Factoring and bounded arithmetic

Emil Jeřábek

jerabek@math.cas.cz

http://math.cas.cz/~jerabek/

Institute of Mathematics of the Czech Academy of Sciences, Prague

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Outline

1. Bounded arithmetic and witnessing theorems
2. Search problems and classes
3. Quadratic reciprocity
4. Factoring and PPA

Bounded arithmetic and witnessing theorems

Proof complexity

Background: [Cook&Nguyen'10], [Krajíček'19]

The big picture: a loose 3-way correspondence between

- Propositional proof systems
- Complexity classes
- Theories of arithmetic

Proof complexity

Background: [Cook&Nguyen'10], [Krajíček'19]

The big picture: a loose 3-way correspondence between

- Propositional proof systems
  - proof systems P for classical propositional logic: a formula is a tautology $\iff$ it has a P-proof
  - P-proofs recognizable in polynomial time
  - main measure: length of proofs
    polynomial? exponential?
  - examples: resolution, sequent calculus, polynomial calculus, ...
- Complexity classes
- Theories of arithmetic

Proof complexity

Background: [Cook&Nguyen'10], [Krajíček'19]

The big picture: a loose 3-way correspondence between

- Propositional proof systems
- Complexity classes
  - depending on setup, language or search problem classes
  - AC^0, TC^0, NC^1, P, NP, Σ^p_k, PSPACE, ...
- Theories of arithmetic

Proof complexity

Background: [Cook&Nguyen'10], [Krajíček'19]

The big picture: a loose 3-way correspondence between

- Propositional proof systems
- Complexity classes
- Theories of arithmetic
  - weak first-order theories, valid in $\langle \mathbb{N}, +, \cdot \rangle$
  - one-sorted (numbers) or two-sorted (1: unary/index numbers, 2: sets $\approx$ strings $\approx$ binary numbers)
  - induction/comprehension for restricted class of formulas
  - bounded quantifiers: $\exists x < t\, \varphi(x)$, $\forall x < t\, \varphi(x)$

Theories vs. classes

Theory T corresponds to complexity class C:

- C-problems definable in $\mathbb{N}$ by formulas from Φ_C
- these definitions "provably total" in T (and "well behaved")
- Φ_C-definable problems provably total in T are in C
  - witnessing theorems
- T can "reason" with C-concepts
  - it proves induction/comprehension schemata for Φ_C

Witnessing theorems

Theorem template:

If T proves $\forall x\, \exists y\, \varphi(x, y)$, where $\varphi \in \Phi$

$\implies$ $\exists$ a function/search problem $f \in C$ s.t.

$$\mathbb{N} \vDash \forall x\, \varphi(x, f(x))$$

Prototypical example:

Theorem (Buss): If $S^1_2 \vdash \forall x\, \exists y\, \varphi(x, y)$, where $\varphi \in \Sigma^b_1$, then there is $f \in \mathrm{FP}$ s.t. $\mathbb{N} \vDash \forall x\, \varphi(x, f(x))$

- S^1_2: a theory corresponding to P
- Σ^b_1: $\approx$ NP

Applications of witnessing theorems

Common argument: T does not prove XYZ

- Assume it does
- By FGH witnessing theorem, there is $f \in C$ that computes UVW
- UVW is not computable in C because ...
  $\Longrightarrow$ contradiction

Typically, C is relativized or there are further assumptions

Constructive applications?

In principle, witnessing may work in the forward direction:

- Prove XYZ in T
- Infer the existence of a C-algorithm computing UVW

This is uncommon:

- Formalizing proofs in T is hard:
  to get anything done, we usually need to start with C-algorithms for the main steps, and then some ...
- Algorithms extracted by cut elimination from T-proofs tend to be crude and bloated
  $\Longrightarrow$ direct construction is usually more efficient
- More people work on algorithms than on bounded arithmetic $\Rightarrow$ someone would have already thought of it

Room to make a difference

Bounded arithmetic offers one advantage:
it supports (a little bit of) abstract reasoning

Illustration: two theories corresponding to P

- PV_1 (or VPV): bare-bones P-theory
  - FP-function symbols
  - defining equations, induction for P-predicates
- S^1_2 (or V^1):
  - $+$ a form of NP-induction
- Both: provably total NP-search problems $=$ FP
- S^1_2 proves: every integer has a prime factorization!

This talk

The complexity of integer factoring

- [Buresh-Oppenheim'06] Factoring of integers N s.t.

$$N \equiv 1 \pmod 4, \qquad -1 \not\equiv \Box \pmod N,$$

  can be done in the class PPA

- [J'16] Factoring of general integers has a randomized reduction to PPA
  - [J'10] Prove the quadratic reciprocity theorem in a theory corresponding to PPA
  - Apply a witnessing theorem and an easy reduction

Search problems and classes

NP search problems

This talk: complexity of search problems, not languages

NP search problems:

- defined by a poly-time relation $R(x, y)$, $|y| \leq |x|^c$
- task: input $x \longmapsto$ output some y s.t. $R(x, y)$
- FNP $=$ class of all NP search problems
- TFNP $=$ subclass of total problems: $\forall x \, \exists y \, R(x, y)$

Examples

- Factoring: composite $N \in \mathbb{N} \longmapsto$ a proper factor of N
- FullFactoring: $N \in \mathbb{N} \longmapsto N = \prod_{i<k} P_i$, P_i primes
- ModSqRoot: $N, A \in \mathbb{N} \longmapsto B$ s.t. $B^2 \equiv A \pmod{N}$

PPA

PPA [Papadimitriou'94]:
Class of TFNP problems total due to a "parity argument"

Complete problem LEAF:

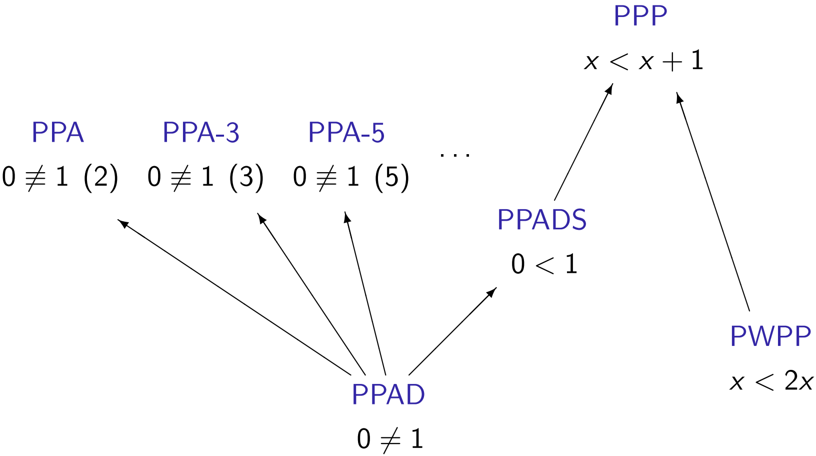
- Succinct graph G of degree ≤ 2, leaf vertex v_0
  $\longmapsto$ leaf vertex $v_1 \neq v_0$
- Total because $\sum_v \deg(v)$ is even

Alternative PPA-complete problem:

LONELY [BCEIP'98]

- Succinct partial matching on $\{0,1\}^n \smallsetminus \{0^n\}$
  $\longmapsto$ unmatched vertex

TFNP classes by counting arguments



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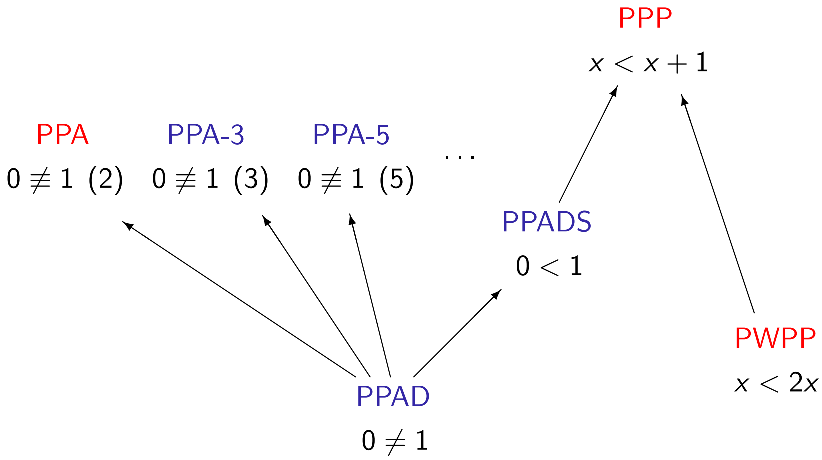

FACTORING has randomized reductions to PPA and PWPP

A theory for PPA

Recall: theory S^1_2 corresponds to FP

Axiom $Count_2(f)$: variant of LONELY

$$\forall a, p\, \exists x\, \bigl(f(x,{}_p) > 2a \lor f(f(x,{}_p),{}_p) \ne x \lor f(x,{}_p) = x\bigr)$$

"If $f(-,p)$ is an involution on $[2a+1]$, it has a fixpoint"

Theory $TPPA = S^1_2 + Count_2(PV)$ corresponds to PPA:

Theorem: If $TPPA \vdash \forall x\, \exists y\, \varphi(x,y)$, where $\varphi \in \Sigma^b_1$, then the search problem $x \longmapsto y$ s.t. $\varphi(x,y)$ is in PPA

Relies on [Buss&Johnson'12]: $\mathrm{FP}^{\mathrm{PPA}} = \mathrm{PPA}$

Quadratic reciprocity

Legendre symbol

$a \in \mathbb{Z}$, p odd prime:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a \equiv 0 \pmod{p} \\ 1 & \text{if } a \text{ quadratic residue} \pmod{p} \\ -1 & \text{if } a \text{ quadratic nonresidue} \pmod{p} \end{cases}$$

$a \mapsto \left(\frac{a}{p}\right)$ is a Dirichlet character:
p-periodic and completely multiplicative

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$$

Euler's criterion:

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$$

Quadratic reciprocity theorem

Notation: $p^* = (-1)^{\frac{p-1}{2}} p \implies p^* = \pm p$ and $p^* \equiv 1 \ (4)$

Theorem [Gauß 1801]:

- (quadratic reciprocity) If p, q are odd primes then

$$\left(\frac{q}{p}\right) = \left(\frac{p^*}{q}\right)$$

- (supplementary laws)

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv -1 \pmod 4 \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & p \equiv \pm 1 \pmod 8 \\ -1 & p \equiv \pm 3 \pmod 8 \end{cases}$$

Reciprocity laws

- quadratic reciprocity conjectured by L. Euler
- incomplete proof attempt by A.-M. Legendre
- C. F. Gauß: gives 8 proofs of "aureum theorema"
  over 240 proofs published by now (cf. [Lemmermeyer'00])
- generalized and more abstract reciprocity laws
  central topic in algebraic number theory ever since
  (Hilbert r., Artin r., Langlands program, . . .)

Jacobi symbol

$a \in \mathbb{Z}$, $n > 0$ odd, $n = \prod_{i<k} p_i^{e_i}$ prime factorization:

$$\left(\frac{a}{n}\right) = \prod_{i<k} \left(\frac{a}{p_i}\right)^{e_i}$$

- $a \mapsto \left(\frac{a}{n}\right)$ still n-periodic, completely multiplicative
- ~~Euler's criterion,~~ $\cancel{\left(\frac{a}{n}\right) = 1 \implies a \equiv \square \pmod{n}}$
- $\left(\frac{a}{n}\right) \neq 0 \iff \gcd(a, n) = 1$
- quadratic reciprocity: $n, m > 0$ odd $\implies$

$$\left(\frac{m}{n}\right) = \left(\frac{n^*}{m}\right)$$

- supplementary laws

Efficient computation of $\left(\frac{A}{N}\right)$

Definition of the Jacobi symbol $\left(\frac{A}{N}\right)$ requires factorization of N

Reciprocity $\implies$ polynomial-time algorithm ($\approx$ gcd):

$r \leftarrow 1$
while $A \neq 0$ **do**:
  if $A < 0$ **then**:
    $A \leftarrow -A$
    $r \leftarrow -r$ **if** $N \equiv -1 \pmod 4$
  while A is even **do**:
    $A \leftarrow A/2$
    $r \leftarrow -r$ **if** $N \equiv \pm 3 \pmod 8$
  swap A and N
  $A \leftarrow -A$ **if** $A \equiv -1 \pmod 4$
  reduce A modulo N so that $|A| < N/2$
if $N > 1$ **then** output 0 **else** output r

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Gauß's lemma in *TPPA*

Several proofs of QRT (Gauß 3&5, Eisenstein 3, ...) based on

Gauß's lemma: p odd prime, $P^+ = \{1, \dots, \frac{p-1}{2}\} \implies$

$$\left(\frac{a}{p}\right) = (-1)^{\#\{m \in P^+ : (ma \bmod p) \notin P^+\}}$$

- Only the parity of the set matters
- Key insight: One can witness the parity by explicit poly-time involutions!

Related work:

- [H-B'84], [Zag'90]: Fermat's 2-□ theorem by involutions
- [BI'91]: Supplementary laws by mod 8 counting principles

Quadratic reciprocity in *TPPA*

Theorem [J'10]: *TPPA* proves

- Quadratic reciprocity theorem + supplementary laws, for Legendre and Jacobi symbols
- Multiplicativity of Legendre and Jacobi symbols

Corollary:
TPPA proves soundness of the poly-time algorithm

There is a *PV*-function $J(a, n)$ s.t.

$$TPPA \vdash n > 0 \text{ odd} \implies J(a, n) = \left(\frac{a}{n}\right)$$

Factoring and PPA

Witnessing argument

Legendre symbol special case of Jacobi symbol
$\Longrightarrow$ the poly-time algorithm applies to it:

$$TPPA \vdash J(a,n) = 1 \wedge n \text{ prime} \implies a \equiv \Box \pmod{n}$$

Reformulate as a $\forall\Sigma^b_1$ statement:

$$TPPA \vdash J(a,n) = 1 \implies \exists b \left(b \text{ factor of } n \text{ or } a \equiv b^2 \pmod{n} \right)$$

$\Longrightarrow$ can apply the witnessing theorem for $TPPA$

Witnessing argument (cont'd)

Corollary

The problem FACROOT is in PPA:

- given A and odd $N > 0$ s.t. $\left(\frac{A}{N}\right) = 1$, find
  - a proper factor of N, or
  - a square root of A modulo N

Note:

- direct construction possible, but complicated (messy dynamic programming)
- [Buresh-Oppenheim'06]: the special case $A = -1$

Randomized reduction

Lemma: $N > 0$ odd, not a prime power $\Longrightarrow$ with prob. $\geq \frac{1}{4}$:
a random A is a quadratic nonresidue mod N s.t. $\left(\frac{A}{N}\right) = 1$

Corollary:

- $\textsc{Factoring}$ has a randomized poly-time reduction to $\textsc{FacRoot} \in \mathsf{PPA}$
- $\textsc{FullFactoring}$ has a randomized poly-time reduction to $\mathsf{FP}^{\textsc{FacRoot}} \subseteq \mathsf{PPA}$

Derandomization: $\textsc{FullFactoring} \in \mathsf{PPA}$
assuming a generalized/extended Riemann hypothesis
(for quadratic Dirichlet L-functions)

Unconditional deterministic results

Theorem: The following problems are in PPA:

- $N, A \in \mathbb{N} \longmapsto B$ s.t. $B^2 \equiv A \pmod{N}$ if it exists
- N odd, not a perfect square $\longmapsto A$ s.t. $\left(\frac{A}{N}\right) = -1$
- $N \geq 3 \longmapsto$ quadratic nonresidue $A \in (\mathbb{Z}/N\mathbb{Z})^\times$

Definition: N is M-strongly composite if N is a product of two quadratic nonresidues modulo M

Factoring of M-strongly composite numbers is in PPA for

- M constant, or
- $M = (\log N)^c!_{\longleftarrow \text{factorial, not exclamation}}$

Open problems

- Full unconditional derandomization:
  is FACTORING in PPA?
- Other classes:
  does FACTORING reduce to PPA-3 or PPAD?

Thank you for attention!

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