreme points. By Theorem 1, both $D(f, x; \chi_n)$ and $D(\varphi, x; \chi_n)$ exist and are finite except in a set N_n such that $m\chi_n(N_n) = 0$. The argument now proceeds as in Theorem 2. We observe that at a point where $D(f, x; \chi_n)$ and $D(\varphi, x; \chi_n)$ both exist and $D(\varphi, x; \chi_n) \neq 0$, both $D(f, x; \varphi)$ and $df(x)/d\varphi(x)$ must exist (and be equal).

The corollary follows at once from lemma 4.

2. The Perron-Stieltjes Integral.

The Perron integral is defined by means of major and minor functions, which in turn are usually defined by inequalities relating to derivates [14]). There would be obvious difficulties in extending such a definition to the case of integrals with respect to a general function $\varphi(x)$, which may attain the same value at an infinite set of points. For this reason we define the Perron-Stieltjes integral by means of inequalities concerning the increments $M(x+h) - M(x)$, $\varphi(x+h) - \varphi(x)$ *directly*, and not in terms of the derivates of M with respect to φ. The resulting integral is found to include the ordinary Stieltjes integral, whether $\varphi(x)$ is of bounded variation or not. It also includes the Lebesgue-Stieltjes integral with respect to an increasing function; but it does not include the 'Lebesgue-Stieltjes' integral with respect to a function of bounded variation (defined by Mlle N. Bary and Menchoff), which may exist in an interval (a, b) without existing in a smaller interval (a, x) [15]).

2.1. Given any functions $f(x)$, $\varphi(x)$ we say that $M(x)$ is a major function of $f(x)$ with respect to $\varphi(x)$ if $M(a) = 0$, and for any point x of $\langle a, b\rangle$ there exists $\delta(x) > 0$ such that

(5) $M(\xi) \geqq M(x) + f(x)\{\varphi(\xi) - \varphi(x)\}$ if $0 \leqq \xi - x \leqq \delta(x)$,

(6) $M(\xi) \leqq M(x) + f(x)\{\varphi(\xi) - \varphi(x)\}$ if $0 \geqq \xi - x \geqq -\delta(x)$.

(If x is equal to a or b, we consider only the one inequality which is appropriate.)

The upper Perron-Stieltjes integral of $f(x)$ with respect to $\varphi(x)$ is defined as

$$(PS)\overline{\int_a^b} f(x)\,d\varphi(x) = \underline{\text{bound}}\ M(b) \quad \text{(all major functions)}$$

if finite major functions exist; otherwise we write $+\infty$ for the value of the upper integral.

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[14]) Ridder (loc. cit.) applies the same method in defining his Perron-Stieltjes integral.

[15]) N. Bary and D. Menchoff, Ann. Mat. pura appl. (4) (1928), S. 19—54.

(1911-1984)

1936

The Perron-Stieltjes integral

- 1936 : A.J. WARD, *The Perron-Stieltjes integral*, Math. Z. 41, 578-604
  - $f : [a,b] \to \mathbb{R}, \ \varphi : [a,b] \to \mathbb{R}$
  - aim : to define a **Perron-Stieltjes integral** of f with respect to φ

The Perron-Stieltjes integral

- 1936 : A.J. WARD, *The Perron-Stieltjes integral*, Math. Z. 41, 578-604
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  - aim : to define a **Perron-Stieltjes integral** of f with respect to φ

- *The Perron integral is defined by means of major and minor functions, which in turn are usually defined by inequalities relating to derivates. There would be obvious difficulties in extending such a definition to the case of integrals with respect to a general function* φ*, which may attain the same value at an infinite set of points. For this reason we define the Perron-Stieltjes integral by means of inequalities concerning the increments of the minor and major functions and of* φ *directly, and not in terms of the derivates of the minor and major functions with respect to* φ

Ward's minor and major functions

- $f,\ \varphi : [a,b] \to \mathbb{R},\ \ W_{\pm} : [a,b] \to \mathbb{R},\ W_{-}(a) = W_{+}(a) = 0$
- W_{-} is a **Ward minor function** of f with respect to φ on $[a,b]$ if
$\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x - \delta(x), x + \delta(x)] \cap [a,b]:$
$(y-x)[W_{-}(y) - W_{-}(x)] \le (y-x) f(x)[\varphi(y) - \varphi(x)]$
- W_{+} is a **Ward major function** of f with respect to φ on $[a,b]$ if
$\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x - \delta(x), x + \delta(x)] \cap [a,b]:$
$(y-x)[W_{+}(y) - W_{+}(x)] \ge (y-x) f(x)[\varphi(y) - \varphi(x)]$

Ward's minor and major functions

- $f,\ \varphi : [a,b] \to \mathbb{R},\ \ W_\pm : [a,b] \to \mathbb{R},\ W_-(a) = W_+(a) = 0$
- W_- is a **Ward minor function** of f with respect to φ on $[a,b]$ if
  $\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x-\delta(x), x+\delta(x)] \cap [a,b]$:
  $(y-x)[W_-(y) - W_-(x)] \leq (y-x)f(x)[\varphi(y) - \varphi(x)]$
- W_+ is a **Ward major function** of f with respect to φ on $[a,b]$ if
  $\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x-\delta(x), x+\delta(x)] \cap [a,b]$:
  $(y-x)[W_+(y) - W_+(x)] \geq (y-x)f(x)[\varphi(y) - \varphi(x)]$
- apparition of the concept of **gauge** $\delta : [a,b] \to (0, +\infty)$
  (was hidden in the definition of $\underline{D}, \overline{D}$ in Perron's definition)
- $\mathcal{W}_-(f,\varphi)$ (resp. $\mathcal{W}_+(f,\varphi)$) : set of Ward minor functions (resp. Ward major functions) of f with respect to φ on $[a,b]$

Lower and upper W-integrals

- **W-lower integral** of f with respect to φ on $[a,b]$:

  $(W)\underline{\int_a^b} f\,d\varphi := \sup_{W_- \in \mathcal{W}_-(f,\varphi)} W_-(b)$ if $\mathcal{W}_-(f,\varphi) \neq \emptyset$

  $(W)\underline{\int_a^b} f\,d\varphi := -\infty$ if $\mathcal{W}_-(f,\varphi) = \emptyset$

- **W-upper integral** of f with respect to φ on $[a,b]$:

  $(W)\overline{\int_a^b} f\,d\varphi := \inf_{W_+ \in \mathcal{W}_+(f,\varphi)} W_+(b)$ if $\mathcal{W}_+(f,\varphi) \neq \emptyset$

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- **W-upper integral** of f with respect to φ on $[a,b]$:

  $(W)\overline{\int_a^b} f\,d\varphi := \inf_{W_+\in\mathcal{W}_+(f,\varphi)} W_+(b)$ if $\mathcal{W}_+(f,\varphi)\neq\emptyset$

  $(W)\overline{\int_a^b} f\,d\varphi := +\infty$ if $\mathcal{W}_+(f,\varphi)=\emptyset$

- **lemma** : $(W)\underline{\int_a^b} f\,d\varphi \leq (W)\overline{\int_a^b} f\,d\varphi$

  proof analogous to the case of Perron's integral

W-integral

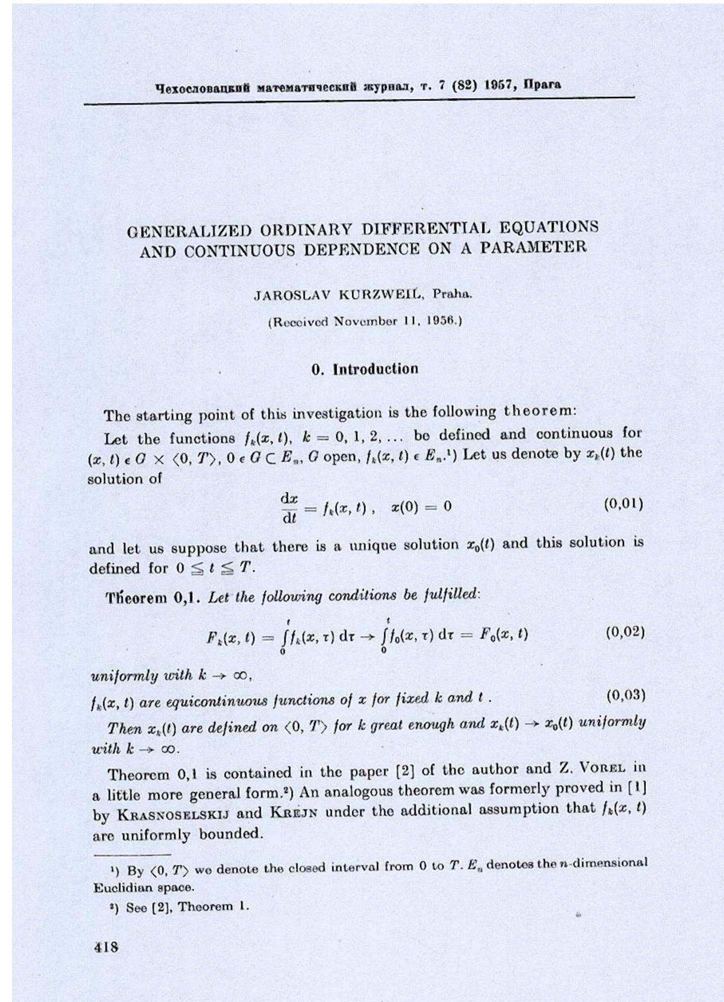
- f is **W-integrable** with respect to φ on $[a, b]$ (or $f\, d\varphi$ is **W-integrable** on $[a, b]$) if $(W)\overline{\int_a^b} f\, d\varphi$ and $(W)\underline{\int_a^b} f\, d\varphi$ are finite and equal

W-integral

- f is **W-integrable** with respect to φ on $[a,b]$
  (or $f\,d\varphi$ is **W-integrable** on $[a,b]$) if
  $(W)\overline{\int_a^b} f\,d\varphi$ and $(W)\underline{\int_a^b} f\,d\varphi$ are finite and equal
- the **W-integral** $(W)\int_a^b f\,d\varphi$ is their common value
  - $f \in P[a,b]$
    $\Leftrightarrow f\,dI \in W[a,b]$, *and* $(W)\int_a^b f\,dI = (P)\int_a^b f$
  - S = Stieltjes, PS = Pollard-Stieltjes, LS = Lebesgue-Stieltjes
  - $f\,d\varphi \in S[a,b]$ *or* $f\,d\varphi \in PS[a,b]$
    $\Rightarrow \; f \in W[a,b]$ *with the same integral*
  - $\varphi \in BV[a,b]$ *and* $f\,d\varphi \in LS[a,b]$
    $\Rightarrow \; f \in W[a,b]$, *with related integrals*

Jaroslav Kurzweil

born in 1926



Чехословацкий математический журнал, т. 7 (82) 1957, Прага

GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS AND CONTINUOUS DEPENDENCE ON A PARAMETER

JAROSLAV KURZWEIL, Praha.

(Received November 11, 1956.)

0. Introduction

The starting point of this investigation is the following theorem:

Let the functions $f_k(x, t)$, $k = 0, 1, 2, \ldots$ be defined and continuous for $(x, t) \in G \times \langle 0, T \rangle$, $0 \in G \subset E_n$, G open, $f_k(x, t) \in E_n$.[1]) Let us denote by $x_k(t)$ the solution of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f_k(x, t)\,, \quad x(0) = 0 \qquad (0{,}01)$$

and let us suppose that there is a unique solution $x_0(t)$ and this solution is defined for $0 \leqq t \leqq T$.

Theorem 0,1. *Let the following conditions be fulfilled:*

$$F_k(x, t) = \int_0^t f_k(x, \tau)\,\mathrm{d}\tau \to \int_0^t f_0(x, \tau)\,\mathrm{d}\tau = F_0(x, t) \qquad (0{,}02)$$

uniformly with $k \to \infty$,

$f_k(x, t)$ *are equicontinuous functions of* x *for fixed* k *and* t. (0,03)

Then $x_k(t)$ *are defined on* $\langle 0, T \rangle$ *for* k *great enough and* $x_k(t) \to x_0(t)$ *uniformly with* $k \to \infty$.

Theorem 0,1 is contained in the paper [2] of the author and Z. Vorel in a little more general form.[2]) An analogous theorem was formerly proved in [1] by Krasnoselskij and Krejn under the additional assumption that $f_k(x, t)$ are uniformly bounded.

[1]) By $\langle 0, T \rangle$ we denote the closed interval from 0 to T. E_n denotes the n-dimensional Euclidian space.

[2]) See [2], Theorem 1.

1957

Generalizing averaging method

- 1957 : JAROSLAV KURZWEIL, *Generalized ODE's and continuous dependence on a parameter*, Czech. Math. J. 7 (82), 418-449
  - **aim** : generalization of averaging method in ODE's
    $u'(x) = f(u(x), x)$
    with conditions on
    $F(u, x) := \int_a^x f(u, s)\, ds$
  - for y near x :
    $u(y) - u(x) = \int_x^y f(u(s), s)\, ds \simeq \int_x^y f(u(x), s)\, ds$
    $= F(u(x), y) - F(u(x), x)$

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    $u(y) - u(x) = \int_x^y f(u(s), s)\, ds \simeq \int_x^y f(u(x), s)\, ds$
    $= F(u(x), y) - F(u(x), x)$
- suggests to associate an integral to functions of two variables
  - $U : [a, b] \times [a, b] \to \mathbb{R},\ (x, y) \mapsto U(x, y)$
  - P-integral : $U(x, y) = f(x)y$
    W-integral : $U(x, y) = f(x)\varphi(y)$

Kurzweil minor and major functions

- $U : [a,b] \times [a,b] \to \mathbb{R}$
- $V_- : [a,b] \to \mathbb{R}$ is a **Kurzweil minor function** of U on $[a,b]$ if
  $\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x - \delta(x), x + \delta(x)] \cap [a,b] :$
  $(y-x)[V_-(y) - V_-(x)] \leq (y-x)[U(x,y) - U(x,x)]$
- $V_+ : [a,b] \to \mathbb{R}$ is a **Kurzweil major function** of U on $[a,b]$ if
  $\forall\, x \in [a,b],\ \exists\, \delta(x) > 0,\ \forall\, y \in [x - \delta(x), x + \delta(x)] \cap [a,b] :$
  $(y-x)[V_+(y) - V_+(x)] \geq (y-x)[U(x,y) - U(x,x)]$
- $\mathcal{K}_-(U)$: set of Kurzweil minor functions of U on $[a,b]$
  $\mathcal{K}_+(U)$: set of Kurzweil major functions of U on $[a,b]$

Lower and upper KW-integrals

- **lower KW-integral** of U on $[a,b]$:

  $(KW)\underline{\int_a^b} DU := \sup_{V_- \in \mathcal{K}_-(U)} V_-(b)$ if $\mathcal{K}_-(U) \neq \emptyset$

  $(KW)\underline{\int_a^b} DU := -\infty$ if $\mathcal{K}_-(U) = \emptyset$

- **upper KW-integral** of U on $[a,b]$:

  $(KW)\overline{\int_a^b} DU := \inf_{V_+ \in \mathcal{K}_+(U)} V_+(b)$ if $\mathcal{K}_+(U) \neq \emptyset$

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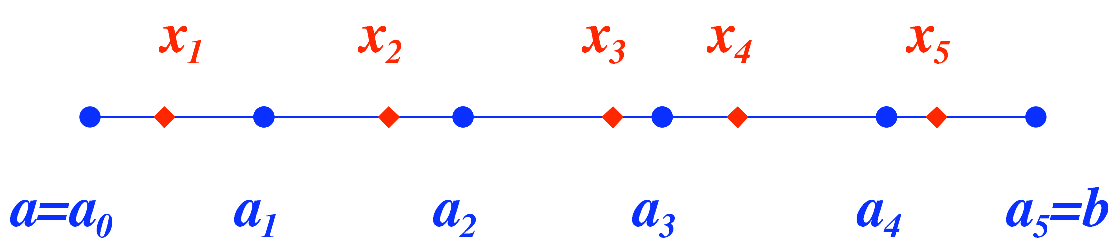
- **lemma** : *if $\mathcal{K}_-(U)$ and $\mathcal{K}_+(U)$ are non empty, then*

  $(KW)\underline{\int_a^b} DU \leq (KW)\overline{\int_a^b} DU$

- with respect to PERRON and WARD, KURZWEIL introduced a **new technique** to prove this lemma

Subdivision subordinate to a gauge

- let $\delta : [a,b] \to (0,+\infty)$ be a gauge on $[a,b]$
- $\mathcal{S} = \{(a_0, a_1, \ldots, a_{m-1}, a_m;\ x_1, x_2, \ldots, x_m) :$
  $a = a_0 < a_1 < \ldots < a_m = b,\ a_{j-1} \le x_j \le a_j\ (1 \le j \le m)\}$
  is a (Riemann) **subdivision of** $[a,b]$ **subordinate to** δ if
  $\forall\, j = 1, 2, \ldots, m : [a_{j-1}, a_j] \subset [x_j - \delta(x_j), x_j + \delta(x_j)]$



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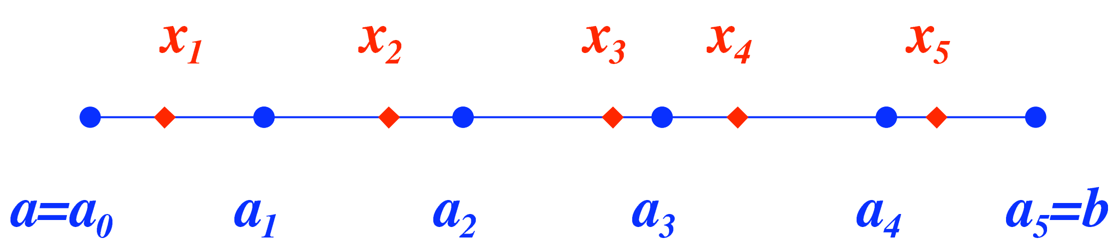

- KURZWEIL proves (with Borel-Lebesgue's lemma) what is called **Cousin's lemma** : *For each gauge δ on $[a,b]$, there exists a subdivision $\mathcal{S}$ of $[a,b]$ subordinate to δ*

Kurzweil's proof of $\underline{\int_a^b} DU \le \overline{\int_a^b} DU$

- let $V_- \in \mathcal{K}_-(U), V_+ \in \mathcal{K}_+(U)$, gauges $\delta_-,\ \delta_+$ on $[a,b]$:
  $(y-x)[V_\pm(y) - V_\pm(x)] \overset{\geq}{\underset{\leq}{}} (y-x)[U(x,y) - U(x,x)]$
  $\forall\, y \in [x - \delta_\pm(x), x + \delta_\pm(x)]$
- take $\delta = \min(\delta_-, \delta_+)$ and $\mathcal{S}$ subordinate to $\delta : \forall\, 1 \le j \le m$
  - $V_-(a_j) - V_-(x_j) \le U(x_j, a_j) - U(x_j, x_j)$
    $V_-(x_j) - V_-(a_{j-1}) \le U(x_j, x_j) - U(x_j, a_{j-1})$
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- summing : $V_-(b) \le V_+(b)$, $(KW)\underline{\int_a^b} DU \ge (KW)\overline{\int_a^b} DU$

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- hidden generalized Riemann sums
  $\sum_{j=1}^m [U(x_j, a_j) - U(x_j, a_{j-1})]$ appear in this proof !

KW-integral and KS-integral

- $U : [a,b] \times [a,b] \to \mathbb{R}$ is **K-integrable** on $[a,b]$ if $(KW)\overline{\int_a^b} DU$ and $(KW)\underline{\int_a^b} DU$ are finite and equal
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- U is **KS-integrable** on $[a,b]$ if $\exists\, J \in \mathbb{R}$: $\forall\, \varepsilon > 0, \exists\, \delta : [a,b] \to (0,\infty), \forall\, \mathcal{S}$ subordinate to δ : $|\sum_{j=1}^{m}[U(x_j,a_j) - U(x_j,a_{j-1})] - J| \leq \varepsilon$
  - $\sum_{j=1}^{m}[U(x_j,a_j) - U(x_j,a_{j-1})]$: **generalized Riemann sum** for U and subdivision $\mathcal{S}$
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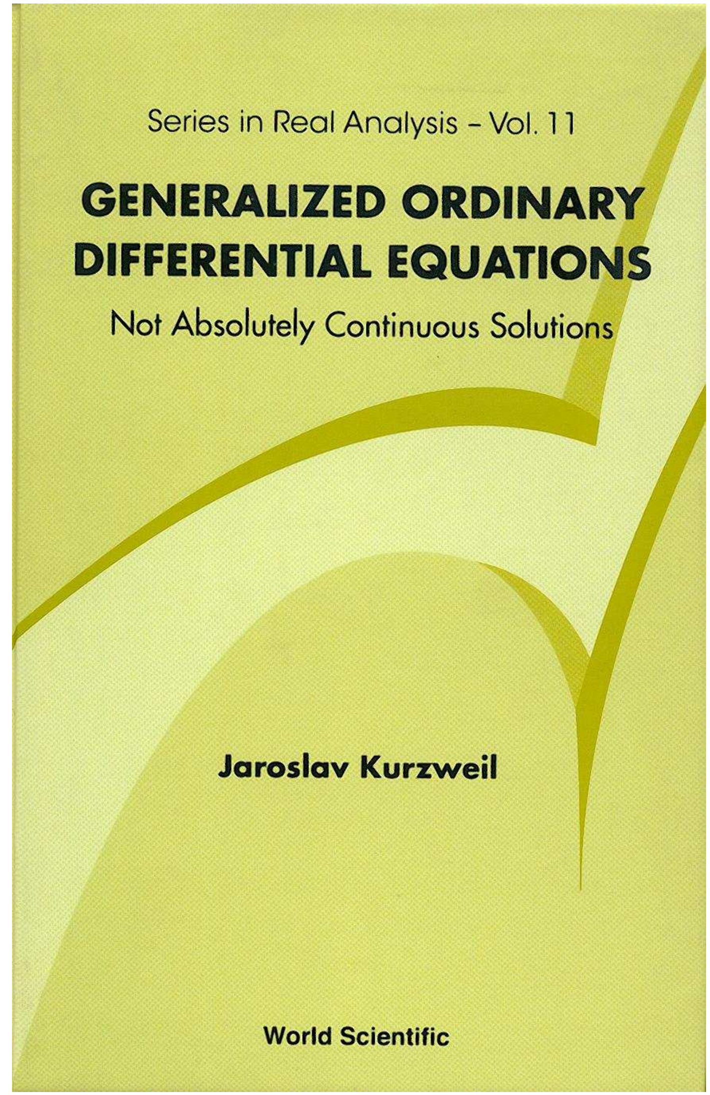
- a definition of **Riemann-type** for the generalized Perron integral !

A most fruitful integral

- the KS-integral is nowadays a widely used tool known under the names of **generalized Perron**, **S-**, **sum-**, **Riemann-complete**, **Kurzweil**, **Kurzweil-Stieltjes**, **Henstock**, **Kurzweil-Henstock**, **Henstock-Kurzweil**, **generalized Riemann**, **Riemann-type**, **gauge integral**,...
- it has inspired many variants and generalizations
- it has completely changed the picture of real analysis in the second half of XXth century
- it has provided striking applications in differential and integral equations, harmonic analysis, probability theory and quantum mechanics
- it has renovated and refreshed the teaching of integration
- it has been the subject of more than 50 monographs

Two books to read

1980



2012

Kurzweil's integral has surpassed
de La Vallée Poussin and Perron's ones.

I wish to Jaroslav also to surpass
La Vallée Poussin and Perron in longevity !

Happy 90th birthday anniversary
on behalve of all your Belgian friends !