# On the complexity of addition

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# Addition algorithms

1 Addition algorithms

2 Amortized analysis

# Computational complexity of arithmetic

Time complexity of integer arithmetic operations:

- Standard computational complexity model:
  - multitape Turing machines
    (RAM model has $+$ built in $\implies$ trivial cheat)
  - integers $X$, $Y$, $\dots$ written in binary (or decimal)
  - how many steps does it take, measured in terms of the size of input: $n = \|X\| + \|Y\| + \cdots$, $\|X\| \approx \log X$
- $X + Y$, $X - Y$, $X < Y$
  - linear time $O(n)$
  - optimal: need to read the input
- $X \cdot Y$, $\lfloor X/Y \rfloor$
  - still not quite settled after many decades of research
  - best known upper bound: $O(n \log n)$ [HvdH21]
  - lower bounds?

    (network coding conjecture $\implies$ circuit LB: $\Omega(n \log n)$ wires [ACKL19])

# School-book addition algorithm

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Input tape 0: | ⋯ | | | | | | | 1 | 1 | 0 | 1 |
| Input tape 1: | ⋯ | | | | | 1 | 1 | 0 | 1 | 0 | 1 |
| Output tape: | ⋯ | | | | | | | | | | |

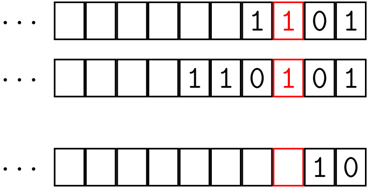
State: carry 0

# School-book addition algorithm

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Input tape 0: | ··· | | | | | | | 1 | 1 | 0 | 1 |
| Input tape 1: | ··· | | | | | 1 | 1 | 0 | 1 | 0 | 1 |
| Output tape: | ··· | | | | | | | | | | 0 |

State: carry 1

# School-book addition algorithm



Input tape 0: ⋯ | | | | | | 1 | 1 | 0 | 1

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Output tape: ⋯ | | | | | | | | 1 | 0

State: carry 0

# School-book addition algorithm

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Input tape 0: | ⋯ | | | | | | | 1 | 1 | 0 | 1 |
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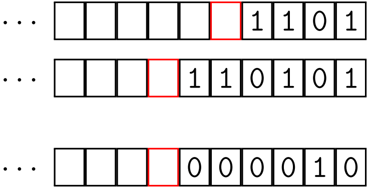

Input tape 0: ··· 1 1 0 1

Input tape 1: ··· 1 1 0 1 0 1

Output tape: ··· 0 0 0 0 1 0

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State: halt

# Sequence sum

What if we want to add more than two numbers?

$\textsc{SeqSum}$

- input: sequence of integers $\langle X_i : i < k \rangle$ separated with "+"
- output: $\sum_{i<k} X_i$

Size of input: $n = k + \sum_{i<k} n_i, \ n_i = \|X_i\|$

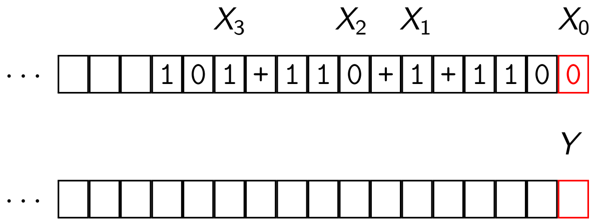
Question:

- What is the time complexity of $\textsc{SeqSum}$?
- Can we do it in time $O(n)$?

# Simple SeqSum algorithm

Use one tape as an accumulator $Y$:

```
$Y \leftarrow 0$
for $i < k$ do:
    $Y \leftarrow Y + X_i$
```



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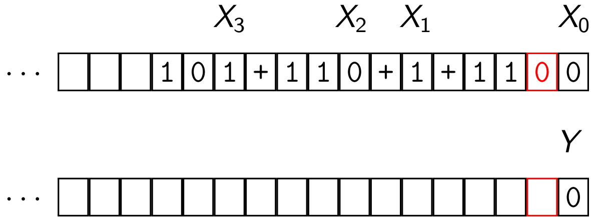

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: $\cdots$ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |

$Y$

Output tape: $\cdots$ | | | | | | | | | | | | | | | | 0 |

State: carry 0

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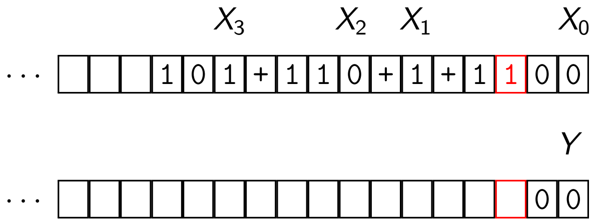

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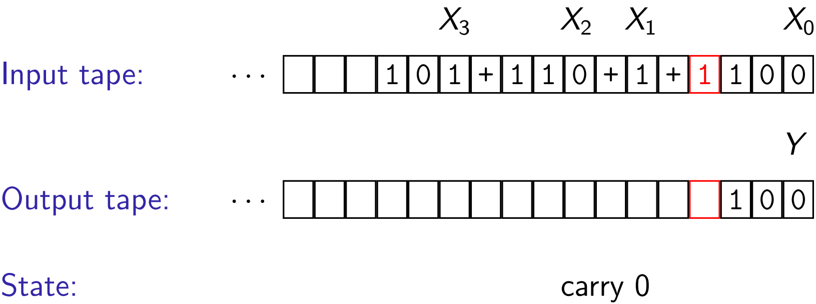

$X_3$ $X_2$ $X_1$ $X_0$

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$Y$

Output tape: $\cdots$ 1 0 0

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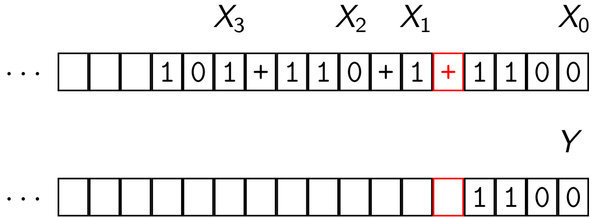

Input tape: · · · | | | |1|0|1|+|1|1|0|+|1|+|1|1|0|0|

Output tape: · · · | | | | | | | | | | | | | |1|1|0|0|

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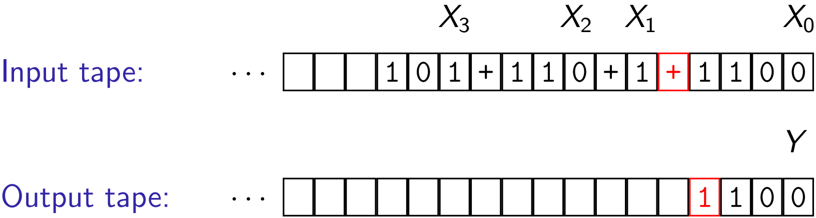

$X_3$ $X_2$ $X_1$ $X_0$

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$Y$

Output tape: $\cdots$ 1 1 0 0

State: rewind

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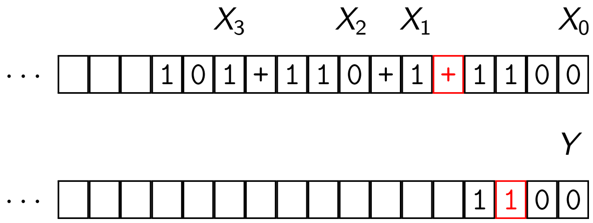

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| Input tape: | ··· | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |
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| Input tape: | $\cdots$ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |
| | | | | | | | | | | | | | | | | | | $Y$ |
| Output tape: | $\cdots$ | | | | | | | | | | | | | | 1 | 1 | 0 | 0 |

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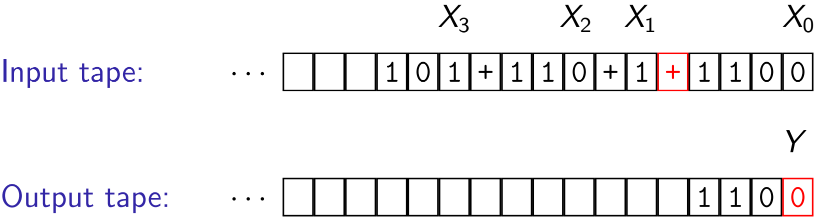

Input tape: $\cdots$ | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0

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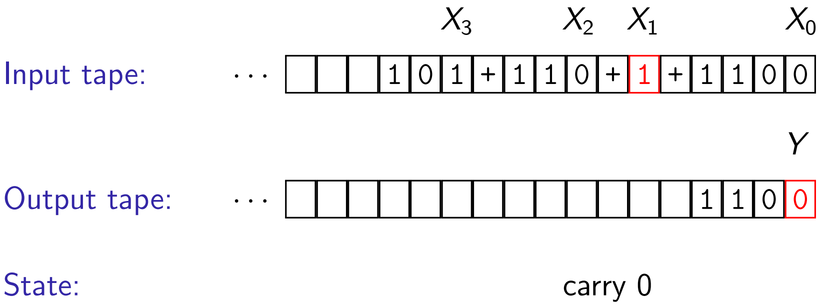

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: $\cdots$ 1 0 1 + 1 1 0 + 1 + 1 1 0 0

$Y$

Output tape: $\cdots$ 1 1 0 0

State: carry 0

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| | | | | | | | $X_3$ | | | | $X_2$ | | $X_1$ | | | | | $X_0$ |
| Input tape: | ⋯ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |
| | | | | | | | | | | | | | | | | | | $Y$ |
| Output tape: | ⋯ | | | | | | | | | | | | | | 1 | 1 | 0 | 1 |

State: carry 0

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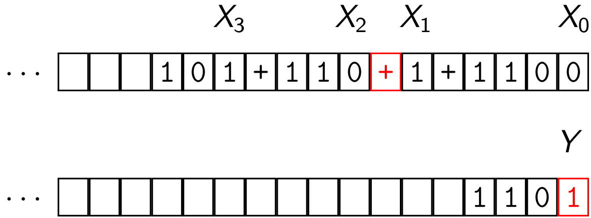

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Input tape: $\cdots$ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |

$Y$

Output tape: $\cdots$ | | | | | | | | | | | | | | 1 | 1 | 0 | 1 |

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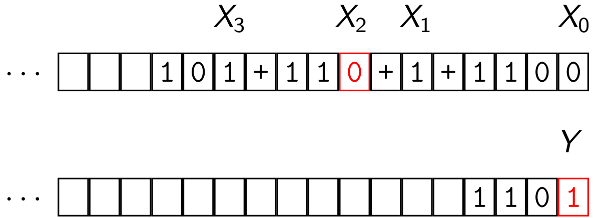

Input tape: $\cdots$ | | | |1|0|1|+|1|1|0|+|1|+|1|1|0|0|

Output tape: $\cdots$ | | | | | | | | | | | | | |1|1|0|1|

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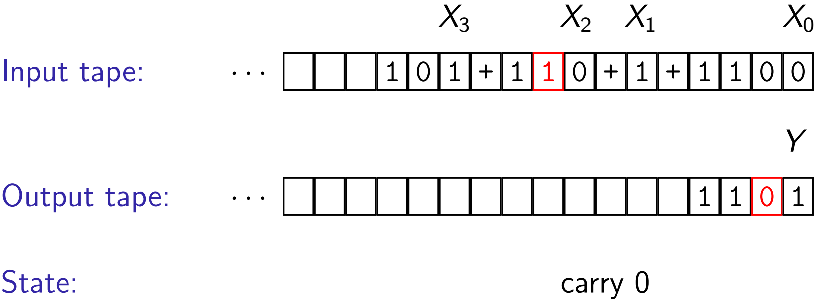

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: $\cdots$ | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |

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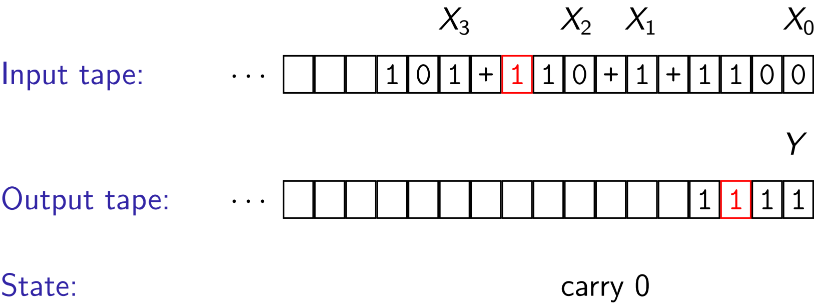

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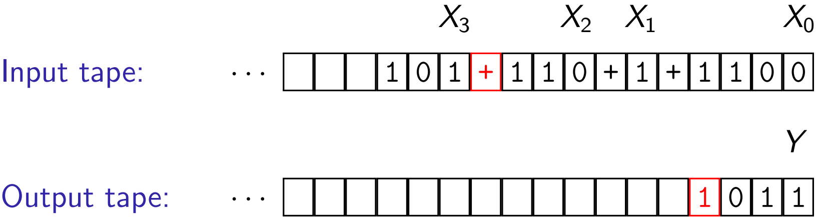

Input tape: ⋯ 1 0 1 + 1 1 0 + 1 + 1 1 0 0

Output tape: ⋯ 1 0 1 1

State: carry 1

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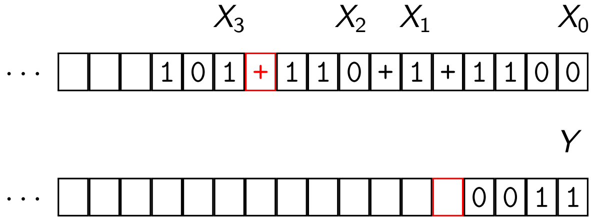

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: ⋯ | | | |1|0|1|+|1|1|0|+|1|+|1|1|0|0|

$Y$

Output tape: ⋯ | | | | | | | | | | | | | |0|0|1|1|

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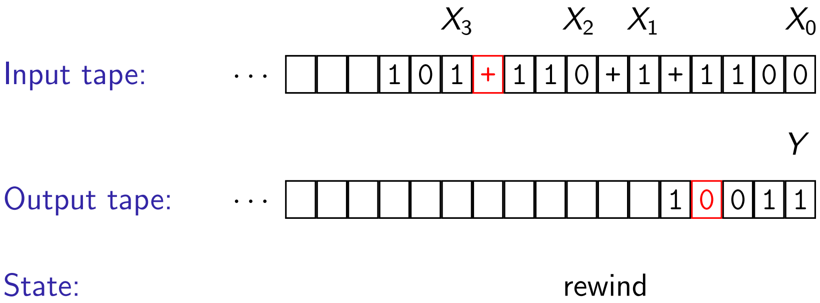

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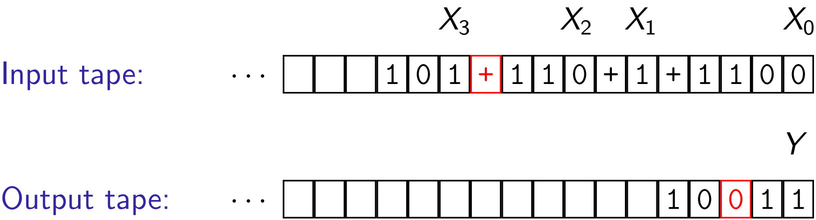

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: ⋯ □ □ □ 1 0 1 + 1 1 0 + 1 + 1 1 0 0

$Y$

Output tape: ⋯ □ □ □ □ □ □ □ □ □ □ □ □ 1 0 0 1 1

State: rewind

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| | | | | | | | | | | | | | | | | | | |
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| Input tape: | ⋯ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |
| | | | | | | | | | | | | | | | | | | $Y$ |
| Output tape: | ⋯ | | | | | | | | | | | | | 1 | 0 | 0 | 1 | 1 |

State: rewind

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```

Input tape: $\cdots$

| | | | | | $X_3$ | | | | $X_2$ | | $X_1$ | | | | | $X_0$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |

Output tape: $\cdots$

| | | | | | | | | | | | | | | | | $Y$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | 1 | 0 | 0 | 1 | 1 |

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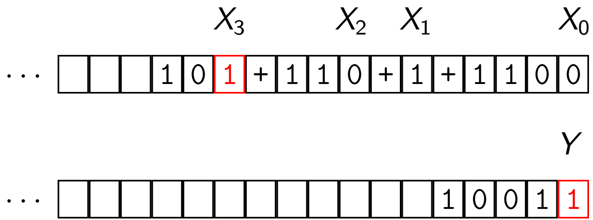

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: ⋯

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  |  |  | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |

$Y$

Output tape: ⋯

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
|  |  |  |  |  |  |  |  |  |  |  |  | 1 | 0 | 0 | 1 | 1 |

State: carry 0

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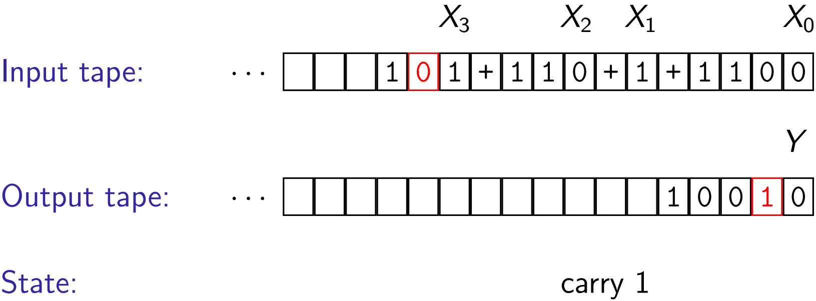

$X_3$ $X_2$ $X_1$ $X_0$

Input tape: $\cdots$ 1 0 1 + 1 1 0 + 1 + 1 1 0 0

$Y$

Output tape: $\cdots$ 1 0 0 1 0

State: carry 1

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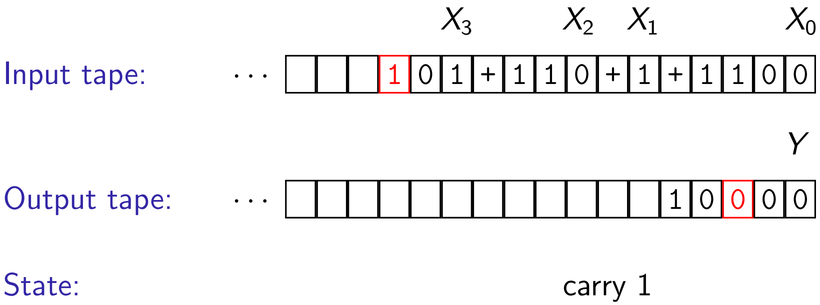

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Input tape: $\cdots$ 1 0 1 + 1 1 0 + 1 + 1 1 0 0

$Y$

Output tape: $\cdots$ 1 0 0 0 0

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| | | | | | | | | | | | | | | | | | | |
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| | | | | | | | $X_3$ | | | | $X_2$ | | $X_1$ | | | | | $X_0$ |
| Input tape: | $\cdots$ | | | | 1 | 0 | 1 | + | 1 | 1 | 0 | + | 1 | + | 1 | 1 | 0 | 0 |
| | | | | | | | | | | | | | | | | | | $Y$ |
| Output tape: | $\cdots$ | | | | | | | | | | | | | 1 | 0 | 0 | 0 | 0 |

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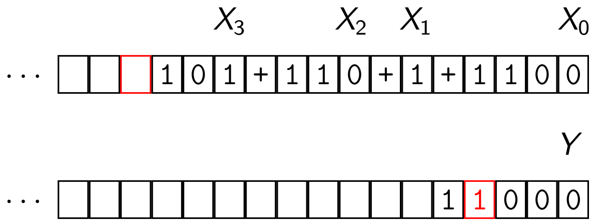

State: halt

# Time complexity analysis

The content of $Y$ before adding $X_i$: $Y_i = \sum_{j<i} X_j$, $m_i = \|Y_i\|$

$Y \leftarrow Y + X_i$ takes time $O(n_i + m_i) \subseteq O(n)$ as $m_i \le n$

$\implies$ total time: $O(nk) \subseteq O(n^2)$

- even if $n_i < m_i$, $Y \leftarrow Y + X_i$ may take time up to $\approx m_i$ due to carry propagation
- we may have $m_i = \Omega(n)$ for all $i > 0$, and $k = \Omega(n)$: take $\text{huge } X_0$ and constant-size $X_1, \dots, X_{k-1}$, $k \approx \|X_0\|$

# The complexity of SeqSum

SeqSum is computable in time $O(n^2)$

Question: Can we do better?

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Question: Can we do better?

Answer:

- Yes, we can! SeqSum is computable in time $O(n)$
- We don’t even need a better algorithm:
  we just need a better analysis!

# Amortized analysis

1. Addition algorithms

2. Amortized analysis

# Amortized complexity

Identified as a concept and named by [Tar85]

- If an operation is used many times in an algorithm, it may happen that its average (amortized) cost is smaller than its maximal cost
- NOT average-case analysis: still worst-case wrt the input!
- Typical use case: data structures
- Example: stack implemented by an array
  - when capacity exhausted, reallocate double size and copy
  - algorithm performs $n$ stack operations (`push`, `pop`)
    $\Longrightarrow$ each operation may cost up to $O(n)$ steps
  - but: the average cost is only $O(1)$!
    total cost of reallocations is $O(n + \frac{n}{2} + \frac{n}{4} + \cdots) = O(n)$
- Basic strategies: aggregate analysis, accounting method, potential method

# Binary counter

Basic example (see e.g. [CLRS22]):

Counter

- holds an integer in binary
- starts with 0, performs $n$ increments
  $0 \to 1 \to \cdots \to n$

Cost of an increment:

- maximal $O(\log n)$: carry propagation
- amortized $O(1)$

0
1
10
11
100
101
110
111
1000
1001
1010
1011
1100

# Binary counter

Basic example (see e.g. [CLRS22]):

Counter

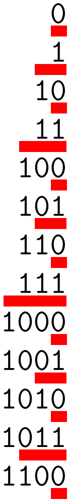
- holds an integer in binary
- starts with 0, performs $n$ increments
  $0 \to 1 \to \cdots \to n$

Cost of an increment:

- maximal $O(\log n)$: carry propagation
- amortized $O(1)$: aggregate analysis

updates: $n \times$ position $0$, $\frac{n}{2} \times$ pos. $1$, $\frac{n}{4} \times$ pos. $2$, $\ldots$

$\implies$ total cost $n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots < 2n$



# Increments → sums?

Counter ≈ accumulator $\textsc{SeqSum}$ algorithm for $1 + 1 + \cdots + 1$

Can we generalize the amortized analysis to the full algorithm?

- direct aggregate analysis not easy
- accounting method:
  - pay the cost of excess carries from "credits" saved earlier
- potential method:
  - define "potential energy" of TM configurations
  - changes of the potential account for work on carries

# Improved analysis of SeqSum

Recall: input $\langle X_i : i < k\rangle$, $n_i = \|X_i\|$, $n = k + \sum_{i<k} n_i$

The cost of one addition $Y \leftarrow Y + X_i$:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_i$: | | | | | | | | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| old $Y$: | | | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| new $Y$: | | | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| | | | | | | | | | | | $n_i$ | | | |

- regular costs: $n_i + 1 \implies$ total: $k + \sum_{i<k} n_i = n$
- paid from credit: carries $1 \to 0$
  - the "1" got there by a regular change $0 \to 1$ earlier!
    $\implies$ cover all credits by charging regular costs twice
- grand total: $2n$ (actually $4n$ due to rewinds)

# Potential method

Potential function $\Phi$ = the number of 1s in $Y$

$\Phi_i$ = the value of $\Phi$ before the addition $Y \leftarrow Y + X_i$

By the same argument: the cost of $Y \leftarrow Y + X_i$ is at most

$$2(n_i + 1) + \Phi_i - \Phi_{i+1}$$

Since $\Phi_0 = 0$ and $\Phi_k \geq 0$, the total cost is at most

$$\sum_{i<k} \bigl(2(n_i + 1) + \Phi_i - \Phi_{i+1}\bigr) = 2\Bigl(k + \sum_{i<k} n_i\Bigr) + \Phi_0 - \Phi_k \leq 2n$$

# Summary

Computational complexity of $\sum_{i<k} X_i$:

- the obvious algorithm appears to require time $O(n^2)$ on the first sight
- it actually runs in time $4n$
- extension of a common example in amortized complexity
- seems to be missing in standard literature,
  even though it is a fundamental algorithmic problem

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