Neutral functional dynamic equations on time scales

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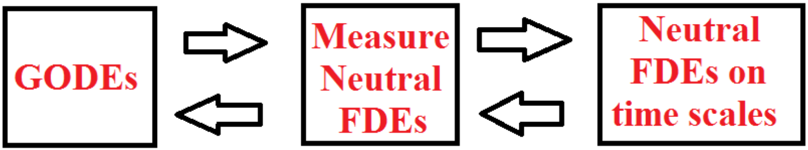
The integral form of the neutral FDE on time scales is given by

$$
\begin{aligned}
x(t) = {} & x(t_0) + \int_{t_0}^{t} f(x_s^*, s)\Delta s + \int_{-r}^{0} \Delta_\theta[\eta(t,\theta)]x^*(t+\theta) \\
& - \int_{-r}^{0} \Delta_\theta[\eta(t_0,\theta)]\varphi^*(t_0+\theta) \\
x(t) = {} & \varphi(t), \;\; t \in [t_0 - r, t_0]_{\mathbb{T}}.
\end{aligned}
$$

Our goal: Establish a correspondence between the solutions of the following equations:



and to obtain the results about:

- Existence and uniqueness of solutions
- Continuous dependence on parameters

Generalized ODEs

Definition

A **tagged division** of $[a,b] \subset \mathbb{R}$ is a finite collection of point-interval pairs $(\tau_i, [s_{i-1}, s_i])$, with

$$a = s_0 \leq s_1 \leq \ldots \leq s_k = b \quad \text{and} \quad \tau_i \in [s_{i-1}, s_i],$$

for $i = 1, 2, \ldots, |D|$.

Definition

Given a function $\delta : [a,b] \to (0, +\infty)$ (called **gauge** of $[a,b]$), a tagged division $D = (\tau_i, [s_{i-1}, s_i])$ is **δ-fine**, whenever

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)),$$

for $i = 1, 2, \ldots, |D|$.

Definition

A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is **Kurzweil integrable** over $[a, b]$, if $\exists!\ I \in X$ such that $\forall\ \varepsilon > 0$, $\exists$ a gauge δ of $[a, b]$ such that $\forall$ δ-fine tagged division $d = (\tau_i, [s_{i-1}, s_i])$ of $[a, b]$,

$$\left\| \sum_i [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon.$$

In this case, $I = \int_a^b DU(\tau, t)$.

Let X be a Banach space, $O \subset X$ be open, $[\alpha, \beta] \subset [a, +\infty)$ and $\Omega = O \times [\alpha, \beta]$.

Definition

A function $x : [\alpha, \beta] \to X$ is **a solution** on $[\alpha, \beta]$ of the GODE

$$\frac{dx}{d\tau} = DF(x, t),$$

whenever $(x(t), t) \in \Omega \; \forall \; t \in [\alpha, \beta]$ and

$$x(v) = x(\gamma) + \int_{\gamma}^{v} DF(x(\tau), t), \qquad \gamma, v \in [\alpha, \beta].$$

Measure neutral FDEs as GODEs

A **measure NFDE** has the following integral form

$$
\begin{aligned}
y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\, dg(s) \\
&\quad + \int_{-r}^{0} d_\theta[\eta(t,\theta)]y(t+\theta) - \int_{-r}^{0} d_\theta[\eta(t_0,\theta)]y(t_0+\theta)
\end{aligned}
$$

Hypotheses:

- $O \subset G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ open;
- $P = \{y_t;\ y \in O,\ t \in [t_0, t_0 + \sigma]\} \subset G([-r, 0], \mathbb{R}^n)$;
- $f : P \times [t_0, t_0 + \sigma] \to \mathbb{R}^n$;
- $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ is nondecreasing.

(H1) $\forall\, y \in O$, $\exists$ the Kurzweil-Henstock-Stieltjes integral

$$\int_{t_0}^{t_0+\sigma} f(y_t, t)\, \mathrm{d}g(t).$$

(H2) $\exists\, M : [t_0, t_0+\sigma] \to \mathbb{R}$ Lebesgue-Stieltjes integrable with respect to g s.t.

$$\left| \int_{t_0}^{t} f(y, s) \mathrm{d}g(s) \right| \leq \int_{t_0}^{t} M(s) \mathrm{d}g(s),$$

$\forall\, y \in P$, $\forall\, t \in [t_0, t_0+\sigma]$.

(H3) $\exists\, L : [t_0, t_0+\sigma] \to \mathbb{R}$ Lebesgue-Stieltjes integrable with respect to g s.t.

$$\left| \int_{t_0}^{t} [f(y, s) - f(z, s)] \mathrm{d}g(s) \right| \leq \int_{t_0}^{t} L(s) \|y - z\|_\infty \mathrm{d}g(s)$$

$\forall\, y, z \in P$, $\forall\, t \in [t_0, t_0+\sigma]$.

For the normalized function $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n \times n}$, we assume:

(H4) $\eta(t,\cdot)$ is left continuous on $(-r,0)$, BV on $[-r,0]$ and

$$\mathrm{Var}_{[s,0]}\eta(t,\cdot) \to 0, \text{ as } s \to 0.$$

(H5) $\exists\, C : [t_0, t_0+\sigma] \to \mathbb{R}$ Lebesgue integrable s.t. $\forall$ $s_1, s_2 \in [t_0, t_0+\sigma]$ and $\forall\, z \in O$,

$$\left| \int_{-r}^{0} d_\theta \eta(s_2,\theta) z(s_2+\theta) - \int_{-r}^{0} d_\theta \eta(s_1,\theta) z(s_1+\theta) \right|$$

$$\leq \int_{s_1}^{s_2} C(s) \int_{-r}^{0} d_\theta \eta(s,\theta) \|z(s+\theta)\| ds,$$

The **measure NFDE** has the integral form

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s) \\ &\quad + \int_{-r}^{0} d_\theta[\eta(t,\theta)]y(t+\theta) - \int_{-r}^{0} d_\theta[\eta(t_0,\theta)]y(t_0+\theta), \end{aligned}$$

and a **regulated** solution $y : [t_0 - r, t_0 + \sigma] \to \mathbb{R}^n$, can be regarded as a **GODE** of the form

$$\frac{dx}{d\tau} = DG(x,t),$$

where $x : [t_0, t_0 + \sigma] \to O \subset \mathbf{G}([\mathbf{t_0} - \mathbf{r}, \mathbf{t_0} + \sigma], \mathbb{R}^{\mathbf{n}})$.

The function $G : O \times [t_0, t_0 + \sigma] \to X$ on the right-hand side of $\frac{dx}{d\tau} = DG(x, t)$ is defined by

$$G(y, t)(\vartheta) = F(y, t)(\vartheta) + J(y, t)(\vartheta),$$

where $\forall\ y \in O$ and $\forall\ t \in [t_0, t_0+\sigma]$

$$F(y,t)(\vartheta) = \begin{cases} 0,\ t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(y_s, s)\, dg(s),\ t_0 \leq \vartheta \leq t \leq t_0+\sigma, \\ \displaystyle\int_{t_0}^{t} f(y_s, s)\, dg(s),\ t \leq \vartheta \leq t_0+\sigma \end{cases}$$

$$J(y,t)(\vartheta) = \begin{cases} 0,\ t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{-r}^{0} d_\theta[\eta(\vartheta,\theta)]y(\vartheta+\theta) - \int_{-r}^{0} d_\theta[\eta(t_0,\theta)]y(t_0+\theta), \dots \\ \displaystyle\int_{-r}^{0} d_\theta[\eta(t,\theta)]y(t+\theta) - \int_{-r}^{0} d_\theta[\eta(t_0,\theta)]y(t_0+\theta), \dots \end{cases}$$

Theorem - Federson, Frasson, Mesquita, Tacuri

Consider $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, with $c \geq 1$, $\phi \in P_c = \{x_t;\, x \in B_c,\ t \in [t_0, t_0+\sigma]\}$, $g : [t_0, t_0+\sigma] \to \mathbb{R}$ nondecreasing and **(H1), (H2), (H3), (H4), (H5)** fulfilled. Let $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ be as defined and $y \in P_c$ be **a solution of the measure NFDE** in $[t_0, t_0 + \sigma]$. Define, for $t \in [t_0 - r, t_0 + \sigma]$,

$$x(t)(\vartheta) = \begin{cases} y(\vartheta), & \vartheta \in [t_0 - r, t], \\ y(t), & \vartheta \in [t, t_0 + \sigma]. \end{cases}$$

Then $x : [t_0, t_0 + \sigma] \to B_c$ is **a solution of the GODE** $\dfrac{dx}{d\tau} = DG(x, t)$.

Theorem - Federson, Frasson, Mesquita, Tacuri

Let $B_c = \{z \in G([t_0 - r, t_0 + \sigma], \mathbb{R}^n); \|z - \tilde{x}\| < c\}$, with $c \geq 1$, $\phi \in P_c = \{z_t; z \in B_c,\ t \in [t_0, t_0 + \sigma]\}$, $g : [t_0, t_0 + \sigma] \to \mathbb{R}$ nondecreasing and **(H1), (H2), (H3), (H4), (H5)** fulfilled. Let $G : B_c \times [t_0, t_0 + \sigma] \to G([t_0 - r, t_0 + \sigma], \mathbb{R}^n)$ as defined and $x : [t_0, t_0 + \sigma] \to B_c$ be **a solution of the GODE** $\frac{dx}{d\tau} = DG(x, t)$, with initial condition $x(t_0)(\vartheta) = \phi(\vartheta)$ for $\vartheta \in [t_0 - r, t_0]$, and $x(t_0)(\vartheta) = x(t_0)(t_0)$ for $\vartheta \in [t_0, t_0 + \sigma]$. Then $y \in B_c$ given by

$$y(\vartheta) = \begin{cases} x(t_0)(\vartheta), & t_0 - r \leq \vartheta \leq t_0, \\ x(\vartheta)(\vartheta), & t_0 \leq \vartheta \leq t_0 + \sigma. \end{cases}$$

is **a solution of the measure NFDE** in $t \in [t_0 - r, t_0 + \sigma]$.

Neutral functional dynamic equations on time scales

Time scales calculus

Historical overview

1988: Stefan Hilger, *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg.

Bernd Aulbach (1947-2005) Stefan Hilger

Motivation

1. Unify the discrete and continuous cases;
2. Unify other different cases, depending on the time scale set;
3. Applications to economic problems;
4. Applications to population models;

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   F. M. Atici, D. C. Biles, A. Lebedinsky, An application of time scales to economics, *Mathematical and Computer Modelling*, 43, 718-726 (2006)
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   F. B. Christiansen and T. M. Fenchel, *Theories of populations in biological communities*, vol. 20 of Lectures Notes in Ecological Studies, Springer-Verlag, Berlin, 1977.

Definition

A **time scale** is a nonempty and closed subset of the real numbers.

Example

$\mathbb{Z}$, $\mathbb{R}$, Cantor set, closed intervals, among others are examples of time scales.

Time scale - Basic Concepts

Definition

Define the following operators:

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ and } \sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

where ρ is called **backward jump operator** and σ is called **forward jump operator**.

Definitions	t
right-dense	$\sigma(t) = t$
right-scattered	$\sigma(t) > t$
left-dense	$\rho(t) = t$
left-scattered	$\rho(t) < t$

Definition

We define the **graininess function** $\mu : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) := \sigma(t) - t.$$

Definition

A function $f : \mathbb{T} \to \mathbb{R}$ is called **rd-continuous** if it is regulated on $\mathbb{T}$ and continuous at right-dense points of $\mathbb{T}$.

Definition

Define the set $\mathbb{T}^{\kappa}$ as follows

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases} \tag{1}$$

Definition

Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, define $f^{\Delta}(t)$ to be the number (provided it exists) satisfying $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$. We call $f^{\Delta}(t)$ the **delta (or Hilger) derivative** of f at t.

Properties of delta-derivatives

If f is continuous at t and t is **right-scattered**, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If t is **right-dense**, then f is delta-differentiable at t, if and only if, the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Definition - A. Slavík

If $t \in \mathbb{R}$ is such that $t \leq \sup \mathbb{T}$, define:

$$t^* = \inf\{s \in \mathbb{T}; s \geq t\}.$$

Definition - A. Slavík

Define

$$\mathbb{T}^* = \begin{cases} (-\infty, \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ (-\infty, \infty), & \text{otherwise.} \end{cases}$$

Definition - A. Slavík

Given a function $f : \mathbb{T} \to \mathbb{R}^n$, we define its extension $f^* : \mathbb{T}^* \to \mathbb{R}^n$ by:

$$f^*(t) = f(t^*), \quad t \in \mathbb{T}^*.$$

Theorem - Federson, Frasson, Mesquita, Tacuri

Let $-r, 0 \in \mathbb{T}$, $\eta : \mathbb{T} \times [-r, 0]_{\mathbb{T}} \to \mathbb{R}^{n \times n}$ and $\varphi : [-r, 0]_{\mathbb{T}} \to \mathbb{R}^n$ a regulated function, g a nondecreasing function. Then the Kurzweil-Henstock Δ-integral $\int_{-r}^{0} [\Delta_\theta \eta(t, \theta)] \varphi(\theta)$ exists if, and only if, the Kurzweil-Henstock-Stieltjes integral $\int_{-r}^{0} [d_\theta \eta^*(t, \theta)] \varphi^*(\theta)$ exists, for $t \in \mathbb{T}$. In this case, both integrals have the same value.

Neutral functional dynamic equations on time scales as measure NFDEs

Theorem - Federson, Frasson, Mesquita, Tacuri

Let $t_0 \in \mathbb{T}$. If $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ is a **solution of neutral FDE on time scales**

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s^*, s)\,\Delta s + \int_{-r}^{0} \Delta_\theta[\eta(t,\theta)]x^*(t+\theta) \tag{2}$$

$$- \int_{-r}^{0} \Delta_\theta[\eta(t_0,\theta)]x^*(t_0+\theta) \tag{3}$$

$$x(t) = \varphi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \tag{4}$$

then $x^* : [t_0 - r, t_0 + \sigma] \to B$ is a **solution of the measure NFDE**

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s^*)\,\mathrm{d}g(s) + \int_{-r}^{0} d_\theta[\eta^*(t^*,\theta)]y(t+\theta) \tag{5}$$

$$- \int_{-r}^{0} d_\theta[\eta^*(t_0,\theta)]y(t_0+\theta) \tag{6}$$

$$y_{t_0} = \varphi^*. \tag{7}$$

Theorem - Federson, Frasson, Mesquita, Tacuri

Conversely, if $y : [t_0 - r, t_0 + \sigma] \to B$ satisfies (6) and (7), then it must have the form $y = x^*$, where $x : [t_0 - r, t_0 + \sigma]_{\mathbb{T}} \to B$ is a solution of (3) and (4).

Open and Developing Problems

Open and Developing problems

- Stability results for measure NFDEs and for neutral FDEs on time scales
- Nonperiodic and periodic averaging for measure neutral functional differential equations
- Impulsive neutral functional dynamic equations on time scales and impulsive measure neutral FDEs
- Boundedness of solutions of measure NFDEs and for neutral FDEs on time scales

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Congratulations, Jaroslav Kurzweil!

Děkuji za pozornost!