Differential Equations with Small Delays

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Consider the nonautonomous sytem

$$x'(t) = L(t)x_t,$$

where, for every $t \in \mathbb{R}$, $L(t) : C \to \mathbb{R}^n$ is a bounded linear functional on $C = C([-r, 0], \mathbb{R}^n)$ and $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-r, 0].$$

According to the Riesz representation theorem, we have

$$L(t)\phi = \int_{-r}^{0} d[\eta(t, \theta)]\phi(\theta), \qquad \phi \in C$$

where $\eta(t, \cdot) : [-r, 0] \to \mathbb{R}^{n \times n}$ is of bounded variation so that the above system can be written in the form

$$x'(t) = \int_{-r}^{0} d[\eta(t, \theta)]x(t + \theta).$$

Theorem (Yu. Ryabov, R. Driver, J. Jarník and J. Kurzweil)

Suppose that

$$Kre < 1, \qquad K = \sup_{t\in\mathbb{R}} \|L(t)\|.$$

Then the equation $x'(t) = L(t)x_t$ *has n linearly independent special solutions* $x_1, x_2, \dots, x_n$ *on* $(-\infty, \infty)$ *such that*

$$\sup_{t\le 0} |x_j(t)| e^{t/r} < \infty, \qquad 1 \le j \le n,$$

and for every $(\tau, \phi) \in \mathbb{R} \times C$ *there exist constants* $c_1, c_2, \dots, c_n \in \mathbb{R}$ *such that*

$$\sup_{t\ge\tau} \left| x(t;\tau,\phi) - \sum_{j=1}^{n} c_j x_j(t) \right| e^{t/r} < \infty.$$

The special solutions $x_1, x_2, \ldots, x_n$ form a fundamental set of solutions of an n-dimensional linear ordinary differential equation

$$x' = M(t)x$$

where $M : \mathbb{R} \to \mathbb{R}^{n \times n}$ is continuous.

Problem

Is it possible to give the ordinary differential equation

$$x' = M(t)x$$

explicitly in terms of the kernel η of the original delay differential equation

$$x'(t) = \int_{-r}^{0} d[\eta(t,\theta)]x(t+\theta) \ ?$$

Theorem (O. Arino, I. Győri, M.P.)

Define

$$\begin{aligned} M_0(t,s) &= \eta(s,0) - \eta(t,-r), \qquad s \le t, \\ M_{j+1}(t,s) &= -\int_{-r}^{0} d_\theta[\eta(t,\theta)] \int_{s+\theta}^{t} M_j(t,u)\,du, \quad s \le t, \quad j = 0,1,\ldots. \end{aligned}$$

Suppose that

$$Kre < 1, \qquad K = \sup_{t\in\mathbb{R}} \mathrm{Var}_{[-r,0]}\,\eta(t,\cdot).$$

Then the special solutions $x_1, x_2, \ldots, x_n$ *of the delay equation* $x'(t) = L(t)x_t$ *form a fundamental set of solutions of the ordinary differential equation* $x' = M(t)x$, *where*

$$M(t) = \sum_{j=0}^{\infty} M_j(t,t), \qquad t \in \mathbb{R},$$

the convergence of the last series being uniform on $\mathbb{R}$.

Consider the simple equation

$$x'(t) = A(t)x(t-r)$$

where $r \geq 0$ and $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is continuous.

In this case, we have

$$\begin{aligned} M_0(t,s) &= A(s), \qquad s \leq t, \\ M_{j+1}(t,s) &= -A(s) \int_{s-r}^{t} M_j(t,u)\,du, \qquad s \leq t,\ j = 0,1,2,\ldots. \end{aligned}$$

.

Theorem (I. Győri, M.P.)

Suppose that

$$\int_0^\infty \|A(t)\|^k \, dt < \infty \qquad \text{for some } k \in \mathbb{N}.$$

Then the stability (asymptotic stability) of the delay equation

$$x'(t) = A(t)x(t-r)$$

is equivalent to the stability (asymptotic stability) of ordinary equation

$$x' = \sum_{j=0}^{k-1} M_j(t,t)x.$$

Definition

The logarithmic norm of a matrix $A = (a_{ij})_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$ is defined by

$$\mu(A) = \lim_{h \to 0+} \frac{\|I + hA\| - 1}{h}.$$

In case of the standard norms on $\mathbb{R}^n$ $\mu(A)$ can be given explicitly. For example, if

$$|x| = \sum_{j=1}^{n} |x_j|, \qquad x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n,$$

then

$$\mu(A) = \max_{1 \leq j \leq n} \left(\operatorname{Re} a_{jj} + \sum_{i,\, i \neq j} |a_{ij}| \right).$$

Corollary

Suppose that

$$\int_0^\infty \|A(t)\|^k \, dt < \infty \qquad \textit{for some } k \in \mathbb{N}.$$

If

$$\limsup_{t\to\infty} \int_0^t \mu\left(\sum_{j=0}^{k-1} M_j(s,s)\right) ds < \infty,$$

then the delay equation $x'(t) = A(t)x(t-r)$ *is stable, and if*

$$\lim_{t\to\infty} \int_0^t \mu\left(\sum_{j=0}^{k-1} M_j(s,s)\right) ds = -\infty,$$

then it is asymptotically stable.
In the scalar case $(n = 1)$ *the above conditions are not only sufficient, but also necessary for the stability (asymptotic stability) of the equation.*

Example

Consider the scalar equation

$$x'(t) = \frac{\sin t}{t^{\alpha}} x(t-r), \qquad t \geq 1$$

where $\alpha > 0$. By the application of the stability criteria, we obtain:

if $\alpha > 1/2$, then for every $r \geq 0$ the eqution is stable, but it is not asymptotically stable,

if $1/3 < \alpha \leq 1/2$, then the equation is stable if and only if

$$r \in \bigcup_{k \geq 0 \text{ integer}} [2k\pi, (2k+1)\pi],$$

and the equation is asymptotically stable if and only if

$$r \in \bigcup_{k \geq 0 \text{ integer}} (2k\pi, (2k+1)\pi).$$

The stability criteria were presented in

I. Győri and M. Pituk
Stability criteria for linear delay differential equations
Differential Integral Equations 10 (1997), 841-852.

Nonlinear Systems

For $r \geq 0$, let

$$C = C([-r, 0], \mathbb{R})$$

be the space of continuous functions mapping $[-r, 0]$ into $\mathbb{R}$ equipped with the norm

$$\|\phi\| = \max_{\theta \in [-r,0]} |\phi(\theta)|, \qquad \phi \in C.$$

Consider the scalar retarded FDE

$$x'(t) = f(x_t),$$

where $f : C \to \mathbb{R}$ is Lipschitz continuous and $x_t \in C$ is defined by

$$x_t(\theta) = x(t + \theta), \qquad \theta \in [-r, 0].$$

Theorem (Ryabov, Jarník and Kurzweil)

Suppose that $f : C \to C$ is Lipschitz continuous and its Lipschitz constant L satisfies

$$Lre < 1.$$

Then for every $x \in \mathbb{R}$ the delay equation $x'(t) = f(x_t)$ has a unique (special) solution $\tilde{x}$ defined on $(-\infty, \infty)$ such that

$$\tilde{x}(0) = x, \qquad \sup_{t \leq 0} |\tilde{x}(t)| e^{t/r} < \infty.$$

Moreover, for every solution $x : [\tau - r, \infty) \to \mathbb{R}$ of $x'(t) = f(x_t)$ there exists a unique special solution $\tilde{x}$ such that

$$\sup_{t \geq \tau} |x(t) - \tilde{x}(t)| e^{t/r} < \infty.$$

For every $\phi \in C$, the equation

$$x'(t) = f(x_t)$$

has a unique solution $x(\phi) : [-r, \infty) \to \mathbb{R}$ with initial value

$$x_0(\phi) = \phi.$$

The map $\Phi : [0, \infty) \times C \to C$ defined by

$$\Phi(t, \phi) = x_t(\phi), \qquad t \geq 0,\ \phi \in C.$$

is a continuous semiflow on C.

The set of *equilibria* of the semiflow Φ is

$$E = \{\, \hat{x} \mid x \in \mathbb{R},\ f(\hat{x}) = 0 \,\},$$

where for $x \in \mathbb{R}$, $\hat{x} \in C$ is the constant function with value x.

By a positive cone in C, we mean a nonempty closed subset K of C with the properties:

$$K + K \subset K, \quad K \cap (-K) = \{\hat{0}\}, \quad \lambda K \subset K \quad \text{whenever } \lambda \geq 0.$$

Each positive cone K induces a partial order relation $\leq_K$ on C by

$$\phi \leq_K \psi \qquad \text{iff } \psi - \phi \in K.$$

We can ask what conditions will guarantee that the semiflow Φ is monotone with respect to the ordering $\leq_K$, that is,

$$\phi,\ \psi \in C,\ \phi \leq_K \psi \text{ imply that } x_t(\phi) \leq_K x_t(\psi) \text{ for all } t \geq 0\text{-ra.}$$

Recall some results due to Hal Smith and Horst Thieme summarized in

H. Smith.
Monotone Dynamical Systems
Amer. Math. Soc., Providence, 1991.

The Natural Ordering

Let

$$K = C^+ = \{\phi \in C \mid \phi(\theta) \geq 0 \quad \text{for } \theta \in [-r, 0].\}$$

Then the ordering $\leq_{C^+}$ is the usual pointwise ordering,

$$\phi \leq_{C^+} \psi \quad \text{iff} \quad \phi(\theta) \leq \psi(\theta) \text{ for all } \theta \in [-r, 0],$$

and the semiflow Φ is monotone iff the following quasimonotone condition holds:

(Q) $\quad f(\phi) \leq f(\psi) \quad$ whenever $\phi \leq_{C^+} \psi$ and $\phi(0) = \psi(0)$.

As an example, consider the equation with one single delay

$$x'(t) = g(x(t), x(t-r)),$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable with bounded partial derivatives.

Then the quasimonotone condition (Q) holds provided

$$D_2 g(x, y) \geq 0, \qquad (x, y) \in \mathbb{R}^2.$$

Thus, $g(x, y)$ is nondecreasing in y for each fixed x.

The Exponential Ordering

If instead of the standard cone C^+ we use other cones, we can obtain monotonicity under weaker conditions on f and/or g.

For $\mu \geq 0$, define

$$K_\mu = \{\, \phi \in \mathcal{C} \mid \phi \geq 0 \text{ and } \phi(\theta)e^{\mu\theta} \text{ is nondecreasing on } [-r, 0]\,\}.$$

Each K_μ is a positive cone in C.

The ordering induced by K_μ is called the exponential ordering and is it is denoted by $\leq_\mu$.

The semiflow F is monotone with respect to the ordering $\leq_\mu$ iff

$$(\mathrm{M}_\mu) \quad f(\psi) - f(\phi) + \mu[\psi(0) - \phi(0)] \geq 0 \quad \text{whenever } \phi \leq_\mu \psi.$$

(M_μ) holds for some $\mu \geq 0$ if there exists $L > 0$ such that

$$f(\psi) - f(\phi) \geq -L|\psi - \phi| \qquad \text{whenever } \phi \leq \psi$$

and

$$Lre < 1.$$

For the equation $x'(t) = g(x(t), x(t-r))$, suppose that

$$L_1 = \inf_{(x,y)\in\mathbb{R}^2} D_1 g(x,y) \qquad \text{and} \qquad L_2 = \inf_{(x,y)\in\mathbb{R}^2} D_2 g(x,y)$$

are finite. Then (M_μ) holds for some $\mu \geq 0$ iff one of the following holds:

(a) $L_2 \geq 0$, or
(b) $L_2 < 0$ and $L_1 + L_2 \geq 0$, or
(c) $L_2 < 0$, $L_1 + L_2 < 0$, $r|L_2| < 1$, and $rL_1 - \log(r|L_2|) \geq 1$

Generic Convergence

Perharps the most important consequence of the monotone theory developed by Hirsch and Smith and Thieme is that:

Under a condition "slightly stronger" than (M_μ) and an additional compactness assumption the solutions of the equation

$$x'(t) = f(x_t)$$

starting from a dense subset of C converge to equilibria as $t \to \infty$.

Stability

Another interesting result due to Smith és Thieme is that :

If (M_μ) holds and $\hat{v}$ is an equilibrium of the equation

$$x'(t) = f(x_t),$$

then its stability properties are often the same as the stability properties of v as an equilibrium of the ODE

$$x' = F(x) = f(\hat{x}).$$

More precisely,

$F'(v) < 0$ implies that both v and $\hat{v}$ are asymptotically stable

and

$F'(v) > 0$ implies that both v and $\hat{v}$ are unstable equilibria.

The only case not covered by Smith and Thieme is the critical case when

$$F'(v) = 0.$$

In case of the equation

$$x'(t) = g(x(t), x(t-r)),$$

we have that

$$f(\phi) = g(\phi(0), \phi(-r)).$$

By definition $\hat{x}(\theta) = x$ for $\theta \in [-r, 0]$. Therefore

$$F(x) = f(\hat{x}) = g(x, x).$$

Therefore the previos results say that the delay equation has similar convergence properties as the ODE

$$x' = g(x, x)$$

obtained by "ignoring the delay".

The exponential ordering idea and the dynamical consequences of monotonicity were extended to systems of autonomous retarded FDEs by

H. Smith and H. Thieme.
Strongly order preserving semiflows generated by functional differential equations
J. Differential Equations, 93 (1991), 332-363,

to systems of autonomous neutral FDEs by

T. Krisztin and J. Wu.
Monotone semiflows generated by neutral equations with different delays in neutral and retarded parts
Acta Math. Univ. Comenian. 63 (1994), 207-220.

and to nonautonomous neutral FDEs with infinite delay by

S. Novo, R. Obaya and V. M. Villarragut.
Exponential ordering for nonautonomous neutral functional differential equations with infinite delays
SIAM Journal on Mathematical Analysis, 41 (2009), 1025-1053.

Main Results

Theorem (Stability Criterion)

Suppose (M_μ) *hold for some* $\mu > 0$. *The equilibrium* $\hat{v}$ *of*

$$x'(t) = f(x_t)$$

and the equilibrium v *of the ODE*

$$x' = F(x) = f(\hat{x})$$

are asymptotically stable iff there exists $\eta > 0$ *such that*

$$F(x) < 0 \qquad \text{for all } x \in (v, v+\eta)$$

and

$$F(x) > 0 \qquad \text{for all } x \in (v-\eta, v).$$

In contrast to the result due to Smith és Thieme the theorem applies also in the case when $F'(v) = 0$.

Theorem (Boundedness)

Suppose that (M_μ) *holds for some* $\mu > 0$ *and the set of equilibria* E *of the equation* $x'(t) = f(x_t)$ *is nonemty. Let*

$$S = \sup\{\, v \in \mathbb{R} \mid \hat{v} \in E \,\} \qquad \textit{and} \qquad I = \inf\{\, v \in \mathbb{R} \mid \hat{v} \in E \,\}.$$

Then all solutions of the equation $x'(t) = f(x_t)$ *are bounded on* $[0, \infty)$ *if and only if both conditions below hold.*

(B^+) *Either* $S = \infty$ *or* $S < \infty$ *and* $f(\hat{x}) < 0$ *for all* $x > S$.

(B^-) *Either* $I = -\infty$ *or* $I > -\infty$ *and* $f(\hat{x}) > 0$ *for all* $x < I$.

The key of the proof of the stability theorem is the following

Proposition

Let (M_μ) *hold for some* $\mu > 0$. *Then every constant function* $\hat{k} \in C$ *is a sub-equilibrium or super-equilibrium of the semiflow* Φ *generated by the equation*

$$x'(t) = f(x_t).$$

More precicely,

$f(\hat{k}) \leq 0$ *implies that* $x_t(\hat{k}) \leq_\mu \hat{k}$ *for all* $t \geq 0$,

and

$f(\hat{k}) \geq 0$ *implies that* $x_t(\hat{k}) \geq_\mu \hat{k}$ *for all* $t \geq 0$.

The proposition combined with the Convergence Criterion for Monotone Semiflows implies that under the monotonicity condition (M_μ) the bounded solutions starting from constant initial data are convergent as $t \to \infty$.

Theorem (Global Convergence)

Suppose that the set of equilibria E is nonempty and $f : C \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L satisfying

$$Lre < 1$$

so that (M_μ) holds with $\mu = 1/r$. Then, similarly as for the ODE

$$x' = F(x) = f(\hat{x}),$$

the boundedness of all solutions of

$$x'(t) = f(x_t)$$

implies their convergence.

More precicesly, all solutions of the delay equation $x'(t) = f(x_t)$ are convergent as $t \to \infty$ if and only if (B^+) and (B^-) hold.

Note that the result due to Smith and Thieme does not guarantee the convergence of all solutions, only "most" of them.

The proof of the global convergence result is based on the following

Proposition

Under the smallness condition

$$Lre < 1,$$

where L is the Lipschitz constant of f, the delay equation

$$x'(t) = f(x_t)$$

is asymptotically equivalent to an ODE

$$x' = g(x),$$

whose right-hand side $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and both equations have the same equilibria.

The right-hand side g is given by

$$g(x) = f(\tilde{x}_0(x)), \qquad x \in \mathbb{R},$$

where $\tilde{x}$ is the special solution of $x'(t) = f(x_t)$ described in

Theorem (Ryabov, Jarník and Kurzweil)

Suppose that the Lipschitz constant L of f satisfies

$$Lre < 1.$$

Then for every $x \in \mathbb{R}$ the delay equation $x'(t) = f(x_t)$ has a unique (special) solution $\tilde{x}$ defined on $(-\infty, \infty)$ such that

$$\tilde{x}(0) = x, \qquad \sup_{t \le 0} |\tilde{x}(t)| e^{t/r} < \infty.$$

Moreover, for every solution $x : [\tau - r, \infty) \to \mathbb{R}$ of $x'(t) = f(x_t)$ there exists a unique special solution $\tilde{x}$ such that

$$\sup_{t \ge \tau} |x(t) - \tilde{x}(t)| e^{t/r} < \infty.$$

Example

Consider the equation

$$x'(t) = \delta + \sin x(t-r),$$

where $\delta \geq 0$, $r > 0$, proposed as model of high-frequency oscillators. If $\delta > 1$, this equation has no equilibria, while for $\delta \in [0,1]$ the equilibria are given by

$$v_k = \arcsin \delta + 2k\pi, \qquad w_k = \arcsin \delta + (2k-1)\pi,$$

where k is an integer. Our results applies if

$$\delta \in [0,1] \qquad \text{and} \qquad r < \frac{1}{e}.$$

Under this assumption every solution converges to one of the equilibria as $t \to \infty$.

Note that the previous result implies the convergence of the solutions only for a dense subset of initial data.

The second part of the talk on nonlinear systems was based on the papers

M. Pituk
Convergence to equilibria i scalar nonquasimonotone functional differential equations
J. Differential Equations, 193 (2003), 95-135,

M. Pituk
More on scalar functional differential equations generating a monotone semiflow
Acta Sci. Math. (Szeged), 69 (2003), 633-650.