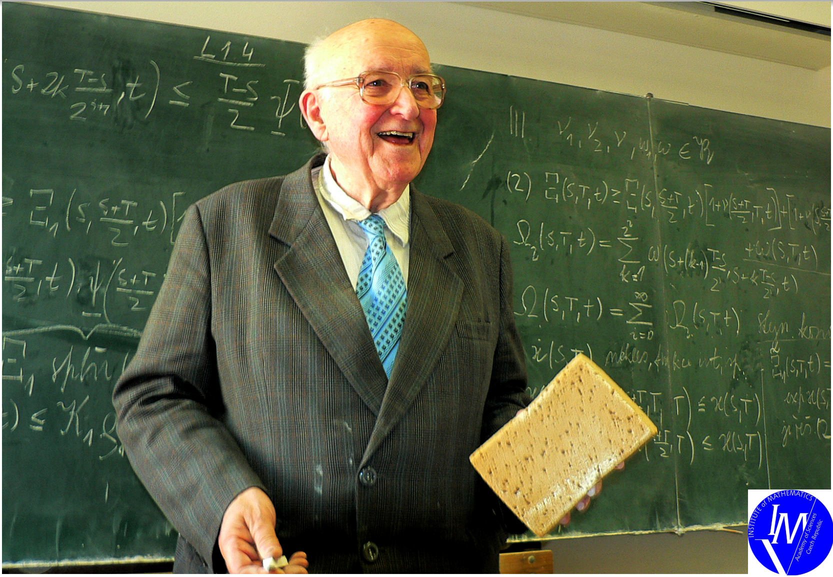
Homage to JAROSLAV KURZWEIL



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Stability of motion (1955 – 1957)

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- Kurzweil resumed the study of the **Bernstein problem** showing that for a uniformly convex Banach space in which every operation F can be uniformly approximated by analytic functions, the condition (A) from his paper published in 1953 is always fulfilled.

  On approximation in real Banach spaces by analytic operations, Studia mathematica.

Continuous dependence of solutions of ODEs on parameters

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Consider: initial value problems

$$\dot{x}_k = f_k(x_k, t), \quad x(0) = 0, \quad k \in \{0\} \cup \mathbb{N}. \qquad (E_k)$$

Assume:

- $f_k : G \times [0, T] \to \mathbb{R}^n$, $G \subset \mathbb{R}^n$ *is open,*
- x_0 *is uniquely determined solution of* (E_0) *on* $[0, T]$,
- *functions* $f_k(x, t)$, $k \in \mathbb{N}$, *are equicontinuous in* x *for fixed* t,
- $\displaystyle\int_0^t f_k(x, \tau)\, \mathrm{d}\tau \rightrightarrows \int_0^t f_0(x, \tau)\, \mathrm{d}\tau$.

Then: solutions x_k *of* (E_k) *are defined on* $[0, T]$ *for* k *sufficiently large and* $x_k \rightrightarrows x_0$ *on* $[0, T]$.

Example

$$\dot{x}_k = x_k \, k^{1-\alpha} \cos kt + k^{1-\beta} \sin kt \,, \quad x_k(0) = 0, \quad k \in \mathbb{N} \,.$$

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But,

- the previously known results (including K&V) justified this convergence effect only for
$$\alpha = 1 \quad \text{and} \quad 0 < \beta \le 1,$$
- just the indefinite integrals $F(x,t) = \int_{t_0}^{t} f(x,\tau)\,\mathrm{d}\tau$ of the right-hand sides are essential.

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Questions:

- To get convergence results covering as much as possible the convergence effects related to Example.
- To describe the notion of a solution of the given differential equation in terms of the indefinite integral of its right hand side.

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- To get convergence results covering as much as possible the convergence effects related to Example.
- To describe the notion of a solution of the given differential equation in terms of the indefinite integral of its right hand side.

Answer: **Generalized Ordinary Differential Equations**

Generalized Ordinary Differential Equations

Generalized ordinary differential equations and continuous dependence on a parameter, Czechoslovak Math. Journal 1957 (issue 3).

Let $f\colon G \times [0,T] \to \mathbb{R}^n$. Then $x\colon [a,b] \to \mathbb{R}^n$ is a solution of $\dot{x} = f(t,x)$ on $[a,b]$ if

- $(x(t),t) \in G \times [0,T]$ for all $t \in [a,b]$,
- $$x(s_2) - x(s_1) \approx \sum_{i=1}^{k} \int_{\alpha_{i-1}}^{\alpha_i} f(x((\tau_i),t))\mathrm{d}t \quad \text{for } s_1, s_2 \in [a,b],$$

  where $D = \{\tau_i, [\alpha_{i-1}, \alpha_i]\}_{i=1}^{k}$ is a sufficiently fine partition of $[s_1,s_2]$ ($\tau_i \in [\alpha_{i-1}, \alpha_i]$).

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Kurzweil integral

Given a **gauge** $\delta\colon[a,b]\to(0,\infty)$, we say that $D=\{\tau_i,[\alpha_{i-1},\alpha_i]\}_{i=1}^{k}$ is δ**-fine** if

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Definition

$\int_a^b \mathrm{D}U(\tau,t)=I$ if for each $\varepsilon>0$ there is a gauge δ such that

$$|S(U,D)-I|<\varepsilon\quad\text{for all } \delta\text{-fine partitions } D \text{ of } [a,b].$$

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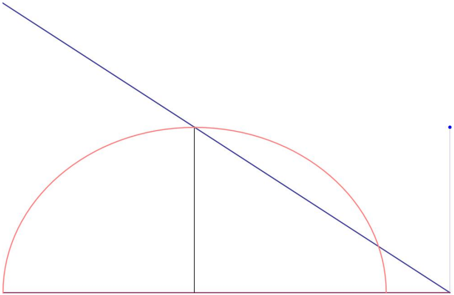

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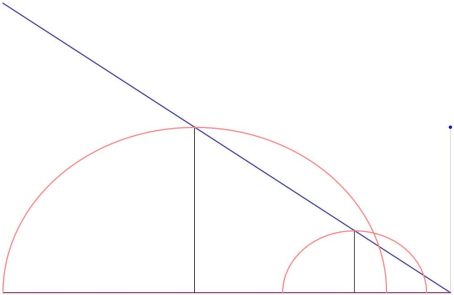

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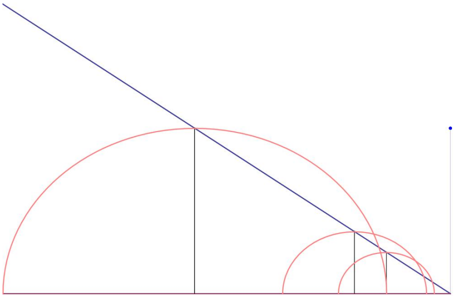

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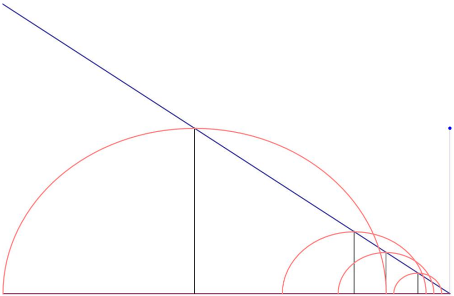

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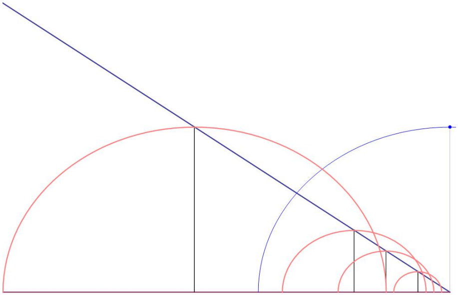

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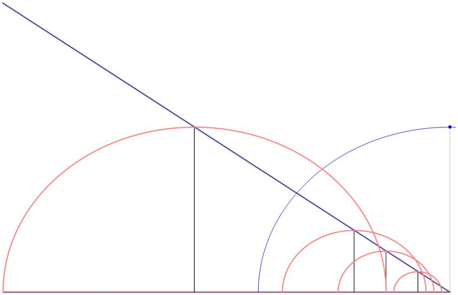

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The essential novelty of the Kurzweil integral is that the tags can be chosen first, while the division points are allowed to vary in a controlled neighborhood of the tag.

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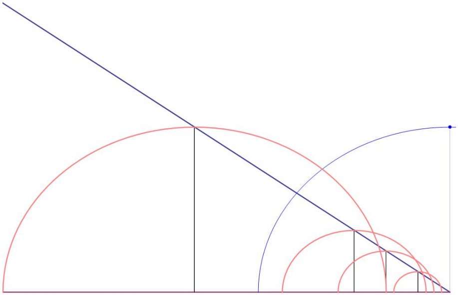

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The essential novelty of the Kurzweil integral is that the tags can be chosen first, while the division points are allowed to vary in a controlled neighborhood of the tag.

This made it possible to control the singularities and integrate very general classes of functions.

- If $U = f(\tau)\, g(t)$ then
$$S(U,D) = \sum_{i=1}^{k} f(\tau_i)\,(g(\alpha_i) - g(\alpha_{i-1}))$$
and
$$\int_a^b \mathrm{D}[f(\tau)\, g(t)] = \int_a^b f\, \mathrm{d}g,$$
where the integral on the right-hand side is the **(Ward-) Perron-Stieltjes** one.

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where the integral on the right-hand side is the **(Ward-) Perron-Stieltjes** one.
- On the contrary to the Perron integral, the definition of the Kurzweil integral can be naturally extended to **abstract valued** functions.
- On the contrary to the Lebesgue-Stieltjes integral, the Kurzweil(-Stieljes) integral admits **regulated** integrators.

- 1958 - Receives the degree of Doctor of Science (DrSc.).

  1958–1963 Series of papers devoted to generalized differential equations.
- 1964 - Awarded the State Prize.

  1960–1965 Contribution to **control theory**, **functional analysis** and **averaging principle for partial differential equations**.
- 1966 - Appointed full professor of mathematics.

  1966–1969 Crucial results on **invariant manifolds** for differential equations in Banach spaces.
- 1968 - Elected corresponding member of the Czechoslovak Academy of Sciences.

  In the academic year 1968-1969 a visiting professor at Dynamic Centre, Warwick, UK.

Invariant manifolds

- *Exponentially stable integral manifolds, averaging principle and continuous dependence on a parameter.* Czechoslovak Math. Journal 16 (91) 1966, 380–423, 463–492.
- *Invariant manifolds for differential systems.* Atti VIII Congr. UMI, Trieste, 1967, 291–292.
- *Van der Pol perturbation of the equation for a vibrating string.* Czechoslovak Math. Journal 17 (92) 1967, 558–608.
- *Invariant manifolds for flows.* Acta Fac. Rerum Nat. U. Comenianae, Proc. Equadiff Bratislava, 1966, Mathematica XVII (1967), 89–92.
- *Invariant manifolds of differential systems.* Proc. of Fourth Conf. on Nonlinear Oscillations, Academia Praha, 1968, 41–49.
- *Invariant manifolds of a class of linear functional differential equations.* Revue Roum. Math. Pures et Appl. XIII (1968), 113–1120.
- *A theory of invariant manifolds for flows* (with A. Halanay). Revue Roum. Math. Pures et Appl. XIII (1968), 1079–1097.
- *Invariant sets of differential systems* (Russian). Differencialnye uravnenija IV (1968), 785–797.
- *Invariant manifolds for flows.* Differential Equations and Dynamical Systems. Proc. Int. Symp. Mayaguez 1967, Academic Press, 431–468.
- *Invariant manifolds of differential systems.* ZAMM 49 (1969), 11–14.
- *On invariant sets and invariant manifolds of differential systems* (with J. Jarník). Journal of Differential Equations. 6 (1969), 247–263.

Seventies

Global solutions of FDEs

- *Existence of global solutions of delayed differential equations on compact manifolds.* Rend. Ist. di Matem. Univ. Trieste, vol. II, fasc. II (1970), 106–108.
- *On solutions of nonautonomous linear delayed differential equations which are exponentially bounded for $t \to -\infty$.* Čas. pěst. mat. 96 (1971), 229–238.
- *Solutions of linear nonautonomous functional differential equations which are exponentially bounded for $t \to -\infty$.* Journal of Diff. Eqs. 11 (1972), 376–384.
- *Ryabov's special solutions of functional differential equations* (with J. Jarník). Boll. U.M.I. (4) 11, Suppl. Fasc. 3 (1975), 198–208.

Invariant manifolds

- *Reducing differential inclusions* (with J. Jarník). Abh. der Akad. der Wissenschaften der DDR, Abt. Mathematik, Naturwissenschaften, Technik No 3 (1977), 477–479.
- *On differential relations and on Filippov's concept of differential equations.* Proc. Uppsala 1977 Int. Conf. on Diff. Eq., Uppsala 1977, 109–113.
- *On conditions on right hand sides of differential relations* (with J. Jarník). Čas. pěst. mat. 102 (1977), 334–349.
- *Extension of a Scorza–Dragoni theorem to differential relations and functional differential relations* (with J. Jarník). Comm. math. spec. in honour of W. Orlicz 1978, 147–158.
- *Kneser's theorem for multivalued differential delay equations* (with P. Krbec). Čas. pěst. mat. 104 (1979), 1–8.
- *Sets of solutions of differential relations* (with J. Jarník). Čas. pěst. mat. 106 (1981), 554–568.

Quasiperiodic solutions

- *On a problem in the theory of linear differential equations with quasi-periodic coefficients* (with A. Vencovská). Proc. of the 9th Int. Conf. on Nonlin. Osc., Kiev 1981, Vol. 1, Kiev 1984, 214–217.
- *On linear differential equations with almost periodic coefficients and the property that the unit sphere is invariant* (with A. Vencovská). Proc. Int. Conf. Equadiff 82, Lecture Notes in Math. 1017, Springer Verlag 1983, 364–368.
- *Linear differential equations with quasiperiodic coefficients* (with A. Vencovská). Czechoslovak Mathematical Journal 37 (1987), 424–470.

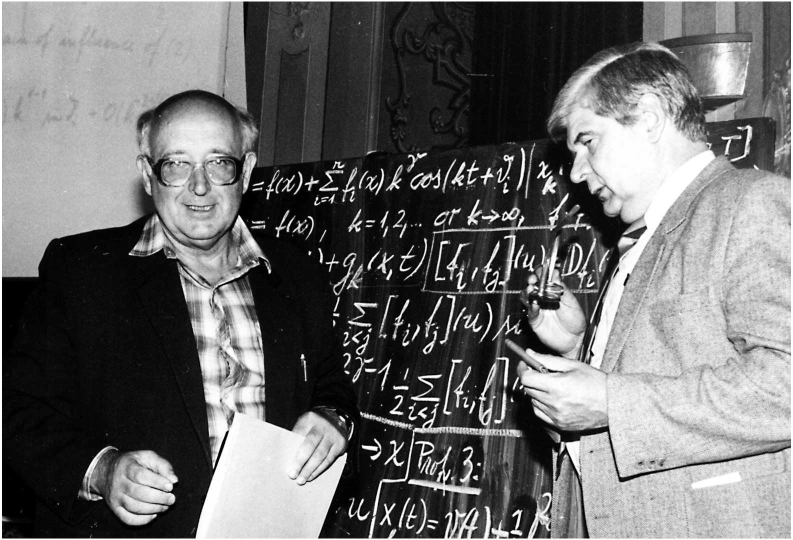
Eighties

Quasiperiodic solutions

- *On a problem in the theory of linear differential equations with quasi-periodic coefficients* (with A. Vencovská). Proc. of the 9th Int. Conf. on Nonlin. Osc., Kiev 1981, Vol. 1, Kiev 1984, 214–217.
- *On linear differential equations with almost periodic coefficients and the property that the unit sphere is invariant* (with A. Vencovská). Proc. Int. Conf. Equadiff 82, Lecture Notes in Math. 1017, Springer Verlag 1983, 364–368.
- *Linear differential equations with quasiperiodic coefficients* (with A. Vencovská). Czechoslovak Mathematical Journal 37 (1987), 424–470.

Integration on multidimensional domains

- *On Mawhin's approach to multiple nonabsolutely convergent integrals* (with J. Jarník and Š. Schwabik). Čas. pěst. mat. L08 (1983), 356–380.
- *The integral as a limit of integral sums.* Jahrbuch Überblicke Mathematik 1984. Bibliographisches Inst. AG 1984, 105–136.
- *A Non-absolutely convergent integral which admits C^1–transformations* (with J Jarník). Čas. pěst. mat. l09 (1984), 157–167.
- *A non-absolutely convergent integral which admits transformation and can be used for integration on manifold* (with J. Jarník). Czechoslovak Math. Journal 35 (110) (1985), 116–139.
- *On regularization of right hand sides of differential relations* (with J. Jarník). Proc. Royal Soc. Edinburgh. 97A (1984), 151–159.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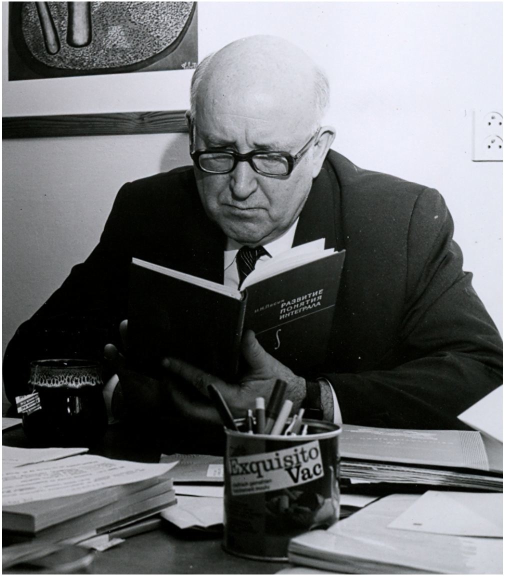
И.Н.Песин
РАЗВИТИЕ
ПОНЯТИЯ
ИНТЕГРАЛА
Exquisito
Vac

PU-integral

- f have a compact support $\operatorname{supp} f$
- Finite system Δ of pairs $(x^j, \theta_j), j \in \{1, 2, \dots, k\}$ is a **PU-partition** if θ_j are functions of class C^1 with compact supports,
$$0 \le \theta_j(x) \le 1 \text{ and } \operatorname{Int}\{x \in \mathbb{R}^n \colon \sum_{j=1}^{k} \theta_j(x) = 1\} \supset \operatorname{supp} f.$$
- For a gauge δ, the PU-partition (x^j, θ_j) is δ-fine if $\operatorname{supp} \theta_j \subset B(x^j, \delta(x^j))$ for all j.
- $$S(f, \Delta) = \sum_{j=1}^{k} f(x^j) \int \theta_j(x) \mathrm{d}\, x$$

$$(\mathrm{PU}) \int f = q \iff \forall \varepsilon > 0\ \exists \delta \colon |q - S(f, \Delta)| < \varepsilon\ \forall \delta - \text{fine } \Delta.$$

- *A new and more powerful concept of the PU–integral* (with J. Jarník). Czechoslovak Mathematical Journal 38 (1988), 8–48.
- *An integral defined by approximating BV partitions of unity* (with J. Mawhin and W. Pfeffer). Czechoslovak Mathematical Journal 41 (1991), 695–712.


Retsina

1991–2004 (Further contributions to the integration theory)

- *Equi-integrability and controlled convergence of Perron–type integrable functions* (with J. Jarník). Real Analysis Exchange 17 (1991–92), 110–139.
- *Differentiability and integrability in n dimensions with respect to α–regular intervals* (with J. Jarník). Results in Mathematics 21 (1992), 138–151.
- *Equivalent definitions of regular generalized Perron integral* (with J. Jarník). Czechoslovak Mathematical Journal 42 (1992), 365–378.
- *Generalized multidimensional Perron integral involving a new regularity condition* (with J. Jarník). Results in Mathematics 23 (1993), 363–373.
- *Perron type integration on n-dimensional intervals as an extension of integration of step functions by strong equiconvergence* (with J. Jarník). Czechoslovak Mathematical Journal 46 (121) (1996), 1–20.
- *A convergence theorem for Henstock-Kurzweil integral and its relation to topology* (with J. Jarník). Bull. Acad. Royal de Belgique, Classe de Sci., 7–12 (1997), 217–223.
- *Convergence cannot be replaced by a locally convex topology* (with J. Jarník). Bulletin de la Classe des Sciences. Acad. Royale Belgique, IX (1998), 331–347.
- *A nonexistence result for the Kurzweil integral* (with P. Krejčí). Mathematica Bohemica, 127 (2002), 571–580.
- *The revival of the Riemannian approach to integration.* Banach Center Publications, Vol. 64 (Orlicz Centenary Volume), Warszawa 2004, 147–158.
- *McShane equi-integrability and Vitali's convergence theorem* (with Š. Schwabik). Mathematica Bohemica, 129 (2004), 141–157.
- *On McShane integrability of Banach space-valued functions* (with Š. Schwabik). Real Analysis Exchange 29(2) (2003/2004), 763–780.

Awards 1978–2006

- 1978 - Elected honorary foreign member of the Royal Society of Edinburgh.
- 1989 - Elected regular member of the Czechoslovak Academy of Sciences.
- 1990 - Elected and appointed Director of the Mathematical Institute, Czechoslovak Academy of Sciences in Prague (he served in this position till 1996).
- 1994 - Member of Learned Society of the Czech Republic (Founding member).
- 1996 - Elected foreign member of the Belgian Royal Academy of Sciences. Awarded the honorary medal "DE SCIENTIA ET HUMANITATE OPTIME MERITIS" of the Academy of Sciences of the Czech Republic. President of the Union of Czech Mathematicians and Physicists (till 2002).
- 1997 - Awarded the State Decoration of the Czech Republic "Medal of Merit (First Grade)" for meritorious service to the state.
- 2006 - Awarded the National Prize of the Government of the Czech Republic “Czech Brain”.

Topology on spaces of integrable functions and new approach to GODEs

- *Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces.* World Scientific, Singapore, 2000.
- *Integration between the Lebesgue Integral and the Henstock-Kurzweil Integral. Its Relation to Local Convex Vector Spaces.* World Scientific, Singapore, 2002.
- *Generalized ordinary differential equations (Not Absolutely Continuous Solutions)*. Series in Real Analysis–Vol. 11, World Scientific, Singapore, 2012.


δ-fine

My personal homage to Jaroslav Kurzweil

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MANY THANKS JAROSLAV !!!
and
strong health, happiness in personal life and the pleasure from new mathematical results !!!


Jaroslav
KURZWEIL
Czech Republic