Admissible rules and their complexity

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24th Applications of Logic in Philosophy and the Foundations of Mathematics
Szklarska Poręba, May 2019

Outline of the talks

Logics and admissibility

Propositional logics

Propositional logic L:

Language: formulas built from atoms $x_0, x_1, x_2, \dots$ using a fixed set of finitary connectives

Consequence relation: a relation $\Gamma \vdash_L \varphi$ between sets of formulas and formulas s.t.

- $\varphi \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\Gamma, \Delta \vdash_L \varphi$
- $\Gamma, \Delta \vdash_L \varphi$ and $\forall \psi \in \Delta\ \Gamma \vdash_L \psi$ imply $\Gamma \vdash_L \varphi$
- $\Gamma \vdash_L \varphi$ implies $\sigma(\Gamma) \vdash_L \sigma(\varphi)$ for every substitution σ

Unifiers and admissible rules

Γ, Δ: finite sets of formulas

L-unifier of Γ: substitution σ s.t. $\vdash_L \sigma(\varphi)$ for all $\varphi \in \Gamma$

Single-conclusion rule: $\Gamma \,/\, \varphi$

Multiple-conclusion rule: $\Gamma \,/\, \Delta$

- $\Gamma \,/\, \Delta$ is *L*-derivable (or valid) if $\Gamma \vdash_L \delta$ for some $\delta \in \Delta$
- $\Gamma \,/\, \Delta$ is *L*-admissible (written as $\Gamma \mathrel{|\!\!\sim}_L \Delta$)
  if every *L*-unifier of Γ also unifies some $\delta \in \Delta$

NB: Γ is *L*-unifiable iff $\Gamma \mathrel{|\!\!\not\sim}_L \varnothing$

Examples

- **CPC**: admissible = derivable (structural completeness)
- **IPC** and intermediate logics admit Kreisel–Putnam rule:

$$\neg x \to y \vee z \mathrel{|\!\!\sim} (\neg x \to y) \vee (\neg x \to z)$$

- $\Box x \;/\; x$ admissible in **K**, **K4**, derivable in **KT**, **S4**
- Löb's rule $\Box x \to x \;/\; x$ admissible in **K**, derivable in **GL**
- $\Diamond x \wedge \Diamond \neg x \;/\; \bot$ admissible in all normal modal logics
- $\bot \mathrel{|\!\!\sim}_L \varnothing$ iff L is consistent
- L has the (modal) disjunction property iff

$$\Box x_1 \vee \cdots \vee \Box x_n \mathrel{|\!\!\sim}_L x_1, \dots, x_n \qquad (n \geq 0)$$

- Rule of margins $x \to \Box x \;/\; x, \neg x$ admissible in **KT**, **KTB**

Basic questions

What rules are L-admissible?

- NB: $\mathrel{|\!\!\sim}_L$ forms a (multiple-conclusion) consequence relation
- Semantic characterization of $\mathrel{|\!\!\sim}_L$ by a class of models (algebras, Kripke models, ...)
- Syntactic presentation of $\mathrel{|\!\!\sim}_L$:
  - Basis of admissible rules = axiomatization of $\mathrel{|\!\!\sim}_L$ over $\vdash_L$
  - Can we describe an explicit basis?
  - Are there finite bases? Independent bases?

How to check $\Gamma \mathrel{|\!\!\sim}_L \Delta$?

- Is admissibility algorithmically decidable?
- What is its computational complexity?

Algebraizable logics

L a logic, K a class of algebras (quasivariety)

L is (finitely) algebraizable wrt K if there are

- formulas $E(x, y) = \{\varepsilon_1(x, y), \dots, \varepsilon_n(x, y)\}$
- equations $T(x) = \{t_1(x) \approx s_1(x), \dots, t_m(x) \approx s_m(x)\}$

such that

- $\Gamma \vdash_L \varphi \Leftrightarrow T(\Gamma) \vDash_K T(\varphi)$
- $\Sigma \vDash_K t \approx s \Leftrightarrow E(\Sigma) \vdash_L E(t, s)$
- $x \dashv\vdash_L E(T(x))$
- $x \approx y \mathrel{=\!\mid\!\vDash}_K T(E(x, y))$

In modal logic: $T(x) = \{x \approx 1\}$, $E(x, y) = \{x \leftrightarrow y\}$,
K is a variety of modal algebras

Elementary equational unification

Θ: equational theory (or a class of algebras)

$\Sigma = \{t_1 \approx s_1, \dots, t_n \approx s_n\}$ finite set of equations

Θ-unifier of Σ: a substitution σ s.t.

$$\sigma(t_1) =_\Theta \sigma(s_1), \dots, \sigma(t_n) =_\Theta \sigma(s_n)$$

$U_\Theta(\Sigma)$ = set of Θ-unifiers of Σ

If L is a logic algebraizable wrt a quasivariety K:

- L-unifier of φ = K-unifier of $T(\varphi)$
- K-unifier of $t \approx s$ = L-unifier of $E(t, s)$

Properties of unifiers

Preorder on substitutions:
σ more general than τ ($\sigma \preceq_\Theta \tau$) if $\exists \upsilon\ \upsilon \circ \sigma =_\Theta \tau$

Complete set of unifiers (csu) of Σ: $S \subseteq U_\Theta(\Sigma)$ s.t.
$\forall \tau \in U_\Theta(\Sigma)\ \exists \sigma \in S\ (\sigma \preceq_\Theta \tau)$

Most general unifier (mgu) of Σ: σ s.t. $\{\sigma\}$ csu

Basic questions:

- Is Σ unifiable?
- Does every Σ a finite csu? Or even mgu (if unifiable)?
- Is it decidable if Σ is unifiable? Can we compute a csu?
- What is the computational complexity?

Rules $\to$ algebraic clauses

L logic algebraizable wrt a quasivarety K

For simplicity: assume $|E(x,y)| = |T(x)| = 1$

Clause: (universally quantified) disjunction of atomic (= equations) and negated atomic formulas

Quasi-identity: clause with 1 positive literal

Rule $\Gamma \,/\, \Delta$ translates to a clause $T(\Gamma \,/\, \Delta)$:

$$\bigwedge_{\varphi\in\Gamma} T(\varphi) \to \bigvee_{\psi\in\Delta} T(\psi)$$

$\Gamma \,/\, \Delta$ single-conclusion rule $\implies$ $T(\Gamma \,/\, \Delta)$ quasi-identity

Clauses $\to$ rules

Conversely: clause $C = \bigwedge_{i<n} t_i \approx t'_i \to \bigvee_{j<m} s_j \approx s'_j$ translates to a rule $E(C)$:

$$\{E(t_i, t'_i) : i < n\} \;/\; \{E(s_j, s'_j) : j < m\}$$

C quasi-identity $\implies$ $E(C)$ single-conclusion rule

- $(\Gamma \;/\; \Delta) \dashv\vdash_L E(T(\Gamma \;/\; \Delta))$
- $C \;=\!\!\!|\!\!\models_K T(E(C))$

(abusing the notation)

Admissible rules algebraically

Derivability:

- Single-concl. rules $\iff$ quasiequational theory of K
- Multiple-concl. rules $\iff$ clausal/universal theory of K

$$\Gamma \vdash_L \Delta \iff T(\Gamma/\Delta) \text{ holds in all } K\text{-algebras}$$

Admissibility:

$$\begin{aligned}\Gamma \mathrel{|\!\!\sim}_L \Delta &\iff T(\Gamma/\Delta) \text{ holds in free } K\text{-algebras}\\ &\iff F_K(\omega) \vDash T(\Gamma/\Delta)\\ &\iff F_K(n) \vDash T(\Gamma/\Delta) \text{ for all } n \in \omega\end{aligned}$$

Parameters

In applications, propositional atoms model both "variables" and "constants"

We don't want substitution for constants

Example (description logic):

- (i) $\forall \mathrm{child}.(\neg \mathrm{HasSon} \sqcap \exists \mathrm{spouse}.\top)$
- (ii) $\forall \mathrm{child}.\forall \mathrm{child}.\neg \mathrm{Male} \sqcap \forall \mathrm{child}.\mathrm{Married}$
- (iii) $\forall \mathrm{child}.\forall \mathrm{child}.\neg \mathrm{Female} \sqcap \forall \mathrm{child}.\mathrm{Married}$

Good: Unify (i) with (ii) by $\mathrm{HasSon} \mapsto \exists \mathrm{child}.\mathrm{Male}$, $\mathrm{Married} \mapsto \exists \mathrm{spouse}.\top$

Bad: Unify (ii) with (iii) by $\mathrm{Male} \mapsto \mathrm{Female}$

Admissibility with parameters

In unification theory, it is customary to consider unification with unconstrained constants

We consider setup with two kinds of atoms:

- variables $x_0, x_1, x_2, \dots \in \mathrm{Var}$ (countable infinite set)
- parameters (constants) $p_0, p_1, p_2, \dots \in \mathrm{Par}$ (countable, possibly finite)

Substitutions only modify variables, we require $\sigma(p_n) = p_n$

Adapt accordingly other notions:

- L-unifier, L-admissible rule, ...

Exception: logics are always assumed to be closed under substitution for parameters

Parameters as signature expansion

Admissibility/unification with parameters in L
$\iff$ plain admissibility/unification in L^{Par}:

- language expanded with nullary connectives $p \in \mathrm{Par}$
- $\vdash_{L^{\mathrm{Par}}}$ = least consequence relation that contains $\vdash_L$

L algebraizable wrt $K \implies L^{\mathrm{Par}}$ algebraizable wrt K^{Par}:

- arbitrary expansions of K-algebras with the new constants

L-admissibility with parameters
$\iff$ validity in free K^{Par}-algebras

NB: $|\mathrm{Par}| = m \implies$

$F_{K^{\mathrm{Par}}}(n) \simeq F_K(n+m)$ with fixed valuation of m generators

Transitive modal logics

Transitive modal logics

We consider axiomatic extensions of the logic **K4**:

- Language: Boolean connectives, $\Box$
- Consequence relation:
  - axioms of **CPC**
  - $\varphi, \varphi \to \psi \vdash \psi$
  - $\vdash \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
  - $\vdash \Box\varphi \to \Box\Box\varphi$
  - $\varphi \vdash \Box\varphi$

Algebraizable wrt the variety of **K4**-algebras:
Boolean algebras with operator $\Box$ satisfying $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box a \leq \Box\Box a$

Frame semantics

Kripke frames: $\langle W, < \rangle$, $< \subseteq W \times W$ transitive

$\implies$ dual **K4**-algebra $\langle \mathcal{P}(W), \Box \rangle$, $\Box X = W \smallsetminus (W \smallsetminus X){\downarrow}$

General frames: $\langle W, <, A \rangle$, A subalgebra of $\langle \mathcal{P}(W), \Box \rangle$

$\implies$ dual **K4**-algebra A

Back: **K4**-algebra $A \implies$ dual frame $\langle \mathrm{St}(A), <, \mathrm{CO}(\mathrm{St}(A)) \rangle$

duals of **K4**-algebras $\simeq$ descriptive frames

We will use frame semantics as it is more convenient, but the general algebraic theory still applies

Convention: frame = general frame,
but finite frame = finite Kripke frame

Notation & terminology

$\langle W, <\rangle$ transitive frame, $u, v \in W$

- u reflexive $\iff u < u$, otherwise irreflexive
- $u \le v \iff u < v$ or $u = v$ preorder
- $u \sim v \iff u \le v$ and $v \le u$ equivalence relation
  equivalence classes = clusters:
  - reflexive/irreflexive
  - proper: size ≥ 2 ($\implies$ reflexive)
  - $\mathrm{cl}(u)$ = the cluster containing u
- $u \lnsim v \iff u < v$ and $v \not< u$ strict order
- $X{\downarrow} = \{u : \exists v \in X\ u < v\}$, $X{\downarrow\!\!\!\!-} = \{\dots u \le v\}$, $X{\uparrow}$, $X{\uparrow\!\!\!\!-}$
- W rooted if $W = r{\uparrow\!\!\!\!-}$ for some $r \in W$
  $\mathrm{rcl}(W) = \mathrm{cl}(r)$ root cluster

Examples of transitive logics

logic	axiom (on top of **K4**)	finite rooted frames
S4	$\Box x \to x$	reflexive
D4	$\Diamond\top$	final clusters reflexive
GL	$\Box(\Box x \to x) \to \Box x$	irreflexive
K4Grz	$\Box(\Box(x \to \Box x) \to x) \to \Box x$	no proper clusters
K4.1	$\boxdot\Diamond x \to \Diamond\!\!\!\cdot\,\Box x$	no proper final clusters
K4.2	$\Diamond\boxdot x \to \Box\Diamond\!\!\!\cdot\, x$	unique final cluster
K4.3	$\Box(\boxdot x \to y) \lor \Box(\Box y \to x)$	linear (chain of clusters)
K4B	$x \to \Box\Diamond x$	lone cluster
S5	$= \mathbf{S4} \oplus \mathbf{B}$	lone reflexive cluster

and their various combinations

Shorthands: $\Diamond\varphi = \neg\Box\neg\varphi$, $\boxdot\varphi = \varphi \land \Box\varphi$, $\Diamond\!\!\!\cdot\,\varphi = \neg\boxdot\neg\varphi$

Frame measures

A frame $\langle W, <, A\rangle$ has various invariants in $\mathbb{N} \cup \{\infty\}$:

- depth = maximal length of strict chains
- cluster size = maximal size of clusters
- width = maximal size of antichains in rooted subframes
- branching = maximal number of immediate successor clusters of any point

A logic L has depth (cl. size, width) $\leq k$
$\iff$ all descriptive L-frames have depth (cl. size, width) $\leq k$
$\iff$ $L \supseteq \mathbf{K4BD}_k$ ($\mathbf{K4BC}_k$, $\mathbf{K4BW}_k$)

Branching:
more complicated (directly works only for finite frames)
$L \supseteq \mathbf{K4BB}_k$

Frames for rules

$M = \langle W, <, \vDash\rangle$ Kripke model:

- $M \vDash \varphi \iff u \vDash \varphi$ for all $u \in W$
- $M \vDash \Gamma \,/\, \Delta \iff$

$$M \vDash \varphi \text{ for all } \varphi \in \Gamma \implies M \vDash \psi \text{ for some } \psi \in \Delta$$

$\langle W, <, A\rangle$ frame:

$$W \vDash \Gamma \,/\, \Delta \iff \langle W, <, \vDash\rangle \vDash \Gamma \,/\, \Delta \text{ for all admissible } \vDash$$

Validity of rules preserved by p-morphic images, but not by generated subframes

Only single-conclusion rules preserved by disjoint sums

Parametric frames

K4-algebras are dual to frames

$\mathsf{K4}^{\mathrm{Par}}$-algebras are dual to parametric frames $\langle W, <, A, \vDash_{\mathrm{Par}} \rangle$

- $\langle W, <, A \rangle$ frame
- $\vDash_{\mathrm{Par}}$ fixed admissible valuation of parameters $p \in \mathrm{Par}$

Model based on $\langle W, <, A, \vDash_{\mathrm{Par}} \rangle$:
$\langle W, <, \vDash \rangle$ s.t.

- $\vDash$ admissible valuation in the frame $\langle W, <, A \rangle$
- $\vDash$ extends $\vDash_{\mathrm{Par}}$

Canonical frames

Free L-algebras $F_L(V)$ are dual to canonical L-frames $C_L(V)$:

- points: maximal L-consistent subsets of $\mathrm{Form}(V)$
- $X < Y \iff \forall\varphi\,(\Box\varphi \in X \Rightarrow \varphi \in Y)$
- $A =$ definable sets: $\{X : \varphi \in X\}$, $\varphi \in \mathrm{Form}(V)$

Free L^{Par}-algebras $F_{L^{\mathrm{Par}}}(V)$ are dual to canonical parametric frames $C_L(\mathrm{Par}, V)$:

- underlying frame $C_L(\mathrm{Par} \cup V)$
- $X \vDash p \iff p \in X$

Universal frames of finite rank (1)

Canonical frames are too large
But: their top parts have an explicit description

Universal model $M_{\mathbf{K4}}(V)$, $V \subseteq \mathrm{Var}$ finite:

- start with empty model
- for each finite rooted model F with $C = \mathrm{rcl}(F)$: if
  - points of C are distinguished by valuation of V,
  - $F \smallsetminus C$ is a generated submodel of $M_{\mathbf{K4}}(V)$, and
  - $\neg($ $F \smallsetminus C$ is rooted, $\mathrm{rcl}(F \smallsetminus C)$ is reflexive, and includes a copy of C wrt valuation $)$

  then extend $M_{\mathbf{K4}}(V)$ with a copy of C below $F \smallsetminus C$ (unless there already is one)

Universal frames of finite rank (2)

Characterization:
$M_{\mathbf{K4}}(V)$ = unique model with valuation for V s.t.

- $M_{\mathbf{K4}}(V)$ is locally finite
  (= rooted generated submodels are finite)
- each finite model with valuation for V has a unique p-morphism to $M_{\mathbf{K4}}(V)$

Universal frame $U_{\mathbf{K4}}(V)$ = underlying frame of $M_{\mathbf{K4}}(V)$

$P \subseteq \mathrm{Par}$ finite:
Universal parametric frame $U_{\mathbf{K4}}(P, V)$ = underlying frame of $M_{\mathbf{K4}}(P \cup V)$ with its valuation of P

Universal frames of finite rank (3)

Generalization to $L \supseteq \mathbf{K4}$ with finite model property (fmp):

$M_L(V)$ = the part of $M_{\mathbf{K4}}(V)$ that's based on an L-frame

$\implies U_L(V)$, $U_L(P, V)$

Properties:

- all finite subsets of $M_L(P, V)$ definable
- the dual of $U_L(P, V)$ is $F_{L^P}(V)$
- $U_L(P, V)$ is the top part of $C_L(P, V)$:
  - $U_L(P, V)$ generated subframe of $C_L(P, V)$ (the points of finite depth)
  - all remaining points of $C_L(P, V)$ see points of $U_L(P, V)$ of arbitrarily large depth
- all $\neq \varnothing$ admissible subsets of $C_L(P, V)$ intersect $U_L(P, V)$

Admissibility using universal frames

$P \subseteq \mathrm{Par}$ finite, $\Gamma, \Delta \subseteq \mathsf{Form}(P, \mathsf{Var})$ finite, $L \supseteq \mathbf{K4}$ fmp

Summary:

$$\begin{aligned}
\Gamma \mathrel{|\!\!\sim}_L \Delta &\iff \forall V \subseteq \mathsf{Var} \text{ finite: } F_{L^P}(V) \vDash \Gamma \,/\, \Delta \\
&\iff \forall V \subseteq \mathsf{Var} \text{ finite: } C_L(P, V) \vDash \Gamma \,/\, \Delta \\
&\iff \forall V \subseteq \mathsf{Var} \text{ finite: } \langle U_L(P, V), <, D, \vDash_P \rangle \vDash \Gamma \,/\, \Delta
\end{aligned}$$

where D = subsets definable in $M_L(P, V)$

Typically:
Validity in $U_L(P, V)$ is not difficult to characterize,
but the restriction to D seriously complicates it

Toy model: logics of bounded depth

Avoid the difficulties . . .

If L is a logic of bounded depth:

- $C_L(P, V) = U_L(P, V)$
- $C_L(P, V)$ is a finite frame

$\Longrightarrow$ admissibility easy to analyze

Teaser: Let L be a logic of bounded depth. If

- Par is finite, or
- the set of finite L-frames is decidable,

then L-unifiability is decidable.

Proof: $\Gamma \subseteq \mathrm{Form}(P, \mathrm{Var})$ is unifiable iff

$$\exists \models \langle U_L(P, \varnothing), \models \rangle \models \Gamma.$$

We can compute $U_L(P, \varnothing)$. QED

. . . but some remain

The characterization

$$\Gamma \mathrel{|\!\sim}_L \Delta \iff U_L(P,V) \vDash \Gamma \,/\, \Delta \quad \forall V \text{ finite}$$

is not quite useful:

- $U_L(P,V)$ are too rigidly specified
- $U_L(P,V)$ are too large: $\approx 2^{2^{\cdot^{\cdot^{\cdot^{|P \cup V|}}}}}$ (height $\approx$ depth of L)
- we have no control over V, anyway

$\Longrightarrow$ need more convenient semantical description

L-extensible models

L logic of bounded depth, fix $P \subseteq \mathrm{Par}$ finite

F finite rooted parametric L-frame, $C = \mathrm{rcl}(F)$:

- F has loosely separated root if points of C are distinguished by valuation of parameters
- F has separated root if moreover $\neg\big($ $F \setminus C$ is rooted, $\mathrm{rcl}(F \setminus C)$ is reflexive, and includes a copy of C wrt valuation $\big)$

W finite parametric L-frame:

- W is L-extensible if $\forall F$ with a separated root: if $F \setminus \mathrm{rcl}(F) \subseteq\cdot\, W$, it extends to $F \subseteq\cdot\, W$
- W is strongly L-extensible if $\forall F$ with a loosely separated root ...

Extensibility and canonical frames

Example: $C_L(P, \varnothing)$ is the minimal L-extensible frame

More generally:
$C_L(P, V)$ is L-extensible for any finite $V \subseteq \mathrm{Var}$

Converse:
W L-extensible $\implies$ p-morphic image of some $C_L(P, V)$

Corollary: If W L-extensible,

$$\Gamma \mathrel{|\!\!\sim}_L \Delta \implies W \vDash \Gamma \,/\, \Delta$$

for all $\Gamma, \Delta \subseteq \mathrm{Form}(P, \mathrm{Var})$

Injectivity of extensible frames

W finite parametric L-frame

W is L-injective if $\forall$ finite par. L-frames $F_0 \subseteq\cdot\, F_1$:
any p-morphism $F_0 \to W$ extends to a p-morphism $F_1 \to W$

Proposition: The following are equivalent:

- W is L-extensible
- W is L-injective
- W is a retract of some $C_L(P,V)$: there are p-morphisms

$$C_L(P,V) \underset{g}{\overset{f}{\rightleftarrows}} W$$

s.t. $f \circ g = \mathrm{id}_W$

Connections among the properties

Proposition: The following are equivalent:

- W is a p-morphic image of some $C_L(P, V)$
- $\Gamma \mathrel{|\!\!\sim}_L \Delta \implies W \vDash \Gamma \,/\, \Delta$ for all $\Gamma, \Delta \subseteq \mathrm{Form}(P, \mathrm{Var})$

Warning: In general, $C_L(P, V)$ are not strongly L-extensible

strongly L-ext. $\underset{\not\longleftarrow}{\longrightarrow}$ L-ext. $\underset{\not\longleftarrow}{\longrightarrow}$ image of $C_L(P, V)$

Proposition: Any finite par. L-frame is a generated subframe of a strongly L-extensible frame

Corollary: Any L-extensible frame is a retract of a strongly L-extensible frame

Extensibility and admissible rules

Recall: L logic of bounded depth, $P \subseteq \mathrm{Par}$ finite

Theorem: For any $\Gamma, \Delta \subseteq \mathrm{Form}(P, \mathrm{Var})$, TFAE:

- $\Gamma \mathrel{|\!\!\sim}_L \Delta$
- $\Gamma \mathbin{/} \Delta$ holds in all L-extensible frames
- $\Gamma \mathbin{/} \Delta$ holds in all strongly L-extensible frames

L-extensible frames are structurally important

strongly L-extensible frames are simpler to define and a bit more robust to work with

Application

What to do next depends on the logic

Logics of bounded depth can still be quite wild

Tame subclass: logics of bounded depth and width

- finitely axiomatizable
- polynomial-size model property
- frames recognizable in polynomial time

Theorem: Let L be a logic of bounded depth and width, $P \subseteq \mathrm{Par}$ finite and $\Gamma, \Delta \subseteq \mathrm{Form}(P, \mathrm{Var})$ of size n.

If $\Gamma \not\mathrel{|\!\sim}_L \Delta$, then $\Gamma \mathbin{/} \Delta$ fails in a strongly L-extensible model of size at most $\mathrm{poly}(n2^{2^{|P|}})$.

In particular, $\mathrel{|\!\sim}_L$ is decidable.

Addendum: smaller models

For fixed finite P, the models are polynomial-size, but in general doubly-exponential

Let $\Sigma \subseteq \mathrm{Form}$ finite, closed under subformulas

Σ-pruned L-extensible model: Like L-extensible, but when extending with a cluster C, allow it to shrink to a subset if satisfaction of Σ-formulas is preserved

Theorem: Let L be logic of bounded depth and width, $\Gamma \cup \Delta \subseteq \Sigma$. TFAE:

- $\Gamma \mathrel{|\!\sim}_L \Delta$
- $\Gamma \mathbin{/} \Delta$ holds in Σ-pruned L-extensible models of size $2^{O(n^2)}$

Projective formulas

Historical note

Projective formulas introduced by [Ghilardi'00]:

- semantical characterization of projective formulas
- existence of projective approximations for extensible logics
  $\implies$ unification finitary
- parameter-free case only

We generalize it to the setup with parameters

Motivation

In the case of logics of bounded depth, we saw:

Admissibility closely connected to injective L-frames

These are dual to projective L-algebras

Finitely presented projective L-algebras are described by projective formulas:

Definition: φ is L-projective if it has an L-unifier σ s.t.

$$\varphi \vdash_L \sigma(\psi) \leftrightarrow \psi \qquad \forall \psi \in \mathrm{Form}$$

- it suffices to check $\psi \in \mathrm{Var}$
- general algebraizable logics: $x \leftrightarrow y$ stands for $E(x, y)$
- σ is a mgu of φ

Löwenheim substitutions

If $\sigma_1, \dots, \sigma_m$ are substitutions s.t.

$$\varphi \vdash_L \sigma_i(\psi) \leftrightarrow \psi \qquad \forall \psi \in \mathrm{Form}, \qquad (*)$$

then this also holds for $\sigma_m \circ \cdots \circ \sigma_1$

$\implies$ build projective unifier inductively by small steps

Löwenheim subtitutions satisfy $(*)$:

Fix $\varphi \in \mathrm{Form}(P, V)$, where $P \subseteq \mathrm{Par}$ and $V \subseteq \mathrm{Var}$ finite

Let $F = \langle f_x : x \in V \rangle$, each $f_x \colon \mathbf{2}^P \to \mathbf{2}$ Boolean function of the parameters:

$$\theta_{\varphi,F}(x) = (\boxdot\varphi \wedge x) \vee (\neg\boxdot\varphi \wedge f_x(\vec{p}))$$

θ_φ = composition of all $\theta_{\varphi,F}$ (in any order)

Characterization of projectivity

Theorem: Let $L \supseteq \mathbf{K4}$ fmp, $\varphi \in \mathrm{Form}(P, V)$. TFAE:

- φ is projective
- θ_φ^N is a unifier of φ, where $N = (2^{|P|} + 1)|\varphi|$
- φ has the model extension property:

Definition:

- Mod_L = finite rooted L-models
- $F, F' \in \mathrm{Mod}_L$ are variants
  if they only differ in valuation of variables in root cluster
- $M \subseteq \mathrm{Mod}_L$ has the model extension property
  if any $F \in \mathrm{Mod}_L$ has a variant in M
  whenever its proper rooted submodels belong to M
- φ has m.e.p. iff $\mathrm{Mod}_L(\varphi) = \{F \in \mathrm{Mod}_L : F \vDash \varphi\}$ does

Projective approximations

NB: projective formulas π are admissibly saturated:

$$\pi \mathrel{|\!\!\sim}_L \Delta \iff \pi \vdash_L \Delta$$

Π is a projective approximation of a formula φ if

- Π finite set of projective formulas
- $\varphi \mathrel{|\!\!\sim}_L \Pi$
- $\pi \vdash_L \varphi$ for each $\pi \in \Pi$

If φ has a projective approximation Π:

- the set of proj. unifiers of $\pi \in \Pi$ is a finite csu of φ
- $\varphi \mathrel{|\!\!\sim}_L \Delta \iff \pi \vdash_L \Delta$ for all $\pi \in \Pi$

Price: existence of proj. apx. needs strong assumptions on L

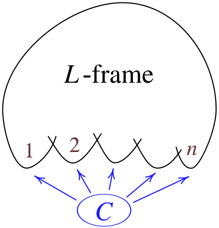
Cluster-extensible logics

$L \supseteq$ **K4** fmp, $n \in \omega$, C finite cluster type:
irreflexive $\bullet$, k-element reflexive ⓚ

A finite rooted frame F is of type $\langle C, n\rangle$ if

- rcl(F) is of type C
- rcl(F) has n immediate successor clusters (= branching n)

L $\langle C, n\rangle$-extensible:
For each type-$\langle C, n\rangle$ frame F, if $F \smallsetminus \mathrm{rcl}(F)$ is an L-frame, then so is F



L cluster-extensible (clx):
$\langle C, n\rangle$-extensible whenever it has some type-$\langle C, n\rangle$ frame

Properties of clx logics

Examples: Any combinations of **K4**, **S4**, **GL**, **D4**, **K4Grz**, **K4.1**, **K4.3**, **K4B**, **S5**, $\mathbf{K4BB}_k$, $\mathbf{K4BC}_k$

Closed under joins and directed intersections
(countable complete lattice)

Nonexamples: **K4.2**, **S4.2**, ...

Theorem: Every clx logic L

- is finitely axiomatizable
- has the exponential-size model property
- is $\forall\exists$-definable on finite frames
- is described by finitely many forbidden types $\langle C, n\rangle$
- is described by finitely many extension conditions:
  $\langle C, n\rangle$ where $n \in \omega \cup \{\infty\}$, C cluster type or $\circledast$

Projective approximation in clx logics

Theorem: L clx logic $\implies$ every formula φ has a projective approximation Π s.t.

- each $\pi \in \Pi$ is a Boolean combination of subformulas of φ
- $|\Pi| \le 2^{2^n}$, $|\pi| = O(n2^n)$ for each $\pi \in \Pi$

Corollary:

- each φ has a finite csu
- we can compute it
- L-admissibility and L-unifiability are decidable

Admissibility in clx logics

Historical note

Admissibility in transitive modal logics studied in depth by [Rybakov'97]

- semantical characterizations
- decidability results
- many results include parameters

We follow a different route, based on Ghilardi's work on projectivity

Tight predecessors

$P \subseteq \mathrm{Par}$ finite, C finite cluster type, $n \in \omega$

L clx logic, W parametric L-frame:

- W is $\langle C, n\rangle$-extensible $\iff$
  $\forall\, E \subseteq \mathbf{2}^P,\ 0 < |E| \leq |C|$
  $\forall\, X = \{w_i : i < n\} \subseteq W$
  $\exists$ tight predecessor (tp) $T = \{u_e : e \in E\} \subseteq W$:

$$u_e \vDash P^e, \qquad u_e{\uparrow} = \begin{cases} X{\Uparrow} & C = \bullet \\ X{\Uparrow} \cup T & C \text{ reflexive} \end{cases}$$

- W is L-extensible if it is $\langle C, n\rangle$-extensible whenever L is

Extension rules

$P \subseteq \mathrm{Par}$ finite, C finite cluster type, $n \in \omega$

$\langle C, n\rangle$-extensible frames axiomatized by extension rules $\mathrm{Ext}^P_{C,n}$:

- $C = \bullet$: for each $e \in \mathbf{2}^P$,

$$P^e \wedge \Box y \to \bigvee_{i<n} \Box x_i \Big/ \{\boxdot y \to x_i : i < n\}$$

- $C = \textcircled{k}$: for each $E \subseteq \mathbf{2}^P$ and $e_0 \in E$, where $|E| \le k$,

$$\frac{\left[\begin{array}{c} P^{e_0} \wedge \boxdot\Big(y \to \bigvee_{e\in E} \Box(P^e \to y)\Big) \\ \wedge \bigwedge_{e\in E} \boxdot\big(\Box(P^e \to \Box y) \to y\big) \end{array}\right] \to \bigvee_{i<n} \Box x_i}{\{\boxdot y \to x_i : i < n\}}$$

Semantics of extension rules

Theorem: Let W be a parametric frame

- If W is $\langle C, n\rangle$-extensible, then $W \vDash \mathrm{Ext}^P_{C,n}$
- If $W \vDash \mathrm{Ext}^P_{C,n}$, then W is $\langle C, n\rangle$-extensible, provided W is descriptive or Kripke

Moreover: If L has fmp, then $L + \mathrm{Ext}^P_{C,n}$ is complete wrt locally finite $\langle C, n\rangle$-extensible Kripke frames

Theorem: If $L \supseteq \mathbf{K4}$ has fmp, TFAE:

- L is $\langle C, n\rangle$-extensible
- $\mathrm{Ext}_{C,n}$ is L-admissible

Characterization of admissibility

Theorem: If L is clx logic, TFAE:

- $\Gamma \mathrel{|\!\sim}_L \Delta$
- $\Gamma \,/\, \Delta$ holds in all L-extensible parametric frames
- $\Gamma \,/\, \Delta$ holds in all locally finite L-extensible parametric Kripke frames
- $\Gamma \,/\, \Delta$ is derivable in $L + \{\mathrm{Ext}^{\mathrm{Par}}_{C,n} : L \text{ is } \langle C, n\rangle\text{-extensible}\}$

NB: To pass from a locally finite L-extensible countermodel to a definable valuation, use projective formulas

Bases of admissible rules

Corollary: If L is clx, the rules $\mathrm{Ext}_{C,n}$ are a basis of L-admissible rules

Variation:

- Explicit single-conclusion bases
- If Par is finite:
  - L has a finite basis $\iff$ L has bounded branching
  - L has explicit independent bases (mc or sc)
- If Par is infinite:
  - No consistent logic has a finite basis
  - Open problem: independent bases?

Smaller models

L-extensible frames are usually infinite

Let $\Sigma \subseteq \mathrm{Form}$ finite, closed under subformulas

Finite models L-pseudoextensible wrt Σ:
Like L-extensible, but instead of tp's, have
tight pseudopredecessors wrt Σ
$\approx$ behave as tp as concerns satisfaction of Σ-formulas

Theorem: If L clx logic and $\Gamma, \Delta \subseteq \Sigma$, TFAE:

- $\Gamma \mathrel{|\!\!\sim}_L \Delta$
- $\Gamma \mathbin{/} \Delta$ holds in all L-pseudoextensible models wrt Σ
- $\Gamma \mathbin{/} \Delta$ holds in all L-pseudoextensible models wrt Σ of size $2^{O(n)}$

Variants of clx logics

Tweak the definition to cover other kinds of logics:

- Logics with a single top cluster (extensions of **K4.2**)
  - Top-restricted cluster-extensible (tclx) logics: extension conditions only for frames with a single top cluster
  - Examples: joins of **K4.2** with clx logics
- Superintuitionistic logics
  - Behave much like their largest modal companions (Blok–Esakia isomorphism)
  - The only (t)clx si logics are **IPC**, $\mathbf{T}_n$, **KC**, $\mathbf{KC} + \mathbf{T}_n$ (NB: $\mathbf{T}_1 = \mathbf{LC}$, $\mathbf{T}_0 = \mathbf{CPC}$)

Problems and complexity classes

Main questions

For a fixed logic L, what is the computational complexity of:

- Given Γ, Δ, is $\Gamma \mathbin{/} \Delta$ L-admissible?
- Is a given Γ L-unifiable?

Here, Γ and Δ may be sets of formulas

- without parameters
- with parameters
- with $O(1)$ parameters

Recall: unifiability is a special case of inadmissibility
$\Longrightarrow$ typically, admissibility and unifiability
captured by dual complexity classes

Ideally:

- lower bounds for unifiability
- upper bounds for admissibility

Complexity classes

Common classes of languages:

- P = deterministic polynomial time
- NP = nondeterministic polynomial time
- $\mathrm{co}X = \{\Sigma^* \smallsetminus L : L \in X\}$, e.g. coNP
- P^X = polynomial time with oracle from X, etc.
- polynomial hierarchy: $\Sigma_0^p = \Delta_0^p = \Pi_0^p = \mathrm{P}$,

$$\Sigma_k^p = \mathrm{NP}^{\Sigma_{k-1}^p}, \qquad \Delta_k^p = \mathrm{P}^{\Sigma_{k-1}^p}, \qquad \Pi_k^p = \mathrm{coNP}^{\Sigma_{k-1}^p}$$

- PSPACE = polynomial space
- EXP = deterministic exponential (2^{n^c}) time
- NEXP = nondeterministic exponential time
- exponential hierarchy:

$$\Sigma_k^{\exp} = \mathrm{NEXP}^{\Sigma_{k-1}^p}, \quad \Delta_k^{\exp} = \mathrm{EXP}^{\Sigma_{k-1}^p}, \quad \Pi_k^{\exp} = \mathrm{coNEXP}^{\Sigma_{k-1}^p}$$

Alternating Turing machines

alternating Turing machine (ATM):

- multiple transitions from a given configuration ($\approx$ NTM)
- states labelled existential or universal
- acceptance defined inductively:
  - configuration in an $\exists$ state is accepting $\iff$ $\exists$ transition to an accepting configuration
  - configuration in a $\forall$ state is accepting $\iff$ $\forall$ transitions lead to accepting configurations
- alternation: go from $\exists$ state to $\forall$ state or vice versa
- Σ_k-TIME$(f(n))$: computable by ATM in time $f(n)$, start in $\exists$ state, make $\le k-1$ alternations
- Π_k-TIME$(f(n))$: start in $\forall$ state
  - Σ_1-TIME $=$ NTIME, Π_1-TIME $=$ coNTIME

Expressing classes with ATMs

- polynomial and exponential hierarchies:

$$\Sigma_k^p = \Sigma_k\text{-TIME}\big(\mathrm{poly}(n)\big) \qquad \Pi_k^p = \Pi_k\text{-TIME}\big(\mathrm{poly}(n)\big)$$
$$\Sigma_k^{\exp} = \Sigma_k\text{-TIME}\big(2^{\mathrm{poly}(n)}\big) \qquad \Pi_k^{\exp} = \Pi_k\text{-TIME}\big(2^{\mathrm{poly}(n)}\big)$$

- PSPACE $=$ AP (alternating polynomial time)
- EXP $=$ APSPACE (alternating polynomial space)

Reductions and completeness

- Y (many-one) reducible to X if there is $f\colon \Sigma^* \to \Sigma^*$ s.t.

  $$w \in Y \iff f(w) \in X,$$

  f efficiently computable:

  - polynomial-time
  - logspace: in space $O(\log n)$, excluding input tape (read-only) and output tape (write-only)
- C a class: X is C-hard if every $Y \in C$ reduces to X
- X C-complete if $X \in C$ and C-hard

Examples:

- SAT is NP-complete, TAUT (ie, **CPC**) is coNP-complete
- QSAT is complete for AP $=$ PSPACE

Completeness in exponential hierarchy

Theorem: Fix $k \geq 1$. The set of true Σ^2_k sentences

$$\exists X_1 \subseteq \mathbf{2}^n \, \forall X_2 \subseteq \mathbf{2}^n \, \dots \, Q X_k \subseteq \mathbf{2}^n \, \overline{Q} t_1, \dots, t_m \in \mathbf{2}^n \, \varphi$$

is a $\Sigma^{\exp}_k$-complete problem, where

- $\mathbf{2} = \{0, 1\}$
- $Q = \exists$ for k odd, $\forall$ for k even
- $\overline{Q}$ = dual of Q
- n given in unary
- φ Boolean combination of atomic formulas $t_\alpha \in X_j$, $t_\alpha(i)$ ($i < n$ constant)

Complexity of derivability

Derivability and tautologicity

Before admissibility, let's consider a baseline problem:

- Given Γ, Δ, is $\Gamma \;/\; \Delta$ L-derivable?

In transitive logics, this is equivalent to L-tautologicity:

- Given φ, is $\vdash_L \varphi$?

NB: Special case of L-admissibility, but also of L-unifiability with parameters:

$$\varphi \in \mathsf{Form}(\mathrm{Par}, \varnothing) \implies (\vdash_L \varphi \iff \varphi \text{ unifiable})$$

coNP cases

Lower bound: By reduction from **CPC**,
L-derivability is coNP-hard for any consistent L

Upper bound: L-derivability is in coNP if:

- L has a polynomial-size model property
- finite L-frames are recognizable in P (or NP)

Corollary: L-derivability coNP-complete for:

- consistent linear (= width 1) clx logics
- consistent logics of bounded depth and width

PSPACE cases

Theorem [Ladner'77]
Derivability in **K**, **T**, **S4** is PSPACE-complete
For any $\mathbf{K} \subseteq L \subseteq \mathbf{S4}$, it is PSPACE-hard

Upper bound:

$\approx$ explore proof tree/countermodel one branch a time
Can be adapted (bounded branching little tricky):

Theorem: Derivability in any (t)clx logic is in PSPACE

Lower bound:

- reduction from QSAT
- easily adapted to all logics with the disjunction property
  - reference?
  - superintuitionistic logics with DP: [Chagrov'85]

PSPACE lower bound

We give another generalization, using reduction from **IPC**

Theorem: Derivability is PSPACE-hard for all logics $L \supseteq \mathbf{K4}$ that are subframe-universal for trees

- subreduction: $\approx$ p-morphism from a subframe
- weak subreduction: ignore reflexivity
- L subframe-universal for trees if $\forall$ finite tree T $\exists$ weak subreduction from an L-frame onto T
- cofinal subreduction: $\mathrm{dom}(f){\uparrow} \subseteq \mathrm{dom}(f){\downarrow}$ $\implies$ cofinal weak subreduction, cofinally subframe-universal for trees

Applications of the lower bound

Theorem: All logics $L \supseteq \mathbf{K4}$ with the disjunction property are cofinally subframe-universal for trees

Corollary:

- Derivability in $L \supseteq \mathbf{K4}$ with DP is PSPACE-hard
- Derivability in nonlinear clx logics is PSPACE-complete
- Derivability is also PSPACE-complete in nonlinear tclx logics: $\mathbf{K4.2}$, $\mathbf{S4.2}$, ...

Complexity of admissibility

Summary: clx logics

Completeness results for complexity of clx logics:

<table>
<tr><th colspan="2">logic</th><th rowspan="3">$\nvdash_L$</th><th colspan="3">unifiability, $\not\approx_L$</th><th rowspan="3">examples</th></tr>
<tr><th rowspan="2">bran-
ching</th><th rowspan="2">clust.
size</th><th colspan="3">parameters:</th></tr>
<tr><th>no†</th><th>$O(1)$</th><th>any</th></tr>
<tr><td rowspan="2">0</td><td>$< \infty$</td><td rowspan="4" colspan="2">NP</td><td rowspan="2"></td><td>Π^p_2</td><td>S5 $\oplus$ Alt$_k$, CPC</td></tr>
<tr><td>∞</td><td>coNEXP</td><td>S5, K4B</td></tr>
<tr><td rowspan="2">1</td><td>$< \infty$</td><td colspan="2">PSPACE</td><td>GL.3, LC</td></tr>
<tr><td>∞</td><td></td><td>coNEXP</td><td>S4.3, K4.3</td></tr>
<tr><td rowspan="2">≥ 2</td><td>$< \infty$</td><td rowspan="2">PSPACE</td><td rowspan="2" colspan="2">NEXP</td><td></td><td>GL, S4Grz, IPC</td></tr>
<tr><td>∞</td><td>$\Sigma^{\exp}_2$</td><td>K4, S4</td></tr>
</table>

† The parameter-free case is for $\not\approx_L$ only

Summary: tclx logics

Completeness results for complexity of tclx logics:

logic		$\nvdash_L$	unifiability, $\mathrel{|\!\!\not\sim}_L$			examples
cluster size:			parameters:			
inner	top		no†	$O(1)$	any	
$< \infty$	$< \infty$	PSPACE	NEXP			**GL**.2, **Grz**.2, **KC**
	∞				Θ_2^{exp}	**S4**.1.4 $\oplus$ **S4**.2
∞					Σ_2^{exp}	**K4**.2, **S4**.2

† The parameter-free case is for $\mathrel{|\!\!\not\sim}_L$ only

NB: branching ≥ 2 by definition

Summary: bounded depth and width

Complexity results for logics of bounded depth and width:

<table>
<tr><th colspan="2">logic</th><th rowspan="3">$\nvdash_L$
(†)</th><th rowspan="2">unif.</th><th colspan="2">$\not\mathrel{|\!\!\sim}_L$</th><th rowspan="3">notes</th></tr>
<tr><th rowspan="2">width</th><th rowspan="2">cluster size</th><th>sing.-c.</th><th>mult.-c.</th></tr>
<tr><th colspan="3">(unrestricted parameters)</th></tr>
<tr><td rowspan="2">1</td><td>$< \infty$</td><td rowspan="6">NP</td><td colspan="3">Π^p_{2d}</td><td>depth d</td></tr>
<tr><td>∞</td><td colspan="3">coNEXP</td><td></td></tr>
<tr><td rowspan="4">≥ 2</td><td>$< \infty$</td><td colspan="3">NEXP</td><td></td></tr>
<tr><td rowspan="3">∞</td><td>DEXP</td><td>$\mathrm{BH}^{\mathrm{exp}}_4$</td><td>$\mathrm{EXP}^{\mathrm{NP}[\log n]}$</td><td rowspan="3">under certain conditions</td></tr>
<tr><td colspan="3">Θ^{exp}_2</td></tr>
<tr><td colspan="3">Σ^{exp}_2</td></tr>
</table>

(†) also the complexity of unifiability and $\not\mathrel{|\!\!\sim}_L$ with $O(1)$ parameters

Upper bounds

Semantic characterization:
pseudoextensible / pruned extensible models (size $2^{\mathrm{poly}(n)}$)
$\Longrightarrow$ inadmissibility in (t)clx or bd-dp-wd logics is in Σ_2^{exp}:

$$\exists\,\mathrm{model}\ \forall\, E \subseteq \mathbf{2}^P \dots$$

Optimization in certain cases:

- bounded cluster size:
  $\forall E \subseteq \mathbf{2}^P$, $|E| \le k$ becomes a poly-size quantifier
- constant number of parameters: same reason
- width 1:
  - the model is an upside-down tree of clusters
  - an ATM can explore it while keeping only one branch

Lower bound conditions

Basic tenet: Hardness of L-unifiability stems from finite patterns that occur as subframes in L-frames

That is, study conditions of the form:
$\exists$ an L-frame that subreduces to $F \implies$ L-unifiability C-hard

Example conditions:

- L has unbounded depth:
  L-frames weakly subreduce to arbitrary finite chains
- L has unbounded cluster size:
  L-frames subreduce to arbitrary reflexive clusters
- L is nonlinear (= width ≥ 2 = branching ≥ 2):
  an L-frame subreduces to a 2-prong fork

Lower bound technique

Reduce a C-complete problem to L-unifiability:

- PSPACE, Σ^p_k/Π^p_k: QSAT, Σ_k/Π_k-SAT
- $\Sigma^{\exp}_k/\Pi^{\exp}_k$: special Σ^2_k/Π^2_k-sentences as above

Encode quantifiers:

- $\exists$ simulated by variables, $\forall$ by parameters
- $t \in \mathbf{2}^n$: directly by n-tuple of atoms
- $\forall X \subseteq \mathbf{2}^n$: parameter assignments realized in a cluster
- $\exists X \subseteq \mathbf{2}^n$: single variable x
  - use antichains to enforce consistency:
  - $u \vDash \sigma(x)$ unaffected by a change of parameters in $v \not\geq u$

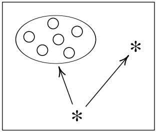
$\Sigma_2^{\exp}$ bounds

Lower bound:

If $\forall n$ an L-frame subreduces to a rooted frame containing

- an n-element cluster, and
- an incomparable point

$\implies$ L-unifiability is $\Sigma_2^{\exp}$-hard



Upper bound:

L (t)clx or bd-dp-wd logic $\implies$ L-inadmissibility is in $\Sigma_2^{\exp}$

Examples:

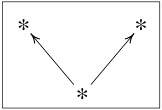
K4, S4, S4.1, S4.2, $K4BB_2$, $S4BB_2BD_2$, ...

NEXP bounds

Lower bound:

L nonlinear $\implies$ L-unifiability is NEXP-hard

($O(1)$ parameters? next slide)



Upper bounds:

- L (t)clx logic of bounded cluster size, or a tabular logic $\implies$ L-inadmissibility is in NEXP
- L (t)clx logic $\implies$ L-inadmissibility with $O(1)$ parameters is in NEXP

Examples:

GL, **K4Grz**, **S4Grz**, **S4Grz.2**, **IPC**, **KC**, ...

($\pm$ bounded branching)

NEXP lower bounds w/ $O(1)$ parameters

Need stronger hypothesis (cf. logics of bounded depth)

Theorem: The following problems are NEXP-hard

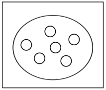
- L-unifiability with 2 parameters
  - if L subframe-universal for trees
- L-unifiability with 1 parameter
  - if L cofinally subframe-universal for trees (includes: logics with disjunction property)
  - if L subframe-universal respecting reflexivity
- L-unifiability with 0 parameters
  - if $\mathbf{K4} \subseteq L \subseteq \mathbf{K4.2GrzBB_2}$
- single-conclusion L-inadmissibility with 0 parameters
  - if L has a certain weak extension property
  - this includes: nonlinear clx logics

coNEXP bounds

Lower bound:

L unbounded cluster size

$\implies$ L-unifiability is coNEXP-hard



Upper bound:

L linear clx or bd-dp logic

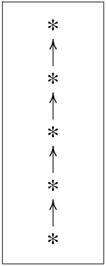
$\implies$ L-inadmissibility is in coNEXP

Examples: **S5**, **K4.3**, **S4.3**, ...

PSPACE bounds

Lower bound:
L unbounded depth $\implies$
L-unifiability with 2 parameters is PSPACE-hard

Corollary:
L-unifiability is PSPACE-hard
unless L linear tabular logic



Upper bound:
L linear clx logic of bounded cluster size
$\implies$ L-admissibility is in PSPACE

Examples:
GL.3, **K4Grz**.3, **S4Grz**.3, **LC**, ...

Polynomial hierarchy

Recall: L-unifiability PSPACE-hard unless L linear tabular

Remaining case:
L linear tabular logic of depth $d \implies$
L-unification and L-inadmissibility are Π^p_{2d}-complete

Examples:
$\mathbf{CPC}$, $\mathbf{G}_{d+1}$, $\mathbf{S5} \oplus \mathbf{Alt}_k$, $\mathbf{K4} \oplus \Box\bot$, ...

L unbounded depth $\implies$ PSPACE-hard with 2 parameters

Theorem:
L bd-dp-wd logic $\implies$
L-inadmissibility with $O(1)$ parameters is NP-complete

More exotic classes

Exponential version of Θ_2^p:

$$\Theta_2^{\mathrm{exp}} = \mathrm{EXP}^{\mathrm{NP[poly]}} = \mathrm{EXP}^{\|\mathrm{NP}} = \mathrm{P}^{\mathrm{NEXP}} = \mathrm{PSPACE}^{\mathrm{NEXP}}$$

Exponential version of the Boolean hierarchy:

- $\mathrm{BH}^{\mathrm{exp}}$ = closure of NEXP under Boolean operations
- Stratified into levels:
  - $\mathrm{BH}_1^{\mathrm{exp}} = \mathrm{NEXP}$
  - $\mathrm{BH}_{k+1}^{\mathrm{exp}} = \{A \smallsetminus B : A \in \mathrm{NEXP}, B \in \mathrm{BH}_k^{\mathrm{exp}}\}$
- Special case:
  $\mathrm{DEXP} = \mathrm{BH}_2^{\mathrm{exp}} = \{A \cap B : A \in \mathrm{NEXP}, B \in \mathrm{coNEXP}\}$

$$\mathrm{NEXP}, \mathrm{coNEXP} \subseteq \mathrm{BH}^{\mathrm{exp}} \subseteq \Theta_2^{\mathrm{exp}} \subseteq \Delta_2^{\mathrm{exp}}$$

$\Theta_2^{\exp}$ bounds

Lower bound:

$\forall n\ \exists$ $_{\text{graph-}}$connected L-frame of cluster size $\geq n$ and width ≥ 2
$\implies$ L-unifiability is $\Theta_2^{\exp}$-hard

Upper bound:

L-admissibility is in $\Theta_2^{\exp}$ if

- L is a tclx logic of bounded inner cluster size, or
- L is a bd-dp-wd logic, doesn't satisfy the $\Sigma_2^{\exp}$ LB condition

Example: $\mathbf{S4.2} \oplus \mathbf{S4.1.4}$

$\mathrm{BH}^{\mathrm{exp}}$ bounds

The NEXP and coNEXP lower bounds imply:

Lower bound:
L nonlinear logic of unbounded cluster size
$\Longrightarrow$ L-unifiability is DEXP-hard

Rare case where unifiability and inadmissibility (with parameters) have different complexity:

Upper bound:
L bd-dp-wd logic, doesn't satisfy the Θ_2^{exp} LB condition $\Longrightarrow$

- L-unifiability is in DEXP
- single-conclusion L-inadmissibility is in $\mathrm{BH}_4^{\mathrm{exp}}$
- multiple-conclusion L-inadmissibility is in $\mathrm{EXP}^{\mathrm{NP}[\log n]}$

Full classification?

Known: complexity of unifiability for (t)clx, bd-dp-wd logics

Could it be determined for all logics $L \supseteq \mathbf{K4}$?

- Hopeless:
  already $\vdash_L$ can be undecidable, arbitrary Turing degree
- The form of results that we've seen:
  - Upper bounds: tame, nicely-behaved logics
  - Lower bounds: logics allowing certain frame patterns

    $\Longrightarrow$ downward-closed classes of logics
- Determine minimal complexity of unifiability among sublogics of L?

Definition: Unifiability has hereditary hardness C below L if

- L'-unifiability is C-hard for all $L' \subseteq L$
- L'-unifiability is in C for some $L' \subseteq L$

Hereditary hardness

Theorem: Below any $L \supseteq \mathbf{K4}$, one of the following applies:

logic L			hereditary hardness of unifiability	witness $L' \subseteq L$
width	cluster size	extra condition		
1	$<\infty$	depth d	Π^p_{2d}	$= L$
		depth ∞	PSPACE	$\mathbf{K4.3BC}_k$
	∞		coNEXP	$\mathbf{K4.3}$
≥ 2	$<\infty$		NEXP	$\mathbf{K4BC}_k$
	∞	certain conditions (as before)	DEXP	$\mathbf{K4BC}_k \cap \mathbf{K4.3BIC}_k$ $\mathbf{D4.3} \cap \overline{\mathbf{D}}\mathbf{4BC}_k$ $\mathbf{D4BC}_k \cap \mathbf{D4.3BIC}_k \cap \overline{\mathbf{D}}\mathbf{4.3}$
			Θ^{exp}_2	
			Σ^{exp}_2	$\mathbf{K4}$

($\overline{\mathbf{D}}\mathbf{4} = \mathbf{K4} \oplus \Diamond\Box\bot$, $\mathbf{BIC}_k$ = bounded inner cluster size)

Except for DEXP, also applies to inadmissibility

Thank you for attention!

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