Well-posedness and maximum principles for lattice reaction-diffusion equations

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Reaction-diffusion equations

Classical reaction-diffusion equation:

$$\partial_t u(x,t) = k\partial_{xx} u(x,t) + f(u(x,t))$$

Spatial discretization (lattice differential equation; $x \in \mathbb{Z}$, $t \in \mathbb{R}_0^+$):

$$\partial_t u(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t))$$

Spatial and temporal discretization: ($x \in \mathbb{Z}$, $t \in \mathbb{N}_0$)

$$u(x,t+1) - u(x,t) = k(u(x+1,t) - 2u(x,t) + u(x-1,t)) + f(u(x,t))$$

Some references

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Time scales – basic definitions

- Time scale: closed set $\mathbb{T} \subseteq \mathbb{R}$
- $\sigma_{\mathbb{T}}(t) = \inf\{s \in \mathbb{T}; s > t\}$, $t \in \mathbb{T}$
- $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T}; a \leq t \leq b\}$
- Δ-derivative:

$$f^{\Delta}(t) = \begin{cases} \lim\limits_{s \to t} \frac{f(t)-f(s)}{t-s} & \text{if } \sigma_{\mathbb{T}}(t) = t, \\ \frac{f(\sigma_{\mathbb{T}}(t))-f(t)}{\sigma_{\mathbb{T}}(t)-t} & \text{if } \sigma_{\mathbb{T}}(t) > t. \end{cases}$$

Lattice reaction-diffusion equation with general time

We study the equation

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t),$$

where $a, b, c \in \mathbb{R}$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$, $\mathbb{T} \subseteq \mathbb{R}$ is a time scale, and u^{Δ} denotes the delta derivative with respect to time.

We consider the initial-value problem with the condition

$$u(x,t_0) = u_x^0, \quad x \in \mathbb{Z},$$

where $u^0 = \{u_x^0\}_{x\in\mathbb{Z}}$ is a bounded real sequence.

Conditions (H1)–(H3)

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t)$$

(H1) f is bounded on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.

(H2) f is Lipschitz-continuous in the first variable on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.

(H3) For each bounded set $B \subset \mathbb{R}$ and each choice of $\varepsilon > 0$ and $t \in [t_0, T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [t_0, T]_{\mathbb{T}}$, then $|f(u,x,t) - f(u,x,s)| < \varepsilon$ for all $u \in B$, $x \in \mathbb{Z}$.

Examples

1) **Fisher-type equation**

$$u^{\Delta}(x,t) = au(x+1,t)+bu(x,t)+cu(x-1,t)+\lambda u(x,t)\,(1-u(x,t))$$

2) **Nagumo-type equation**

$$u^{\Delta}(x,t) = au(x+1,t)+bu(x,t)+cu(x-1,t)+\lambda u(x,t)\left(1-u(x,t)^2\right)$$

3) **Logistic population model with variable carrying capacity**

$$u^{\Delta}(x,t) = au(x+1,t)+bu(x,t)+cu(x-1,t)+\lambda u(x,t)(d(x,t)-u(x,t))$$

Conditions (H1)–(H3) hold e.g. in the following cases:

- $d(x,t)=e(t)$, where $e:\mathbb{R}\to\mathbb{R}$ is continuous and periodic
- $d(x,t)=e(x-\gamma t)$, where $e:\mathbb{R}\to\mathbb{R}$ is continuous, monotone, and bounded

Local existence

Theorem

Assume that $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ *satisfies* (*H*1)–(*H*3)*. Then for each* $u^0 \in \ell^\infty(\mathbb{Z})$*, the initial-value problem*

$$\begin{aligned} u^\Delta(x,t) &= au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t), \\ u(x,t_0) &= u^0_x, \quad x \in \mathbb{Z}, \end{aligned}$$

has a bounded local solution defined on $\mathbb{Z} \times [t_0, t_0+\delta]_{\mathbb{T}}$*. The solution is obtained by letting* $u(x,t) = U(t)_x$*, where* $U : [t_0, t_0+\delta]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ *is a solution of the abstract dynamic equation*

$$U^\Delta(t) = \Phi(U(t),t), \quad U(t_0) = u^0,$$

with $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_{\mathbb{T}} \to \ell^\infty(\mathbb{Z})$ *being given by*

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{au_{x+1} + bu_x + cu_{x-1} + f(u_x,x,t)\}_{x\in\mathbb{Z}}.$$

Global uniqueness

Even in the linear case $f \equiv 0$, the solutions of the initial-value problem are not unique. To get uniqueness, we restrict ourselves to the class of bounded solutions.

Theorem

Assume that $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ satisfies $(H1)$ and $(H2)$. Then for each $u^0 \in \ell^\infty(\mathbb{Z})$, the initial-value problem has at most one bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.

Towards the maximum principle

Given an initial condition $u^0 \in \ell^\infty(\mathbb{Z})$, let

$$m = \inf_{x \in \mathbb{Z}} u_x^0, \quad M = \sup_{x \in \mathbb{Z}} u_x^0.$$

If $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of our initial-value problem, is it true that

$$m \leq u(x,t) \leq M$$

for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$?

Additional assumptions are needed to derive this result.

Conditions (H4)–(H6)

Denote

$$\overline{\mu}_{\mathbb{T}} = \max_{t \in [t_0, T)_{\mathbb{T}}} (\sigma(t) - t).$$

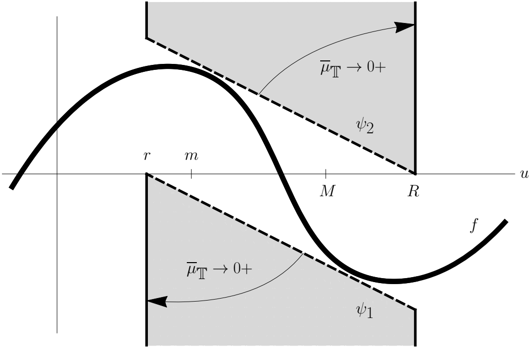
(H4) *a, b, $c \in \mathbb{R}$ are such that $a, c \geq 0$, $b < 0$, and $a + b + c = 0$.*

(H5) $\overline{\mu}_{\mathbb{T}} \leq -1/b$.

(H6) *There exist $r, R \in \mathbb{R}$ such that $r \leq m \leq M \leq R$, and one of the following statements holds:*

1. $\overline{\mu}_{\mathbb{T}} = 0$ *and*
$$f(R, x, t) \leq 0 \leq f(r, x, t)$$
for $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.
2. $\overline{\mu}_{\mathbb{T}} > 0$ *and*
$$(1/\overline{\mu}_{\mathbb{T}} + b)(r - u) \leq f(u, x, t) \leq (1/\overline{\mu}_{\mathbb{T}} + b)(R - u)$$
for $u \in [r, R]$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.

Condition (H6) – geometric meaning



- $\overline{\mu}_{\mathbb{T}} > 0$, $(1/\overline{\mu}_{\mathbb{T}} + b)\,(r - u) \le f(u, x, t) \le (1/\overline{\mu}_{\mathbb{T}} + b)\,(R - u)$
- $\overline{\mu}_{\mathbb{T}} = 0$, $f(R, x, t) \le 0 \le f(r, x, t)$

Note: (H4), (H5) $\Rightarrow 1/\overline{\mu}_{\mathbb{T}} + b \ge 0$

Maximum principle and global existence

Theorem

Assume that (H1)–(H6) *hold. If* $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ *is a bounded solution of the initial-value problem, then*

$$r \leq u(x,t) \leq R \quad \text{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}.$$

Theorem

If $u^0 \in \ell^\infty(\mathbb{Z})$ *and* (H1)–(H6) *hold, then the initial-value problem has a unique bounded solution* $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.

Maximum principle – proof outline

1. Establish the maximum principle for discrete time scales (isolated points only)
2. Prove that solutions depend continuously on the choice of the time scale
3. Use continuous dependence to extend the maximum principle to all time scales

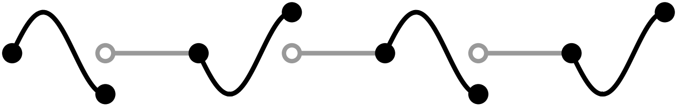
Extension of time scale functions

Let $g_{\mathbb{T}} : [t_0, T] \to \mathbb{R}$ be given by

$$g_{\mathbb{T}}(t) = \inf\{s \in [t_0, T]_{\mathbb{T}};\ s \geq t\}, \quad t \in [t_0, T].$$

Each function $x : [t_0, T]_{\mathbb{T}} \to X$ can be extended to a function $x^* : [t_0, T] \to X$ by letting

$$x^*(t) = x(g_{\mathbb{T}}(t)), \quad t \in [t_0, T].$$



Continuous dependence with respect to time scale

Theorem

Let X be a Banach space, $\mathcal{B} \subseteq X$. Consider an interval $[t_0, T] \subset \mathbb{R}$ and a sequence of time scales $\{\mathbb{T}_n\}_{n=0}^{\infty}$ such that $t_0 \in \mathbb{T}_n$ and $\sup \mathbb{T}_n \geq T$ for each $n \in \mathbb{N}_0$, $T \in \mathbb{T}_0$, and $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$. Denote $\mathbb{T} = \overline{\bigcup_{n=0}^{\infty} \mathbb{T}_n}$.
Suppose that $\Phi : \mathcal{B} \times [t_0, T]_{\mathbb{T}} \to X$ is continuous on its domain and Lipschitz-continuous with respect to the first variable. Let $x_n : [t_0, T]_{\mathbb{T}_n} \to \mathcal{B}$, $n \in \mathbb{N}_0$, be a sequence of functions satisfying

$$x_n^{\Delta}(t) = \Phi(x_n(t), t), \quad t \in [t_0, T]_{\mathbb{T}_n}, \quad n \in \mathbb{N}_0,$$

and $x_n(t_0) \to x_0(t_0)$. Then $x_n^* \rightrightarrows x_0^*$ on $[t_0, T]$.

Continuous dependence – a more general result

Theorem

Let X be a Banach space, $\mathcal{B} \subseteq X$. Consider a sequence of nondecreasing left-continuous functions $g_n : [t_0, T] \to \mathbb{R}$, $n \in \mathbb{N}_0$, such that $g_n \rightrightarrows g_0$ on $[t_0, T]$. Assume that $\Phi : \mathcal{B} \times [t_0, T] \to X$ is Lipschitz-continuous in the first variable. Let $x_n : [t_0, T] \to \mathcal{B}$, $n \in \mathbb{N}_0$, be a sequence of functions satisfying $x_n(t_0) \to x_0(t_0)$ and

$$x_n(t) = x_n(t_0) + \int_{t_0}^{t} \Phi(x_n(s), s)\, dg_n(s), \quad t \in [t_0, T], \quad n \in \mathbb{N}_0,$$

where the integral is the Kurzweil-Stieltjes integral. Suppose finally that the function $s \mapsto \Phi(x_0(s), s)$, $s \in [t_0, T]$, is regulated. Then $x_n \rightrightarrows x_0$ on $[t_0, T]$.

Approximation by discrete time scales

Theorem

If $\mathbb{T}_0 \subset \mathbb{R}$ is a time scale with $t_0, T \in \mathbb{T}_0$, there exists a sequence of discrete time scales $\{\mathbb{T}_n\}_{n=1}^{\infty}$ with $\mathbb{T}_n \subseteq \mathbb{T}_0$, $\min \mathbb{T}_n = t_0$, $\max \mathbb{T}_n = T$, and such that $g_{\mathbb{T}_n} \rightrightarrows g_{\mathbb{T}_0}$ on $[t_0, T]$. The sequence $\{\mathbb{T}_n\}_{n=1}^{\infty}$ can be chosen so that

- *if $\overline{\mu}_{\mathbb{T}_0} > 0$, then $\overline{\mu}_{\mathbb{T}_n} = \overline{\mu}_{\mathbb{T}_0}$ for all $n \in \mathbb{N}$;*
- *if $\overline{\mu}_{\mathbb{T}_0} = 0$, then $\lim_{n\to\infty} \overline{\mu}_{\mathbb{T}_n} = 0$.*

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